Symmetries and Conservation Laws
for Differential Equations of Mathematical Physics

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ABSTRACT. This book is devoted to the geometric theory of differential equations. It covers: ordinary differential equations and their solving by quadratures from the geometric viewpoint; the theory of classical (point) symmetries; contact geometry and its application to first-order partial differential equation; the theory of higher (generalized) symmetries with emphasis on computational techniques and demonstration of their use in solving concrete problems; conservation laws and their computation; Lagrangian formalism; Noether's theorem and relationship between symmetries and conservation laws; Hamiltonian structures on evolutionary equations; nonlocal symmetries; coverings over differential equations; symmetries of integro-differential equations.

The book is rendered as self-contained as possible and includes detailed motivations, extensive examples and exercises, along with careful proofs of all presented results.

The book is intended for readers who wish to learn the basics on applications of symmetry methods to differential equations of mathematical physics, but will also be useful for the experts because it assembles a lot of results previously dispersed in numerous journal publications. The text is accessible to advanced graduate students in mathematics, applied mathematics, mathematical and theoretical physics, mechanics, etc.

Contents

Preface

The classical symmetry theory for general systems of partial differential equations was created by Sophus Lie more than 100 years ago. The concepts of Lie groups and Lie algebras, so fundamental for modern mathematics, were discovered by S. Lie [68] during these studies. Most of S. Lie basic ideas and results on transformation groups were later worked out in numerous papers and books, while his heritage in differential equations remained aside of these developments. The first, after Lie, attempts to apply systematically the Lie theory to mechanics of continuous media were made by L. V. Ovsiannikov and his collaborators in 1950–60 (see [92]).

A new, non-Lie epoch in the symmetry theory for partial differential equations began with discovering of "completely integrable" systems and with subsequent development of the inverse scattering problem method [1, 22, 89, 28]. As it is well known, any completely integrable equation generates a whole hierarchy consisting of "higher analogs" of the initial equation. Studying these analogs made possible to understand them as symmetries of some equation. Nevertheless, this approach did not comply with the Lie theory and it was the theory of infinite jet spaces which allowed to construct the concept of "higher symmetries".

Informally speaking, classical (i.e., Lie type) symmetries are analytically described in terms of independent and dependent variables and first-order derivatives of the latter, while non-Lie symmetries may depend on arbitrary order derivatives. What is more essential is that classical infinitesimal symmetries are vector fields on the submanifold determined in the corresponding jet manifold by the initial equation, while "higher" (i.e., nonclassical) ones are cohomology classes of a natural differential complex related to the so-called infinite prolongation of the equation at hand. By this reason, a higher infinitesimal symmetry does not, in general, generate a one-parameter group of (local) diffeomorphisms on the space of its solutions. In other words, the usual relations between groups and Lie algebras cease to exist in this context. Nevertheless, it still exists "virtually" and always materializes when additional appropriate conditions arise in the problem (e.g., when boundary problems are considered). This nonclassical cohomological approach becomes even more essential when the theory of conservation laws is constructed. Note that *a priori* it was difficult even to assume that the theory of conserved quantities (integrals, fluxes, etc.) admitted by a given system of partial differential equations may be based on homology algebra and uses the theory of spectral sequences as its main technique.

Foundations of the theory of higher symmetries and conservation laws were developed by one of us in 1975–77 [60, 127, 129]. Later on, it was tested in particular computations and sufficiently efficient computational methods were obtained.

These methods were applied to compute higher symmetries and conservation laws for particular systems of nonlinear partial differential equation [**142**, **137**].

Besides, the most important theoretical details were established [**130**] and revealed striking parallels between the calculus of vector fields and differential forms on smooth manifolds, on one hand, and that of symmetries and conservation laws, on the other hand. This observation gave birth to a hypothesis that these parallels are of a much more general nature and can be extended to all concepts of differential calculus. Subsequent analysis led to discovery of *Secondary Differential Calculus*, which is both a powerful instrument to study general systems of partial differential equations and a natural language to construct modern quantum field theory on a nonperturbative basis.

Higher symmetries and conservation laws are "local" quantities, i.e., they depend on unknown functions (or "fields", if one uses physical terminology) and on their derivatives. This framework, however, becomes insufficient to describe some important concepts, such as Bäcklund transformations or recursion operators: the reason is that in most interesting cases they operate with "nonlocal" quantities (e.g., integrals of local objects). Note also that integro-differential equations, where nonlocal quantities are present from the very beginning, are of independent interest for the theory of symmetries and conservation laws. The desired extension of the theory is achieved by introducing the notion of coverings [**62**] in the category of differential equations.

The fundamentals of the higher symmetry theory[1] and the corresponding computational algorithms are elaborated sufficiently well now. Concerning the latter, one needs to distinguish between algorithmic techniques oriented to computer applications (see, for example, the collection [**137**]) and analytical methods of studying various model systems. It this context, it is necessary to mention works of the "Ufa school" (see, for example, [**85**, **116**, **112**] and many other publications). The results of analysis for particular equations and systems are scattered over numerous publications. The most representable (but unfortunately becoming more and more obsolete) digest can be found in [**44**, **45**, **46**].

There exists now a lot of books dealing with geometrical aspects of differential equations and, in particular, with their symmetries (see [**13**, **4**, **148**, **43**, **92**, **91**, **114**]). The most consistent exposition of geometrical and algebraic foundations of symmetry and conservation law theory for partial differential equations contains in the monograph [**60**] and paper [**132**], but hardly can be considered as "user-friendly" texts. Their deficiencies are partially compensated by the paper [**131**] and by the works [**61**, **62**] (the last two deal with the nonlocal theory). But they do not fill the gap completely.

Therefore the idea to write a more popular and at the same time mathematically rigorous text on symmetries and conservation laws aroused immediately after publication of the book [**60**]. Some chapters of the book you hold now were written then: S. V. Duzhin wrote a text on symmetries of ordinary differential equations and a draft version of the chapter on classical symmetries, A. V. Bocharov wrote the chapter on symmetries of first-order equations, while I. S. Krasil'shchik prepared a text on higher symmetries. By a number of "historical" reasons, this project could not come to reality then and we returned back to it in ten years only.

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[1]In modern literature, the terms *generalized symmetries* (see [**91**]) and *Lie–Bäcklund symmetries* ([**43**]) are also used.

A lot of things changed during this period, our attitudes to some aspects of algebro-geometric foundations of the symmetry theory included. By this reason, some old texts were considerably updated: using A. V. Bocharov's text, Yu. N. Torkhov wrote the final version of Chapter 2, Chapter 3, in its present form, belongs to A. V. Samokhin, I. S. Krasil′shchik prepared a renewed version of Chapter 4. We also decided that the book would be incomplete without mentioning conservation law theory and the corresponding chapter was written by A. M. Verbovetsky. Finally, Chapter 6 was written by N. G. Khor′kova and V. N. Chetverikov (the section on symmetries of integro-differential equations belongs to him). We also supplied the book with an Appendix containing adequately adapted text of A. M. Vinogradov's paper on Secondary Differential Calculus describing a deeper perspective on geometric theory of partial differential equations.

In this book, we expose the basics on the theory of higher and nonlocal symmetries for general systems of partial differential and integro-differential equations. The topic of the book concentrates on the following questions:

- What are higher symmetries and conservation laws admitted by a given system of differential equation?
- What are efficient methods of their computation?
- If we found a symmetry or a conservation law, how to use them?

Concerning the last question, we had to restrict ourselves with the simplest and most straightforward applications of the theory constructed. A more detailed exposition would need another couple of volumes (which we hope to write in future).

We tried to take into account interests of two groups of readers of this book: first, of those who are mainly interested in theoretical aspects and, second, of those ones who are interested in applications. To the latter (and we believe that those who work in theoretical and mathematical physics, field theory, continuous media mechanics, belong to this group) we advise to skip proofs and general conceptual discussions and mostly pay attention to algorithms and techniques needed in particular computations. We hope that exposition of these matters is sufficiently clear. On the other hand, we saw no possibility to make exposition more popular by passing to the standard in mathematical physics language of local coordinates. For understanding the conceptual part of the theory, as well as for efficient use of the corresponding algorithms, such an approach would be murderous. We can state here that a lot of works concerning with the topics of this book are fighting with coordinate difficulties rather than with real problems.

We hope that having read this book you will not only be able to look from a different point of view at various problems related to nonlinear differential equations, but will take a pen and paper (or will switch on your computer and load one of accessible packages for symbolic computations) to find something new and interesting in your own professional problems. Formally speaking, to start looking for symmetries and conservation laws, it suffices to know mathematics in the limits of the standard university course and use the formulas exposed in the book. To understand material deeper, one needs knowledge on:

- geometry of smooth manifolds and vector bundles over these manifolds [**3, 95, 25, 115, 146**],
- symplectic geometry [**143, 7**],
- theory of Lie groups and Lie algebras [**122, 94, 105**],
- commutative algebra [**9**],

- homology algebra [**15, 31, 35, 79, 84**].

In addition, by "philosophical reasons", we also recommend to get at least preliminary knowledge in algebraic geometry [**108**] and category theory [**31, 79**].

Since the fall of sixties, the authors of this book are participants of the seminar on algebraic and geometric foundations of nonlinear differential equations at the Faculty of Mathematics and Mechanics of Moscow State University. On behalf of all authors, we would like to express our gratitude to all the participants who helped us both in forming a general approach and in clarifying some particular problems. We are especially grateful to: D. M. Gessler, V. V. Lychagin, V. E. Shemarulin, M. M. Vinogradov, and V. A. Yumaguzhin. We are also grateful to one of our first Russian readers, A. V. Shepetilov, for his useful remarks on some errors in the book.

Finally, we would like to point out that this book can be considered as an introduction to the monograph series on the foundations of secondary calculus launched by the Diffiety Institute of Russian Academy of Natural Sciences. Hot information on the Diffiety Institute activities can be found on the World Wide Web starting at URL:

                    `http://www.botik.ru/~diffiety/`

or

                    `http://ecfor.rssi.ru/~diffiety/`

                                                        A. M. Vinogradov
                                                        I. S. Krasil′shchik

Ordinary Differential Equations

Of all differential equations, two classes, ordinary differential equations and scalar partial differential equations of order one, stand out by a number of features facilitating their study and making the theory more readily applicable. This chapter discuss the geometric approach to ordinary differential equations. Here we first introduce, in the simplest situation, the concepts that will be extensively used throughout the book. We also show that the geometric theory of symmetries makes it possible to understand and generalize the standard procedures of explicit integration of ordinary differential equations and to obtain new results in this direction as well.

1. Ordinary differential equations from the geometric viewpoint

It is well known (see, e.g., [**6**]) that a first-order ordinary differential equation resolved for the derivative

$$u' = f(x, u) \tag{1.1}$$

can be geometrically interpreted as a vector field on the (x, u)-plane. To this end, at each point (x_0, u_0) one has to consider the vector $(1, f(x_0, u_0))$ and the operator $\partial/\partial x + f(x_0, u_0)\partial/\partial u$ of derivation in the direction of this vector. The trajectories of the field $\partial/\partial x + f(x, u)\partial/\partial u$ are called the *integral curves* of equation (1.1). They are graphics of solutions of the equation under consideration (see Figure 1.1).

To interpret the equation

$$F(x, u, u') = 0 \tag{1.2}$$
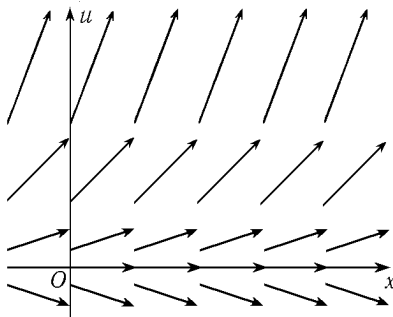
FIGURE 1.1. The vector field $X = \partial/\partial x + u\partial/\partial u$ corresponding to the differential equation $u' = u$
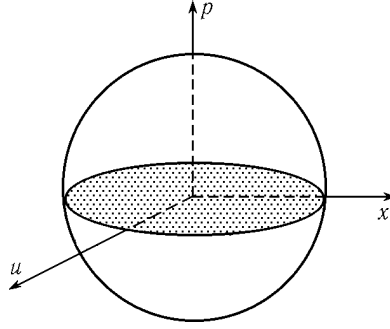
FIGURE 1.2

in the same fashion, one needs to solve it for the derivative u'. When doing so, the following difficulty is encountered: several values of u' (or no values at all) may correspond to some values of x and u by (1.2). Moreover, in a neighborhood of a singular point (that is, of a point where $\partial F/\partial u' = 0$ and, hence, the implicit function theorem is not valid), the function expressing u' in terms of x and u, even though it is well-defined, may not be smooth.

EXAMPLE 1.1. Let

$$F(x, u, u') = u'^2 + u^2 + x^2 - 1.$$

Then $u' = \pm\sqrt{1 - x^2 - u^2}$. This function is defined on the disk $x^2 + u^2 \leq 1$. It is two-valued inside the disk and its derivatives become infinite on the boundary (see Figure 1.2).

These difficulties can be overcome in the following way. Consider the space $\mathbb{R}^3$ with the coordinates x, y, p and the surface $\mathcal{E}$ given by $F(x, u, p) = 0$. To every solution $u = f(x)$ of equation (1.2) there corresponds the curve on this surface defined by the equations

$$u = f(x), \qquad p = f'(x). \tag{1.3}$$

The coordinate x can be taken as a parameter, i.e., the projection of this curve to the x-axis is a diffeomorphism. Furthermore, the functions expressing the coordinate u and p in terms of x are not arbitrary: the latter is the derivative of the former. Consequently, not any curve in $\mathbb{R}^3$ can be written in the form (1.3).

Since $p_0 = f'(x_0)$, the tangent vector to the curve (1.3) at a point $a = (x_0, u_0, p_0)$ takes the form

$$\frac{\partial}{\partial x} + p_0 \frac{\partial}{\partial u} + f''(x_0) \frac{\partial}{\partial p}. \tag{1.4}$$

Therefore this vector lies in the plane given by the equation

$$u - u_0 = p_0(x - x_0), \tag{1.5}$$

i.e., it belongs to the kernel of the 1-form

$$\omega = du - p\, dx \tag{1.6}$$

at the point a. It is also obvious that the span of all vectors of the form (1.4) coincides with the entire plane (1.5).

The inverse statement holds as well: a curve in $\mathbb{R}^3$ projecting diffeomorphically to the x-axis and integral for the 1-form ω (or, which is the same, for the distribution $\mathcal{C}$ of codimension 1 given by the form ω) has the form (1.3).

Thus, the two-dimensional distribution $\mathcal{C}$ given by 1-form (1.6) is the geometric structure which distinguishes in a natural way the class of curves corresponding to solutions of ordinary differential equations of the first order. This distribution $\mathcal{C}$ is called the *Cartan distribution*.

By means of the Cartan distribution, solutions of a differential equation $\mathcal{E} \subset \mathbb{R}^3$ can be interpreted as integral curves of the distribution $\mathcal{C}$ belonging to the surface $\mathcal{E}$ and projecting to the x-axis without degeneration.

Note that the 2-form $d\omega = dx \wedge dp$ cannot be written as $\gamma \wedge \omega$, where γ is an 1-form. Hence, by the Frobenius theorem (see, e.g., [**115, 146**]), the Cartan distribution is not completely integrable. Therefore its maximal integral manifolds are one-dimensional, and the set of points where the plane of the Cartan distribution is tangent to the surface $\mathcal{E}$ is a closed nowhere dense subset of $\mathcal{E}$. We call such points *singular*. Thus, the set of nonsingular points of the surface $\mathcal{E}$ is open and everywhere dense.

Singular points of equation (1.2) can be found from the condition that the differential dF and the form ω are collinear at these points. This condition can be written as two relations:

$$\frac{\partial F}{\partial p} = 0, \qquad \frac{\partial F}{\partial x} + p\frac{\partial F}{\partial u} = 0.$$

If $a \in \mathcal{E}$ is a nonsingular point, then the intersection of the plane tangent to the surface $\mathcal{E}$ at the point a with the plane $\mathcal{C}_a$ of the distribution $\mathcal{C}$ is a straight line l_a tangent to $\mathcal{E}$ at the point a. Thus on $\mathcal{E}$ there arises the direction field (one-dimensional distribution) $a \mapsto l_a$, which will be denoted by $\mathcal{C}(\mathcal{E})$. A curve Γ is integral for the direction field $\mathcal{C}(\mathcal{E})$ if and only if Γ is an integral curve of the distribution $\mathcal{C}$ and lies on $\mathcal{E}$ (with the exception of the set of singular points). Therefore, it can be concluded that solutions of the equation $\mathcal{E}$ are integral curves of the direction field $\mathcal{C}(\mathcal{E})$ projecting to the x-axis without degeneration.

Note that arbitrary integral curves of the direction field $\mathcal{C}(\mathcal{E})$, which do not necessarily satisfy the condition of nonsingularity of projection to the x-axis, can be interpreted as multi-valued solutions of the equation $\mathcal{E}$.

EXAMPLE 1.2. The curve given by the equations

$$\begin{cases} u^3 - u + x = 0, \\ 3u^2p - p + 1 = 0, \end{cases}$$

lies on the surface

$$(3x - 2u)p = u$$

(see Figure 1.3) and is integral for the Cartan distribution, but it is not of the form (1.3). Hence this curve is a multi-valued solution to the equation $(3x-2u)u' = u$.

EXAMPLE 1.3. From the geometric point of view, the equation

$$\left(\frac{du}{dx}\right)^2 + u^2 = 1$$

is the cylinder $\mathcal{E}$ (see Figure 1.4) in the space $\mathbb{R}^3$ with coordinates x, u, p defined
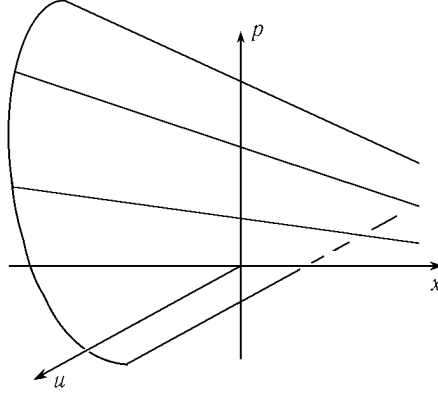
FIGURE 1.3. The graph of the surface $(3x - 2u)p = u$ for $p \geq -1/2$

by the relation $p^2 + u^2 = 1$. Consider the coordinate system (x, φ) on this cylinder, with φ being defined by

$$u = \sin \varphi, \qquad p = \cos \varphi.$$

Then the restriction of $\omega = du - p \, dx$ to $\mathcal{E}$ is

$$\omega_{\mathcal{E}} = \cos \varphi \, d(\varphi - x).$$

The points of tangency of the distribution $\mathcal{C}$ and the cylinder $\mathcal{E}$ are those where $\omega_{\mathcal{E}}$ vanishes. It is obvious that these points are situated on the two straight lines obtained by intersection of $\mathcal{E}$ with the plane $p = 0$. These are singular solutions $u = \pm 1$ to the equation at hand. Further, the equation $\omega_{\mathcal{E}} = 0$ reduces to the equation $d(\varphi - x) = 0$ and implies that the integral curves of the distribution $\mathcal{C}(\mathcal{E})$ are the circular helices

$$\varphi = x + c, \qquad c \in \mathbb{R}.$$

All these curves project to the x-axis without degeneration and give rise to usual solutions. Expressing φ in terms of u, these solutions can be written in the following explicit form:

$$u = \sin(x + c).$$
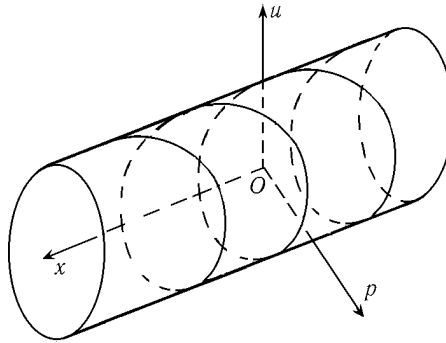
FIGURE 1.4
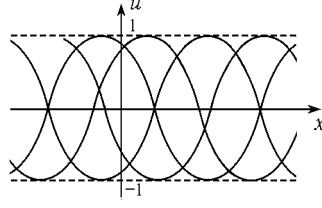
FIGURE 1.5
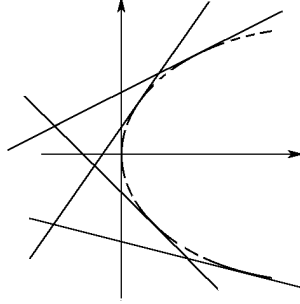
FIGURE 1.6

Projecting to the (x, u)-plane, the solutions fill the region $|u| \leq 1$ (see Figure 1.5).

The singular solutions (the straight lines $u = \pm 1$) are envelopes of the sine curves.

EXAMPLE 1.4. Consider the Clairaut equation

$$u - x\frac{du}{dx} = f\left(\frac{du}{dx}\right),$$

where f is a smooth function. The form of the surface $\mathcal{E} = \{u - xp = f(p)\}$ depends on the choice of f. As coordinates on $\mathcal{E}$, we can take the variables x, p. The form ω in these coordinates is written as

$$\omega_{\mathcal{E}} = d(xp + f(p)) - p\,dx = (x + f'(p))\,dp.$$

This form vanishes at the points of the surface $\mathcal{E}$ belonging to the curve $x = -f'(p)$. The projection of this curve to the (x, u)-plane will be the graph of a singular solution of the Clairaut equation. This solution can be obtained in analytic form by solving the equation $x = -f'(p)$ for p and substituting the result to the equation $u - xp = f(p)$. For example, if $f(p) = p^\alpha$, $\alpha < 0$, then

$$u = (1 - \alpha)\left(-\frac{x}{\alpha}\right)^{\frac{\alpha}{\alpha - 1}}.$$

Note that a singular solution can be multi-valued (e.g., $u = \pm 2\sqrt{x}$ for $\alpha = -1$, see Figure 1.6).

The integral curves of the distribution $\mathcal{C}(\mathcal{E})$ at nonsingular points can be found from the equation $\omega_{\mathcal{E}} = 0$ subject to the condition $x + f'(p) \neq 0$. This yields $dp = 0$, $p = c$, and $u = cx + f(c)$, where $c \in \mathbb{R}$.

As in Example 1.3, we see that singular solutions are envelopes of one-parameter families of nonsingular solutions.
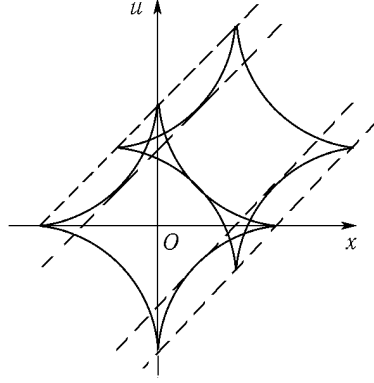
FIGURE 1.7

Let us remark that the formal procedure for searching the envelope, i.e., elimination of p from

$$F(x, u, p) = 0,$$
$$\frac{\partial F(x, u, p)}{\partial p} = 0, \qquad (1.7)$$

can yield a curve with cusps and points of tangency to integral curves[1].

Consider, for instance, the equation [**100**]

$$(x - u)^2 \left(1 + \left(\frac{du}{dx} \right)^2 \right)^3 = a^2 \left(1 + \left(\frac{du}{dx} \right)^3 \right)^2, \qquad a > 0.$$

It can easily be checked that the functions defined by

$$(x - c)^{2/3} + (y - c)^{2/3} = a^{2/3},$$

are solutions to this equation, the solutions of (1.7) being of the form

$$x - u = \pm a \quad \text{or} \quad x - u = \pm a/\sqrt{2}.$$

This family of curves contains envelopes of solutions as well as cusps and points of tangency to integral curves (see Figure 1.7).

2. Ordinary differential equations of arbitrary order

By analogy with §1, we can develop the geometric theory of ordinary differential equations of arbitrary order.

The equation

$$F \left(x, u, \frac{du}{dx}, \ldots, \frac{d^k u}{dx^k} \right) = 0$$

can be interpreted as a hypersurface in the $(k + 2)$-dimensional space $\mathbb{R}^{k+2}$ with coordinates x, u, $p_1, \ldots, p_k$ defined by $F(x, u, p_1, \ldots, p_k) = 0$.

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[1]For more details on finding envelopes see [**27, 97**]. A detailed discussion of singular solutions to differential equations is found in [**113, 100, 47**].

Consider the 2-dimensional distribution in $\mathbb{R}^{k+2}$ given by

$$\omega_0 = du - p_1 \, dx,$$
$$\omega_1 = dp_1 - p_2 \, dx,$$
$$\dots\dots\dots\dots\dots$$
$$\omega_{k-1} = dp_{k-1} - p_k \, dx.$$

For $k = 1$, this distribution coincides with the Cartan distribution discussed in §1, so we shall likewise call it the *Cartan distribution* and denote by $\mathcal{C}$.

Note that a curve in $\mathbb{R}^{k+2}$ is the graph of a function $u = f(x)$ and of its derivatives

$$L_f = \{\, u = f(x), \; p_1 = f'(x), \dots, p_k = f^{(k)}(x) \,\}$$

if and only if it projects to the x-axis without degeneration and is an integral curve for the distribution $\mathcal{C}$ (in §1 we proved this fact for $k = 1$).

Using the Cartan distribution $\mathcal{C}$, one can give a geometric interpretation to the well-known method for integrating of ordinary differential equations not containing the independent variable in explicit form.

EXAMPLE 2.1 (see also [**91**]). Let us consider a second-order equation of the form $F(u, du/dx, d^2u/dx^2) = 0$ or, in other words, a hypersurface $\mathcal{E}$ in $\mathbb{R}^4$ defined by

$$F(u, p_1, p_2) = 0.$$

Restrict the distribution given by 1-forms

$$\omega_0 = du - p_1 \, dx, \quad \omega_1 = dp_1 - p_2 \, dx \tag{2.1}$$

to this hypersurface. The hypersurface $\mathcal{E}$ and the forms ω_0, ω_1 (and, hence, the distribution $\mathcal{C}$) are invariant under translations along the x-axis, i.e., transformations $t_c \colon (x, u, p_1, p_2) \mapsto (x + c, u, p_1, p_2)$, where c is a constant. This makes it possible to construct the quotient equation by the variable x of which the equation is independent (more precisely, by the action of one-parameter group T of parallel translations t_c).

In other words, let us consider the quotient mapping π

$$\pi \colon (x, u, p_1, p_2) \longmapsto (u, p_1, p_2)$$

from $\mathbb{R}^4$ to the 3-dimensional space $\mathbb{R}^4/T$. The image of the hypersurface $\mathcal{E}$ under this mapping is the hypersurface $\mathcal{E}/T$ in the space $\mathbb{R}^4/T$ given by the same equation $F(u, p_1, p_2) = 0$. The distribution $\mathcal{C}$ projects without degeneration to the space $\mathbb{R}^4/T$ and, thus, gives rise to the two-dimensional quotient distribution $\mathcal{C}/T$. This distribution can be defined by a 1-form on $\mathbb{R}^4/T$: for example, one can take the following linear combination of the forms (2.1):

$$\widetilde{\omega} = \omega_1 - \frac{p_2}{p_1}\omega_0 = dp_1 - \frac{p_2}{p_1}\, du.$$

Thus, as a result of the factorization, we obtain the 3-dimensional space $\mathbb{R}^4/T$, endowed with the 2-dimensional distribution given by $\widetilde{\omega} = 0$, and the 2-dimensional surface $\mathcal{E}/T \subset \mathbb{R}^4/T$, i.e., an ordinary first-order differential equation. This will be

in complete agreement with §1 if we identify the form $\widetilde{\omega}$ with the canonical form ω. To this end, let us introduce the coordinates

$$x' = u, \quad u' = p_1, \quad p' = \frac{p_2}{p_1} \tag{2.2}$$

on $\mathbb{R}^4/T$. Then we have $\widetilde{\omega} = du' - p'dx'$. In the new coordinates, the equation of $\mathcal{E}/T$ takes the form

$$F'(x', u', p') = 0,$$

where $F'(x', u', p') = F(x', u', p'u')$.

Thus, the quotient mapping π takes the initial equation of order two to a first-order equation

$$F'\left(x', u', \frac{du'}{dx'}\right) = 0,$$

which is called the *quotient equation*. Therein lies the geometric meaning of the change of coordinates (2.2).

REMARK 2.1. The distribution $\mathcal{C}/T$ can be defined by another form. Consider, for example, the form

$$\widetilde{\omega} = \omega_0 - \frac{p_1}{p_2}\omega_1 = du - \frac{p_1}{p_2}dp_1 = du - \frac{1}{p_2}d\left(\frac{p_1^2}{2}\right).$$

To identify it with the canonical form ω, we are to do the following change of coordinates

$$x' = \frac{p_1^2}{2}, \quad u' = u, \quad p' = \frac{1}{p_2},$$

which reduces the initial equation to the first-order equation

$$F\left(u', \pm\sqrt{2x'}, \frac{dx'}{du'}\right) = 0.$$

In a similar manner, one can reduce by 1 the order of any equation not explicitly depending on x. Factorizing by x the space $\mathbb{R}^{k+2}$ with the coordinates $x, u, p_1, \ldots, p_k$, we obtain the space $\mathbb{R}^{k+2}/T$ with the coordinates $u, p_1, \ldots, p_k$. The image of the Cartan distribution $\mathcal{C}$ on $\mathbb{R}^{k+2}$ under this factorization is the two-dimensional distribution $\mathcal{C}/T$ on $\mathbb{R}^{k+2}/T$ given by

$$\widetilde{\omega}_0 = \omega_1 - \frac{p_2}{p_1}\omega_0 = dp_1 - \frac{p_2}{p_1}\,du,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\widetilde{\omega}_{k-2} = \omega_{k-1} - \frac{p_k}{p_1}\omega_0 = dp_{k-1} - \frac{p_k}{p_1}\,du.$$

We claim that the coordinates $u, p_1, \ldots, p_k$ in the space $\mathbb{R}^{k+2}/T$ can be changed for new coordinates $x', u', p_1', \ldots, p_{k-1}'$, such that the quotient distribution $\mathcal{C}/T$ is given by

$$\widetilde{\omega}_0' = du' - p_1'\,dx', \ldots, \widetilde{\omega}_{k-2}' = dp_{k-2}' - p_{k-1}'\,dx'.$$

Indeed, define x', u', and $p' = p_1'$ by formulas (2.2). Since

$$dp_1' = d\left(\frac{p_2}{p_1}\right) = \frac{1}{p_1}\,dp_2 - \frac{p_2}{p_1^2}\,dp_1,$$

we can put

$$\widetilde{\omega}'_1 = \frac{1}{p_1}\widetilde{\omega}_1 - \frac{p_2}{p_1^2}\widetilde{\omega}_0 = dp'_1 - \left(\frac{p_3}{p_1^2} - \frac{p_2^2}{p_1^3}\right) du.$$

Therefore

$$p'_2 = \frac{p_3}{p_1^2} - \frac{p_2^2}{p_1^3}.$$

Continuing this line of reasoning, we obtain the desired change of variables. If we replace the variables $u, \ldots, p_k$ in the initial k-order equation by $x', u', \ldots, p'_{k-1}$

$$F(u, p_1, \ldots, p_k) = 0,$$

we get the quotient equation of order $k - 1$.

As it was noted above, the restriction $\mathcal{C}(\mathcal{E})$ of the two-dimensional Cartan distribution on $\mathbb{R}^{k+2}$ to the hypersurface $\mathcal{E}$ is one-dimensional at all points except for a nowhere dense set of singular points where the tangent plane to $\mathcal{E}$ contains the plane of $\mathcal{C}$. Therefore, the distribution $\mathcal{C}(\mathcal{E})$ can be defined by one vector field. Let us describe explicitly such a field and establish the conditions distinguishing between singular and nonsingular points.

Let

$$Y = \alpha\frac{\partial}{\partial x} + \beta_0\frac{\partial}{\partial u} + \beta_1\frac{\partial}{\partial p_1} + \cdots + \beta_k\frac{\partial}{\partial p_k}$$

be a field lying in the distribution $\mathcal{C}$ and tangent to the surface $\mathcal{E}$. Since the field Y belongs to the Cartan distribution, we have

$$\omega_i(Y) = 0, \qquad i = 0, 1, \ldots, k - 1,$$

i.e., for all i except for $i = k$ we have $\beta_i = p_{i+1}\alpha$. The requirement that the field Y is tangent to $\mathcal{E}$ is equivalent to the condition $Y(F)|_{\mathcal{E}} = 0$, i.e., to existence of a function λ such that

$$\alpha\left(\frac{\partial F}{\partial x} + p_1\frac{\partial F}{\partial u} + p_2\frac{\partial F}{\partial p_1} + \cdots + p_k\frac{\partial F}{\partial p_{k-1}}\right) + \beta_k\frac{\partial F}{\partial p_k} = \lambda F. \qquad (2.3)$$

Obviously, we can satisfy this condition by putting

$$\lambda = 0, \quad \alpha = -\frac{\partial F}{\partial p_k}, \quad \beta_k = \frac{\partial F}{\partial x} + p_1\frac{\partial F}{\partial u} + \cdots + p_k\frac{\partial F}{\partial p_{k-1}}.$$

Thus, the desired vector field can be written as

$$Y_F = -\frac{\partial F}{\partial p_k}\left(\frac{\partial}{\partial x} + p_1\frac{\partial}{\partial u} + \cdots + p_k\frac{\partial}{\partial p_{k-1}}\right)$$
$$+ \left(\frac{\partial F}{\partial x} + p_1\frac{\partial F}{\partial u} + \cdots + p_k\frac{\partial F}{\partial p_{k-1}}\right)\frac{\partial}{\partial p_k}.$$

In particular, for a first-order equation $F(x, u, p) = 0$, the field Y_F takes the form

$$Y_F = -\frac{\partial F}{\partial p}\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial u}\right) + \left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial u}\right)\frac{\partial}{\partial p}. \qquad (2.4)$$

If the equation is solved with respect to the derivative, i.e., $F = -p + f(x, u)$, then $\mathcal{E} = \{p = f(x, u)\}$ and

$$Y_F = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + (f_x + pf_u)\frac{\partial}{\partial p}.$$
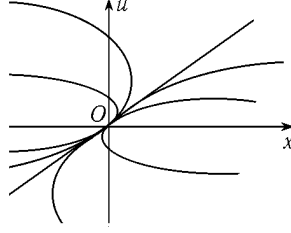
FIGURE 1.8

The projection of this field to the (x, u)-plane has the form

$$\frac{\partial}{\partial x} + f(x, u)\frac{\partial}{\partial u}.$$

The trajectories of the field obtained are, as it is well known, integral curves for the equation at hand.

The point P of the equation $\mathcal{E}$ is singular if relation (2.3) at this point holds for all α and β_k. Since the right-hand side of this relation vanish on $\mathcal{E}$, the equalities

$$F = \frac{\partial F}{\partial x} + p_1\frac{\partial F}{\partial u} + \cdots + p_k\frac{\partial F}{\partial p_{k-1}} = \frac{\partial F}{\partial p_k} = 0$$

are necessary and sufficient condition for the point under consideration to be singular. Thus, the singular points of the hypersurface $\mathcal{E}$ are those where the field Y_F vanishes.

The field Y_F is called the *characteristic field* of the equation $\mathcal{E}$. The projections of its trajectories to the (x, u)-plane are the graphs of solutions of this equation.

EXAMPLE 2.2. Consider the equation $(3x - 2u)u' = u$. Then $F = (3x - 2u)u' - u$, and by formula (2.4) we have

$$Y_F = (2u - 3x)\frac{\partial}{\partial x} + p(2u - 3x)\frac{\partial}{\partial u} + (2p - 2p^2)\frac{\partial}{\partial p}$$

$$= (2u - 3x)\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} + (2p - 2p^2)\frac{\partial}{\partial p}$$

(the last equality holds on the equation, i.e., on the surface $\mathcal{E} = \{(3x - 2u)p = u\}$). Let us take x and u as coordinates on $\mathcal{E}$ and find the trajectories of the field Y_F from the system of equations

$$\begin{cases} \dot{x} = 2u - 3x, \\ \dot{u} = -u. \end{cases}$$

Solving this system, we get $x = C_1 e^{-t} + C_2 e^{-3t}$, $u = C_1 e^{-t}$. Hence $x = u + au^3$, $a = \text{const}$. The family of trajectories of Y_F is shown in Figure 1.8. Note that at the point $(0, 0, 0)$, which is a singular point of the equation under consideration, the uniqueness theorem for solutions of ordinary differential equations is not valid.

3. Symmetries of distributions

Since solutions of ordinary differential equations (including the multi-valued ones) are integral curves for the corresponding Cartan distributions, we begin the study of symmetries of differential equations with discussing symmetries in a more general context of distributions.
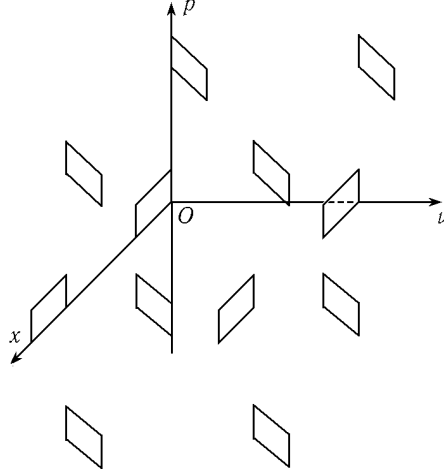
FIGURE 1.9. The Cartan distribution $\mathcal{C}$ on $\mathbb{R}^3$

Recall (see [**115, 146**]) that a p-dimensional *distribution* P on a manifold M of dimension n is a correspondence $a \mapsto P_a$ that takes each point $a \in M$ to a p-dimensional subspace P_a of the tangent space $T_a M$. The distribution P is called *smooth*, if for any point $a \in M$ there exist a neighborhood $U \ni a$ and p smooth vector fields $X_1, \ldots, X_p$ generating the distribution at every point of the neighborhood.

Let P be a smooth distribution on a smooth manifold M.

DEFINITION 3.1. A diffeomorphism $\varphi \colon M \to M$ is called a (finite) *symmetry* of the distribution P if it preserves this distribution, i.e., $\varphi_*(P_a) \subset P_a$ for all $a \in M$.

EXAMPLE 3.1. The Cartan distribution on the space $\mathbb{R}^3$ (see Figure 1.9) with coordinates x, u, p is given by the 1-form $\omega = du - p\,dx$. It is obviously invariant under translations along x and along u, i.e., the diffeomorphism $\varphi_1 = (x+a, u+b, p)$ is a symmetry of $\mathcal{C}$ for all a and b. Note that $\varphi_1^*(\omega) = \omega$. Translations $\varphi_2 = (x, u, p+c)$ along p are not symmetries. Indeed, $\varphi_1^*(\omega) = du - (p+c)\,dx$ and this form is not proportional to ω.

The distribution under study has less trivial symmetries as well.

For example, it can easily be checked that the so-called Legendre transformation $\varphi(x, u, p) = (p, u - xp, -x)$ is a symmetry, since it preserves the form ω.

The transformation

$$\psi \colon \mathbb{R}^3 \to \mathbb{R}^3, \qquad \psi(x, u, p) = (p, xp - u, x),$$

resembles the previous one, but does not preserve the form ω, since $\psi^*(\omega) = -\omega$. Nevertheless, the transformation ψ is a symmetry of the distribution $\mathcal{C}$.

The set of all symmetries will be denoted by $\operatorname{Sym} P$. It is obvious that the composition of two symmetries is a symmetry again. The map inverse to a symmetry is also a symmetry. Thus, $\operatorname{Sym} P$ is the group with respect to the composition.

Let P D be the $C^\infty(M)$-module of vector fields X such that the vector X_a belongs to P_a for all $a \in M$.

Suppose that the distribution P is generated by fields $X_1, \ldots, X_l \in P$ D. Definition 3.1 is equivalent to $\varphi_*(X_i) \in P$ D for all $i = 1, \ldots, l$ or, which is the same,

$$\varphi_*(X_i) = \sum_j \mu_{ij} X_j, \qquad i = 1, \ldots, l, \tag{3.1}$$

for some functions μ_{ij} on M.

Assume that the differential forms $\omega_1, \ldots, \omega_k$, where $\omega_i = \sum_j \omega_{ij}\, dx_j$, $\omega_{ij} \in C^\infty(M)$, generate the distribution P. The condition $\varphi \in \operatorname{Sym} P$ means that

$$\varphi^*(\omega_i) = \sum_j \lambda_{ij} \omega_j, \qquad i = 1, \ldots, k, \tag{3.2}$$

for some functions λ_{ij} on M or, equivalently,

$$\sum_{j,s} \omega_{ij}(\varphi(x)) \frac{\partial \varphi_j}{\partial x_s}\, dx_s = \sum_{j,s} \lambda_{ij}(x) \omega_{js}(x)\, dx_s, \qquad i = 1, \ldots, k,$$

where $s = 1, \ldots, n = \dim M$. Hence the symmetries φ of the distribution at hand can be found from the following system of equations

$$\sum_j \omega_{ij}(\varphi(x)) \frac{\partial \varphi_j}{\partial x_s} = \sum_j \lambda_{ij}(x) \omega_{js}(x), \tag{3.3}$$

where λ_{ij} are arbitrary smooth functions, $i = 1, \ldots, k$, $j = 1, \ldots, n$.

The problem of solving these equation is not easier than the problem of finding the integral manifolds of the distribution under consideration. Moreover, the direct attempt to describe the set $\operatorname{Sym} P$ of all symmetries of a given distribution P is doomed to failure. However, passing to the infinitesimal point of view, i.e., passing from above-defined finite symmetries to infinitesimal ones, considerably simplifies the situation.

DEFINITION 3.2. A vector field $X \in \mathrm{D}(M)$ is said to be an *infinitesimal symmetry* of the distribution P, if the flow A_t generated by the field X consists of finite symmetries.

The set of all infinitesimal symmetries will be denoted by D_P.

Definition 3.2 is generally inconvenient to use in concrete computations. But it can be restated in a more constructive way:

THEOREM 3.1. *Let P be a distribution on M. The following conditions are equivalent*:

1. $X \in \mathrm{D}_P$.
2. *If $X_1, \ldots, X_l$ are vector fields generating P, then there exist smooth functions μ_{ij} such that $[X, X_i] = \sum_j \mu_{ij} X_j$ for all $i = 1, \ldots, l$.*
3. *If $\omega_1, \ldots, \omega_k$ are 1-forms defining P, then there exist smooth functions ν_{ij} such that $X(\omega_i) = \sum_j \nu_{ij} \omega_j$ for all $i = 1, \ldots, k$.*

PROOF. $1 \implies 2$. Let X be a symmetry of the distribution P and A_t be the one-parameter transformation group corresponding to the vector field X. Since the diffeomorphism A_t preserves the distribution P for every $t \in \mathbb{R}$, the image of the vector field $X_i \in P$ D under this diffeomorphism belongs to P D:

$$(A_t)_*(X_i) = \sum_j \alpha_{ij}(t) X_j,$$

where $\alpha_{ij}(t)$ is a family of smooth functions on M smoothly depending on the parameter $t \in \mathbb{R}$. Differentiating this expression with respect to t at $t = 0$, we get

$$[X, X_i] = \sum_j \mu_{ij} X_j,$$

with $\mu_{ij} = -d\alpha_{ij}(t)/dt|_{t=0}$.

$2 \implies 3$. Suppose that a vector field X satisfies Condition 2. We claim that if a 1-form ω vanishes on all vector fields $X_1, \ldots, X_l$, then the form $X(\omega)$ has the same property (and, hence, can be represented as a linear combination of $\omega_1, \ldots, \omega_k$). Indeed, for all i we have

$$X(\omega)(X_i) = -\omega([X, X_i]) = 0.$$

$3 \implies 1$. First note that the equality $A_{t+s}^* = A_t^* \circ A_s^*$ implies

$$\frac{d}{dt}\bigg|_{t=s} A_t^*(\omega) = A_s^*(X(\omega)).$$

Further, consider the following $(k+1)$-form dependent upon the parameter t:

$$\Omega_i(t) = A_t^*(\omega_i) \wedge \omega_1 \wedge \cdots \wedge \omega_k.$$

Since $A_0^*(\omega_i) = \omega_i$, we have $\Omega_i(0) = 0$. We claim that $\Omega_i(t) \equiv 0$. Indeed,

$$\frac{d}{dt}\Omega_i(t) = A_t^*(X(\omega_i)) \wedge \omega_1 \wedge \cdots \wedge \omega_k = \sum A_t^*(\nu_{ij})\Omega_j(t).$$

Hence, the vector consisting of $(k+1)$-forms $(\Omega_1(t), \ldots, \Omega_k(t))$ is a solution to a linear homogeneous system of ordinary differential equations with zero initial conditions. Therefore, $\Omega_i(t) \equiv 0$ for all i.

Thus, $A_t^*(\omega_i)$ is a linear combination of $\omega_1, \ldots, \omega_k$ for all t, i.e., A_t is a symmetry of the distribution P. $\square$

COROLLARY 3.2. *If* $X, Y \in \mathrm{D}_P$, *then* $\alpha X + \beta Y \in \mathrm{D}_P$ $(\alpha, \beta \in \mathbb{R})$ *and, moreover,* $[X, Y] \in \mathrm{D}_P$. *In other words,* D_P *is a Lie algebra over* $\mathbb{R}$ *with respect to the commutation operation.*

PROOF. In view of Condition 2 of the theorem proved, we can write $[X, X_i] = \sum_j \mu_{ij} X_j$, $[Y, X_i] = \sum_j \lambda_{ij} X_j$. Then $[\alpha X + \beta Y, X_i] = \sum_j (\alpha \mu_{ij} + \beta \lambda_{ij}) X_j$. This yields $\alpha X + \beta Y \in \mathrm{D}_P$.

Further, by virtue of the Jacobi identity

$$[[X, Y], X_i] = [[X, X_i], Y] - [[Y, X_i], X]$$

$$= \sum_j ([\mu_{ij} X_j, Y] - [\lambda_{ij} X_j, X]) = \sum_{j,k} (\lambda_{ij} \mu_{ik} - \mu_{ij} \lambda_{jk}) X_k + \sum_j (X(\lambda_{ij}) - Y(\mu_{ij})) X_j.$$

This shows that $[X, Y] \in \mathrm{D}_P$. $\square$

Using Theorem 3.1, we can write down coordinate conditions for a field $X = \sum_i X^i \partial/\partial x_i$ to be a symmetry of the distribution given by the system of 1-forms $\omega_1, \ldots, \omega_k$, $\omega_j = \sum_s \omega_{js} \, dx_s$.

We have

$$X(\omega_j) = \sum_{i,s} \left(X^i \frac{\partial \omega_{js}}{\partial x_i} + \frac{\partial X^i}{\partial x_s} \omega_{ji} \right) dx_s.$$
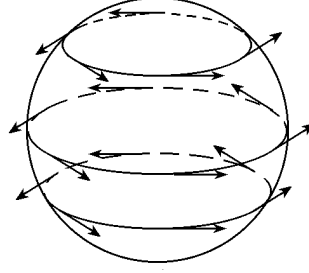
FIGURE 1.10

By Condition 3 of Theorem 3.1, the 1-form $X(\omega_j)$ is represented as $\sum_i \nu_{ji}\omega_i$, i.e., there exist smooth functions ν_{ji} such that

$$\sum_i \left(X^i \frac{\partial \omega_{js}}{\partial x_i} + \frac{\partial X^i}{\partial x_s}\omega_{ji} \right) = \sum_i \nu_{ji}\omega_{is}, \qquad (3.4)$$

where $j = 1, \ldots, k$, $s = 1, \ldots, n$.

Note that in contrast to (3.3), the system of equations (3.4) is linear in the fields $X^1, \ldots, X^k$. It is called the *system of linear Lie equations* corresponding to the *system of nonlinear Lie equations* (3.3). Just as the nonlinear Lie equations serve to find finite symmetries of a given distribution, the linear Lie equations serve to find its infinitesimal symmetries.

Further on we primarily study infinitesimal symmetries, so the word "symmetry" will always mean "infinitesimal symmetry" unless otherwise explicitly specified.

EXAMPLE 3.2. Symmetries of a one-dimensional distribution generated by a vector field Y are vector fields X such that $[X, Y] = \lambda Y$ for some function λ.

Consider, as an example, the vector field Y on the sphere that in the "geographical" coordinates φ, θ has the form $Y = \partial/\partial\varphi$ (see Figure 1.10). Take a field $X = \alpha\partial/\partial\theta + \beta\partial/\partial\varphi$. We have $[X, Y] = -\alpha_\varphi\partial/\partial\theta - \beta_\varphi\partial/\partial\varphi$, so the field X is a symmetry, if $\alpha_\varphi = 0$. Thus, the symmetries of the distribution under study are the fields of the form $\alpha\partial/\partial\theta + \beta\partial/\partial\varphi$, where β is an arbitrary function on the sphere and α is a function constant on the parallels.

The finite symmetries of this distribution are given by pairs of functions $\bar\theta = f(\theta)$, $\bar\varphi = g(\theta, \varphi)$.

EXAMPLE 3.3. Let us discuss the local structure of the symmetry algebra of a completely integrable distribution.

It is clear that for any completely integrable distribution P at a generic point there exists a coordinate system such that P is given by the basis

$$X_1 = \frac{\partial}{\partial x_1}, \ldots, X_l = \frac{\partial}{\partial x_l}.$$

Let $X = \sum_i X^i \partial/\partial x_i$. Then $[X_j, X] = \sum_i (\partial X^i/\partial x_j)\partial/\partial x_i$, so that the condition $[X, X_j] \in P$ D, $j = 1, \ldots, l$, is equivalent to the equalities $\partial X^i/\partial x_j = 0$, $j \leq l < i$.

The field X splits into the longitudinal component $\sum_{i \leq l} X^i \partial/\partial x_i$ and the transversal component $\sum_{i > l} X^i \partial/\partial x_i$. Using this decomposition, we can say that X is a symmetry of the distribution P if and only if the coefficients of its transversal component are constant on the leaves of P, i.e., on the maximal integral manifolds
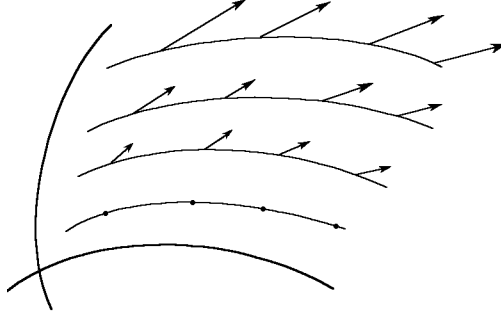
FIGURE 1.11

of the distribution P which are of the form $x_{l+1} = C_{l+1}, \ldots, x_n = C_n$, $C_i = \text{const}$ (see Figure 1.11).

EXAMPLE 3.4. Let us find all (infinitesimal) symmetries of the Cartan distribution $\mathcal{C}$ on $\mathbb{R}^3$. Recall that it is given by the form $\omega = du - p\,dx$. Let

$$X = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial u} + \gamma\frac{\partial}{\partial p} \tag{3.5}$$

be a symmetry of $\mathcal{C}$. We have

$$X(\omega) = (\beta_x - p\alpha_x - \gamma)\,dx + (\beta_u - p\alpha_u)\,du + (\beta_p - p\alpha_p)\,dp.$$

The condition that $X(\omega)$ is proportional to ω reads

$$\begin{cases} \beta_p - p\alpha_p = 0, \\ \beta_x - p\alpha_x - \gamma = -p(\beta_u - p\alpha_u). \end{cases}$$

This system implies that the symmetries of the distribution $\mathcal{C}$ are vector fields of the form (3.5) such that α and β are arbitrary functions related by the equality $\beta_p = p\alpha_p$, and γ is expressed in terms of these functions by the formula $\gamma = \beta_x + p\beta_u - p\alpha_x - p^2\alpha_u$. Consider the function $f = \beta - p\alpha$. It is clear that $\alpha = -f_p$, $\beta = f - pf_p$, $\gamma = f_x + pf_u$, i.e.,

$$X = -f_p\frac{\partial}{\partial x} + (f - pf_p)\frac{\partial}{\partial u} + (f_x + pf_u)\frac{\partial}{\partial p}.$$

Thus, a symmetry X of the Cartan distribution is uniquely determined by the function $f = f(x, u, p)$, which can be chosen arbitrarily[2].

Among symmetries of a given distribution P there is a distinguished class of symmetries that lie in P, i.e., belong to $\mathrm{D}_P \cap P\,\mathrm{D}$. These are called *characteristic* (or *trivial*) symmetries. Symmetries that does not belong to $P\,\mathrm{D}$ will be referred to as *nontrivial*.

PROPOSITION 3.3. *Let* $X, Y \in \mathrm{D}_P$, *the symmetry* Y *being trivial. Then the symmetry* $[X, Y]$ *is also trivial. In other words, the set of all characteristic symmetries is an ideal of the Lie algebra* D_P.

---

[2]This result is a particular case of the general theory of contact fields (see Ch. 2).

PROOF. Let $X_1, \ldots, X_l$ be vector fields generating the distribution P. Suppose that $Y = \sum f_i X_i$. By Theorem 3.1, we have $[X, X_i] = \sum_j \mu_{ij} X_j$. Therefore,

$$[X, Y] = [X, \sum_i f_i X_i] = \sum_i X(f_i) X_i + \sum_{i,j} f_i \mu_{ij} X_j.$$

Thus, the field $[X, Y]$ lies in P. By virtue of Corollary 3.2, it is a symmetry of the distribution P. $\qquad \square$

Let $\operatorname{char} P \subset \mathrm{D}_P$ denote the ideal of trivial (characteristic) symmetries. By definition, $\operatorname{char} P = \mathrm{D}_P \cap P\,\mathrm{D}$.

In view of Proposition 3.3, we can consider the quotient Lie algebra

$$\operatorname{sym} P = \mathrm{D}_P \big/ \operatorname{char} P,$$

which is said to be the *Lie algebra of nontrivial symmetries* of the distribution P. Note that for completely integrable distributions one has $\operatorname{char} P = P\,\mathrm{D}$. Consequently $\operatorname{sym} P = \mathrm{D}_P / P\,\mathrm{D}$ in this case.

Let us consider symmetries of a completely integrable distribution P. Clearly, the one-parameter transformation group generated by a symmetry preserves the leaves of P. By the Frobenius theorem, transformations corresponding to a characteristic symmetry move every leaf of P along itself. By contrast, nontrivial symmetries shuffle the leaves.

These observations can be generalized to nonintegrable distributions.

PROPOSITION 3.4. *A characteristic symmetry of the distribution P is tangent to every maximal integral manifold of P.*

PROOF. Assume that there exists a characteristic symmetry X which is not tangent to an integral manifold L. Consider the flow A_t of X and the manifold $\bar{L} = \bigcup_t A_t(L)$. Clearly, L is strictly embedded to $\bar{L}$.

We claim that the manifold $\bar{L}$ is integral for P. Indeed, the tangent space to $\bar{L}$ at a point x spans the vector X_x and the tangent space to the submanifold $A_t(L)$ passing through x. Since both lie in P_x, the same is true for $T_x(\bar{L})$. $\qquad \square$

REMARK 3.1. Thus, every maximal integral manifold of the distribution P is composed of its characteristics, i.e., of the trajectories of characteristic symmetries of P. A similar construction occurs in the theory of first-order partial differential equations, where our terminology came from.

Similar to the case of completely integrable distributions, transformations corresponding to nontrivial symmetries of an arbitrary distribution preserve the set of maximal integral manifolds, but, unlike trivial symmetries, shuffle them. Note that the action of elements of $\operatorname{sym} P$ on the set of maximal integral manifolds is well-defined, since by Proposition 3.4 such a manifold is invariant under the action of a trivial symmetry.

PROPOSITION 3.5. *The set of characteristic fields is a module over the function ring, i.e., if $X \in \operatorname{char} P$ and $f \in C^\infty(M)$, then $fX \in \operatorname{char} P$.*

PROOF. For an arbitrary vector field $Y \in P\,\mathrm{D}$, we have

$$[fX, Y] = f[X, Y] - Y(f)X \in P\,\mathrm{D}.$$

Hence, $fX \in \mathrm{D}_P$. In addition, $fX \in P\,\mathrm{D}$, therefore $fX \in \operatorname{char} P$. $\qquad \square$

COROLLARY 3.6. *The set* char P *consists of vector fields tangent to the distribution.*

The distribution char P is called *characteristic.* The dimension of this distribution may decrease at some points.

THEOREM 3.7. *The characteristic distribution is completely integrable.*

PROOF. It follows from Proposition 3.3 that char P is a Lie algebra. The theorem is proved. $\square$

Now, let us use Examples 3.2–3.4 to illustrate the above theory.

In Example 3.2, we dealt with a one-dimensional distribution on the sphere. Any one-dimensional distribution is completely integrable, so that in this case the characteristic distribution coincides with the one considered.

The same obviously holds for any completely integrable distribution.

Further, let X be the characteristic symmetry of the Cartan distribution discussed in Example 3.4. Since $X \in \mathrm{D}_P$, we have

$$X = -f_p \frac{\partial}{\partial x} + (f - pf_p)\frac{\partial}{\partial u} + (f_x + pf_u)\frac{\partial}{\partial p}.$$

The fact that $X \in \mathrm{char}\, P$ implies $\omega(X) = 0$. But $\omega(X) = X \lrcorner (du - pdx) = f$. Therefore $f = 0$ and, hence, $X = 0$. Thus, in this case the characteristic distribution is zero-dimensional.

4. Some applications of the symmetry theory to integration of distributions

In this section, we show how to describe maximal integral manifolds of a distribution using its symmetries.

4.1. Distributions and characteristic symmetries. Consider a typical situation. Let P be a distribution and X be a characteristic symmetry of P. Given an integral manifold L transversal to the trajectories of X, we can construct the manifold $N = \bigcup_t A_t(L)$, where A_t is the flow of X. Since maximal integral manifolds of a distribution are composed of its characteristics, the manifold N is integral for P. In particular, taking a point as the manifold L, we get one trajectory of the field X, which is a one-dimensional integral manifold.

EXAMPLE 4.1. In Example 2.2, we discussed the differential equation

$$(3x - 2u)u' = u.$$

For the distribution $\mathcal{C}_{\mathcal{E}}$ given by the form $du - p\,dx$ the characteristic symmetry Y_F was constructed. The trajectories of this field,

$$\begin{cases} x = (x_0 - u_0)e^{-3t} + u_0 e^{-t}, \\ u = u_0 e^{-t}, \end{cases}$$

are solutions of the equation at hand passing through the point (x_0, u_0, p_0).

EXAMPLE 4.2. Consider the upper half-plane

$$H = \{\, x, y \mid y > 0 \,\} \subset \mathbb{R}^2$$

and the manifold M of all unit tangent vectors to H. As coordinates on M take x, y, u, with $u \in \mathbb{R} \mod 2\pi$ being the angle between the upward vertical direction and the given vector. Further, consider the distribution P defined by the 1-form

$$\omega = du + \frac{1 - \cos u}{y}\, dx - \frac{\sin u}{y}\, dy.$$

The reader can readily verify that this distribution satisfies the Frobenius complete integrability condition.

This distribution defines the so-called *oricycle foliation*. Its geometric meaning can be explained as follows.

Recall that the upper half-plane with the metric $ds^2 = (dx^2 + dy^2)/y^2$ is a model of the Lobachevskian plane. The geodesics ("straight lines") in this model are semicircles with centers on the x-axis and vertical rays.

A leaf of the oricycle foliation consists of all vectors tangent to the geodesics passing through the same point of the x-axis.

The lines $L = \{u = \pi, x = c\}$, where c is a constant, are integral manifolds of P, since ω vanishes on them. The field

$$D_x = \frac{\partial}{\partial x} - \frac{1 - \cos u}{y} \frac{\partial}{\partial u}$$

is a characteristic symmetry of P, because the distribution P is completely integrable and the form ω vanishes on the field D_x. Moreover, it is obvious that the field D_x is transversal to the field $\partial/\partial y$, which is tangent to the line L.

Now, we consider all trajectories of the field D_x passing through L to obtain a two-dimensional integral surface of the distribution P.

Solving the system

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = 0, \\ \dot{u} = -\dfrac{1 - \cos u}{y}, \end{cases}$$

yields the trajectories of the field D_x:

$$y = C_1, \qquad x - y \cot \frac{u}{2} = C_2,$$

where C_1 and C_2 are constants.

If points (x, y, u) and $(c, y_0, \pi) \in L$ lie on the same trajectory, then $y = y_0$, $x - y \cot u/2 = c - y_0 \cot \pi/2$. It follows that $x - y \cot u/2 = c$. This is exactly the surface $\bigcup_t A_t(L)$ (see Figure 1.12).

Geometrically, this surface is the set of all unit vectors tangent to the semicircles such that their centers lie on the x-axis and they pass through the same point of this axis (Figure 1.13).

4.2. Symmetries and dynamical systems. Many standard methods for solving differential equations make use of the symmetry theory. Let us discuss an example.

A system of first-order ordinary differential equations resolved with respect to derivatives is interpreted as a vector field X (or, in other words, as a dynamical system). A vector field Y is called a *symmetry* of the field X if $[X, Y] = 0$. It is readily seen that a shift along the trajectories of Y shuffles the trajectories of X. Given a symmetry, one usually changes coordinates to "straighten" the field Y (i.e., chooses new coordinates, $y_1, \dots, y_n$, in such a way that all except one of them,
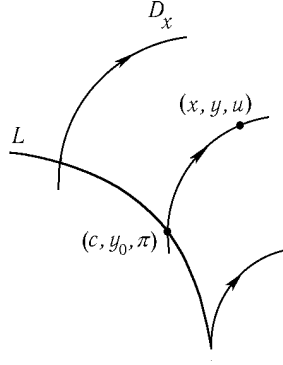
FIGURE 1.12
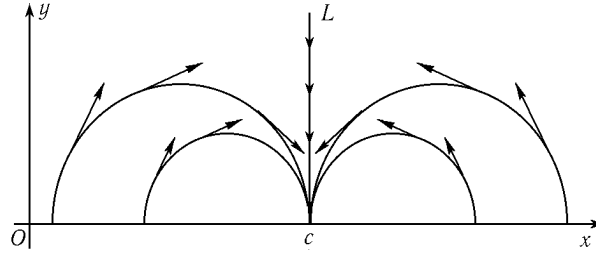
FIGURE 1.13

say y_n, are constant on the trajectories of the field Y). In these new coordinates, the field Y takes the form $\partial/\partial y_n$, so that the condition $[X, Y] = 0$ implies that the coefficients of the field X do not depend on y_n. Thus, finding trajectories of the field X reduces to integration of an $(n-1)$-dimensional dynamical system and one quadrature.

EXAMPLE 4.3. Consider the 3-dimensional dynamical system

$$\begin{cases} \dot{x}_1 = t\left(\dfrac{x_2^2}{x_3} + \dfrac{x_1^2}{x_2^2}x_3\right) + \dfrac{x_1^2}{x_2}, \\ \dot{x}_2 = t\dfrac{x_1 x_3}{x_2} + x_1, \\ \dot{x}_3 = \dfrac{x_1 x_3}{x_2}. \end{cases}$$

The vector field

$$Y = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$$

is a symmetry of this system. Indeed, it can easily be checked that the commutator of the field

$$X = \left(t\left(\frac{x_2^2}{x_3} + \frac{x_1^2}{x_2^2}x_3\right) + \frac{x_1^2}{x_2}\right)\frac{\partial}{\partial x_1} + \left(t\frac{x_1 x_3}{x_2} + x_1\right)\frac{\partial}{\partial x_2} + \frac{x_1 x_3}{x_2}\frac{\partial}{\partial x_3}$$

and the field Y is equal to zero. Now choose the new coordinates

$$\begin{cases} x_1' = \dfrac{x_1}{x_2}, \\ x_2' = \dfrac{x_2}{x_3}, \\ x_3' = \ln x_3. \end{cases}$$

In these coordinates, the field Y takes the form $\partial/\partial x_3'$, while the dynamical system X is

$$\begin{cases} \dot{x}_1' = tx_2', \\ \dot{x}_2' = tx_1', \\ \dot{x}_3' = x_1'. \end{cases}$$

It follows that finding the trajectories of the dynamical system X reduces to the integration of the two-dimensional dynamical system

$$\begin{cases} \dot{x}_1' = tx_2', \\ \dot{x}_2' = tx_1' \end{cases}$$

and one quadrature $x_3' = \int x_1'(t)\, dt$.

We now turn to discuss applications of nontrivial (i.e., noncharacteristic) symmetries to the problem of integrating distributions.

4.3. Distributions and noncharacteristic symmetries. Let X be a symmetry of a distribution P and A_t be the corresponding flow. Given an integral manifold L of the distribution P, we can construct the whole family $A_t(L)$ of such manifolds.

EXAMPLE 4.4. Consider the equation $(3x - 2u)u' = u$ from Example 2.2. The field $X = u^3\partial/\partial x$, written in the local coordinates (x, u) on $\mathcal{E}$, obviously commutes with the field Y_F (see Example 2.2) restricted to $\mathcal{E}$ (check that this restriction has the form $(2u-3x)\partial/\partial x - u\partial/\partial u$). Therefore X is a symmetry of the one-dimensional distribution $\mathcal{C}(\mathcal{E})$ on $\mathcal{E}$. Take the solution $L = \{u = x\}$ of the equation under study. Shifts of this solution along the trajectories of the field X yield all solutions of our equation (except for $u = 0$). In fact, the shift along the trajectories of the field X at time t has the form

$$\begin{cases} x = u_0^3 t + x_0, \\ u = u_0. \end{cases}$$

From these formulas it easily follows that the image of the straight line $L = \{u = x\}$ is the curve $A_t(L) = \{tu^3 + u = x\}$ (see Figure 1.14).

EXERCISE 4.1. Find a symmetry of this equation that yields every (or almost every) solution from the zero solution.

REMARK 4.1. Under shifts along the trajectories of the field X from Example 4.4, the solution $u = 0$ moves along itself. Such a solution is said to be *invariant*.

EXERCISE 4.2. Find the solution of the equation $(3x - 2u)u' = u$ invariant under the infinitesimal homothety $x\partial/\partial x + u\partial/\partial u$.
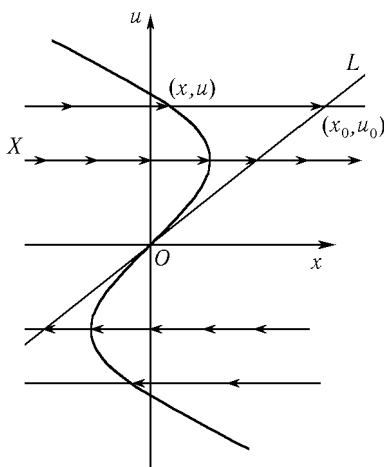
FIGURE 1.14

4.4. Integration of equations by quadratures. The second aspect of applications of nontrivial symmetries to which we would like to call attention, is integration of equations by quadratures. The point is that if for a completely integrable distribution one knows a solvable Lie algebra of nontrivial symmetries and the dimension of this algebra equals the codimension of the distribution, then the integral manifolds of this distribution can be found by quadratures. Before describing of this procedure, let us give the following

DEFINITION 4.1. A function $f \in C^\infty(M)$ is called a *first integral of a distribution P*, if $X(f) = 0$ for any $X \in P$ D.

It is obvious that if f is a first integral of the distribution P, then it is constant on every integral manifold of this distribution, or, in other words, every integral manifold of P lies entirely in a level surface $\{f = c\}$ of the first integral.

Let us find first integrals for distributions from Examples 3.2–3.4.

Clearly, first integrals of the field $X = \partial/\partial\varphi$ on the sphere are functions constant on the parallels of the sphere. This example is easily generalized to the case of an arbitrary one-dimensional distribution[3].

First integrals of a completely integrable distribution P locally given by the fields

$$X_1 = \frac{\partial}{\partial x_1}, \ldots, X_l = \frac{\partial}{\partial x_l},$$

are of the form $f(x_{l+1}, \ldots, x_n)$, where f is an arbitrary function.

Finally, let f be a first integral of the Cartan distribution given by the form $\omega = du - p\,dx$. Since the fields $\partial/\partial p$ and $\partial/\partial x + p\partial/\partial u$ lie in the Cartan distribution, we have $\partial f/\partial p = \partial f/\partial x + p\partial f/\partial u = 0$, therefore $f = $ const. Thus, the Cartan distribution does not possess nontrivial first integrals.

---

[3]Note that the terminology used here is consistent with that used in the theory of first-order partial differential equations: if the distribution P is given by a vector field X, then its first integrals are the first integrals of the equation $X(\varphi) = 0$, i.e., functions constant on the trajectories of the field X. See also Ch. 2.

Assume that we know a first integral of a distribution P on M. Then the problem of finding all maximal integral manifolds reduces to integration of a distribution on a manifold whose dimension is diminished by one with respect to M. Namely, the problem amounts to integrating the restriction of the distribution P to a level surface of the first integral. Each new independent first integral reduces the dimension by one. In the case functionally independent first integrals $f_1, \ldots, f_k$, where $k = \operatorname{codim} P$, are known, the distribution under consideration can be given by the set of exact 1-forms $df_1, \ldots, df_k$. Therefore this distribution is completely integrable and its leaves are the mutual level surfaces $\{\, x \in M \mid f_i(x) = c_i \,\}$ of the functions $f_1, \ldots, f_k$, where $c_1, \ldots, c_k$ are constant.

The problem of finding a first integral can be stated as the problem of finding an exact 1-form $\omega = df$ vanishing on vectors belonging to the distribution P, since in this case $\omega(X) = X(f) = 0$ for all $X \in P$ D. Thus, complete integrability of P is equivalent to existence of $k = \operatorname{codim} P$ such forms, provided that they are independent.

DEFINITION 4.2. A linear subspace $\mathcal{Y} \subset \operatorname{sym} P$ is called *nondegenerate*, if for any point $x \in M$ and any field $X \in \mathcal{Y}$ the condition $X_x \in P_x$ is equivalent to the condition $X = 0$.

Let $x \in M$ and $\mathcal{Y}_x = \{\, X_x \mid X \in \mathcal{Y} \,\}$. If the subspace $\mathcal{Y}$ is nondegenerate, then from Definition 4.2 it follows that

$$\dim \mathcal{Y} = \dim \mathcal{Y}_x \leq \operatorname{codim} P.$$

Let $\omega_1, \ldots, \omega_k$ be a set of forms defining the distribution P and $X_1, \ldots, X_l$ be a basis of a nondegenerate subspace $\mathcal{Y} \subset \operatorname{sym} P$. Clearly, $l \leq k$ and the matrix $\|\omega_i(X_j)\|$ has the rank l at any point of the manifold M. If $l = k$, then $\det \|\omega_i(X_j)\| \neq 0$ and one can choose another set of forms defining P, say $\omega'_1, \ldots, \omega'_k$, such that $\omega'_i(X_j) = \delta_{ij}$. To do this, it suffices to multiply the column $(\omega_1, \ldots, \omega_k)^t$ by the matrix inverse to $\|\omega_i(X_j)\|$. Assume that the space $\mathcal{Y}$ is closed with respect to the commutation; then we have the following

THEOREM 4.1. *Let P be a completely integrable distribution defined by the set of 1-forms $\omega_1, \ldots, \omega_k$. Let $X_1, \ldots, X_k$ be a basis of a nondegenerate Lie algebra $\mathcal{Y}$ of symmetries of this distribution. Suppose that $\omega_i(X_j) = \delta_{ij}$ and*

$$[X_i, X_j] = \sum_s c_{ij}^s X_s,$$

where c_{ij}^s are constant. Then

$$d\omega_s = -\frac{1}{2} \sum_{i,j} c_{ij}^s \omega_i \wedge \omega_j.$$

PROOF. By the Frobenius theorem, there exist 1-forms γ_{ij} such that

$$d\omega_s = \sum_j \gamma_{sj} \wedge \omega_j.$$

Since $X_i \in \operatorname{sym} P$, the 1-forms

$$X_i(\omega_s) = X_i \lrcorner\, d\omega_s + d(X_i \lrcorner\, \omega_s) = X_i \lrcorner\, d\omega_s$$

vanish on vectors from the distribution P. The equality

$$X_i \lrcorner d\omega_s = \sum_j \gamma_{sj}(X_i)\omega_j - \gamma_{si}$$

implies that the forms γ_{sj} are also zero on vectors from P and, therefore, $\gamma_{ij} = \sum_s a_{ij}^s \omega_s$ for some appropriate functions $a_{ij}^s \in C^\infty(M)$. Hence,

$$d\omega_s = \sum_{i<j} a_{ij}^s \omega_i \wedge \omega_j.$$

It follows that

$$d\omega_s(X_i, X_j) = a_{ij}^s, \qquad \text{for } i < j.$$

On the other hand,

$$d\omega_s(X_i, X_j) = X_i(\omega_s(X_j)) - X_j(\omega_s(X_i)) - \omega_s([X_i, X_j]) = -\omega_s\left(\sum_t c_{ij}^t X_t\right) = -c_{ij}^s.$$

Therefore, $a_{ij}^s = -c_{ij}^s$ and

$$d\omega_s = -\sum_{i<j} c_{ij}^s \omega_i \wedge \omega_j = -\frac{1}{2}\sum_{i,j} c_{ij}^s \omega_i \wedge \omega_j. \quad \square$$

COROLLARY 4.2. *Under the conditions of Theorem 4.1, assume that the algebra $\mathcal{Y}$ is commutative. Then all the forms ω_i are closed and, thus, locally exact: $\omega_i = dh_i$ for some smooth functions h_i. These first integrals can be locally found by computing the integrals*

$$h_i(a) = \int_{a_0}^a \omega_i,$$

where a_0 is a fixed point of the manifold M.

In particular, since ordinary differential equations of order k are identified with one-dimensional distributions on $(k+1)$-dimensional manifolds, Corollary 4.2 immediately implies the following

COROLLARY 4.3. *If the distribution corresponding to an ordinary differential equation of order k possesses a k-dimensional commutative nondegenerate Lie algebra of symmetries, then this equation is integrable by quadratures.*

EXAMPLE 4.5. Consider a first-order equation resolved with respect to the derivative. Assume that the right-hand side does not depend on x:

$$\frac{du}{dx} = f(u).$$

The corresponding distribution is defined on the surface $p = f(u)$ by the 1-form $\omega = du - f\,dx$. The vector field

$$X = \frac{\partial}{\partial x}$$

is tangent to this surface and is a nontrivial symmetry of the distribution $\mathcal{C}(\mathcal{E})$.

By virtue of Theorem 4.1 and Corollary 4.2, the form $\omega_\mathcal{E}/f$ is exact. Indeed,

$$\frac{\omega_\mathcal{E}}{f} = \frac{1}{f}(du - f\,dx) = d\left(\int \frac{du}{f(u)} - x\right).$$

In multiplication by $1/f$, one easily recognizes the well-known method of "separation of variables" for solving equations of the considered type. Thus, this method essentially amounts to transforming the 1-form defining the equation to an exact one.

EXERCISE 4.3. Show that on the equation $\mathcal{E}$ the field X coincides with the field

$$f\frac{\partial}{\partial u} + f'p\frac{\partial}{\partial p}.$$

The previous example serves to illustrate the following general fact: with a knowledge of one nontrivial symmetry of a completely integrable distribution of codimension 1 (in particular, of a first-order ordinary differential equation), it is possible to describe leaves of this distribution (in particular, solutions to this differential equation) by means of quadratures. If X is a nontrivial symmetry of a distribution P given by 1-form ω, then the form $f\omega$, with $f = 1/\omega(X)$, which also defines the distribution P, is closed[4]. In this case, the leaves of P coincide with the level surfaces of the integral of this form.

EXAMPLE 4.6. Consider the distribution P from Example 4.2. Since the coefficients of the form

$$\omega = du + \frac{1 - \cos u}{y}\,dx - \frac{\sin u}{y}\,dy$$

do not depend on x, the field $X = \partial/\partial x$ is a symmetry of P. We have $\omega(X) = (1 - \cos u)/y \neq 0$, so that X is a nontrivial symmetry. Thus, the integrating factor equals $f = 1/\omega(X) = y/(1 - \cos u)$ and

$$f\omega = dx - \frac{\sin u}{1 - \cos u}\,dy + \frac{y}{1 - \cos u}\,du = d(x - y\cot\frac{u}{2}).$$

The equation $x - y\cot(u/2) = c$, $c = \text{const}$, defines all the leaves of the distribution at hand, except for the leaf $\{u = 0\}$ (the function f is undefined for $u = 0$).

EXAMPLE 4.7. Any homogeneous first-order equation $\mathcal{E}$

$$u' = \varphi\left(\frac{u}{x}\right)$$

has the symmetry $X = x\partial/\partial x + u\partial/\partial u$. Indeed, in the coordinates x and u the Cartan distribution on $\mathcal{E}$ is given by the form

$$\omega_{\mathcal{E}} = du - \varphi\left(\frac{u}{x}\right)\,dx.$$

It can easily be checked that $X(\varphi) = 0$ and $X(\omega_{\mathcal{E}}) = \omega_{\mathcal{E}}$, hence X is a symmetry. For an integrating factor one can take the function

$$f = \frac{1}{\omega_{\mathcal{E}}(X)} = \frac{1}{u - x\varphi\left(\frac{u}{x}\right)}.$$

In particular, for the equation $(3x - 2u)u' = u$ (see Example 2.2), which is equivalent to the equation

$$u' = \frac{u}{3x - 2u},$$

---

[4]Recall that the function f is said to be an *integrating factor*.

everywhere except for the line $3x - 2u = 0$, we have

$$\varphi(\xi) = \frac{\xi}{3 - 2\xi}, \qquad f = \frac{3x - 2u}{2u(x - u)},$$

and, therefore,

$$\omega'_{\mathcal{E}} = \frac{3x - 2u}{2u(x - u)}\omega_{\mathcal{E}} = d\left(\frac{1}{2}\ln\frac{u^3}{x - u}\right).$$

As above, this yields the solutions of equation under study: $u^3/(x - u) = c$, $c = $ const.

EXAMPLE 4.8. Consider a linear nonhomogeneous equation

$$a_k u^{(k)} + \cdots + a_1 u' + a_0 u = f,$$

where $a_0, a_1, \ldots, a_k$ and f are given functions of x. It is easy to prove that the field

$$X_\varphi = \varphi\frac{\partial}{\partial u} + \varphi'\frac{\partial}{\partial p_1} + \cdots + \varphi^{(k)}\frac{\partial}{\partial p_k},$$

where $\varphi(x)$ is an arbitrary solution of the corresponding homogeneous equation

$$a_k u^{(k)} + \cdots + a_1 u' + a_0 u = 0,$$

is a symmetry of the initial equation. Such symmetries make a k-dimensional commutative Lie algebra, and if one knows its basis, i.e., a fundamental system $\varphi_1, \ldots, \varphi_k$ of solutions of the homogeneous equation, then, by Corollary 4.3, the nonhomogeneous equation can be integrated by quadratures. Note that $\omega_i(X_\varphi) = \varphi^{(i)}$, where, as before, $\omega_i = dp_i - p_{i+1}\,dx$, Therefore, the matrix $\|\omega_i(X_{\varphi_j})\|$, which in the course of solution must be inverted to find an integrating factor, is nothing but the usual Wronski matrix of the fundamental system involved.

Now, suppose that we know a nondegenerate noncommutative Lie algebra $\mathcal{Y}$ of symmetries of a completely integrable distribution P. Consider its commutator subalgebra

$$\mathcal{Y}^{(1)} = [\mathcal{Y}, \mathcal{Y}] = \left\{\sum[X, Y] \mid X, Y \in \mathcal{Y}\right\}$$

and assume that $\mathcal{Y}^{(1)} \neq \mathcal{Y}$. Then one can choose a basis $X_1, \ldots, X_k$ of $\mathcal{Y}$ in such a way that $X_1, \ldots, X_r \notin \mathcal{Y}^{(1)}$, while $X_{r+1}, \ldots, X_k \in \mathcal{Y}^{(1)}$. In this case, for any two fields X_i and X_j we have $[X_i, X_j] \in \mathcal{Y}^{(1)}$, so that $c_{ij}^s = 0$ for all i, j, if $s \leq r$. Hence, the forms $\omega_1, \ldots, \omega_r$ are closed and, therefore, $\omega_1 = dh_1, \ldots, \omega_r = dh_r$ in some open domain. The level surfaces

$$H_c = \{\, h_1 = c_1, \ldots, h_r = c_r \,\}, \qquad c = (c_1, \ldots, c_r), \quad c_i \in \mathbb{R},$$

are invariant under the commutator subalgebra $\mathcal{Y}^{(1)}$, since $X_j(h_i) = \omega_i(X_j) = 0$ if $i \leq r$, $j \geq r + 1$.

Let P_c be the restriction of the distribution P to the surface H_c. The distribution P_c is completely integrable. Indeed, the foliation of M whose leaves are maximal integral manifolds of P cuts a foliation on the surface H_c. The same fact can be proved analytically. By Theorem 4.1,

$$d\omega_s = -\frac{1}{2}\sum_{i,j > r} c_{ij}^s \omega_i \wedge \omega_j, \qquad s \geq r + 1,$$

The forms ω_i vanish on H_c for $i \leq r$. Hence, by the Frobenius theorem, the distribution P_c is completely integrable.

Observe now that $\mathcal{Y}^{(1)}$ is a nondegenerate Lie algebra of symmetries of the distribution P_c. Actually, a shift along the trajectories of any field $X \in \mathcal{Y}^{(1)}$ shuffles the leaves of P and preserves the manifold H_c; hence it must also shuffle the leaves of P_c, which is the restriction of P to H_c. Nondegeneracy of $\mathcal{Y}^{(1)}$ follows from the fact that $\|\omega_i(X_j)\|$, $r < i, j \leq k$, is the unit matrix and, therefore, is nondegenerate.

Because of this, we can subject $\mathcal{Y}^{(1)}$ to the same procedure that was formerly applied to $\mathcal{Y}$. Namely, let

$$\mathcal{Y}^{(2)} = [\mathcal{Y}^{(1)}, \mathcal{Y}^{(1)}]$$

be the commutator subalgebra of $\mathcal{Y}^{(1)}$. If $\mathcal{Y}^{(2)} \neq \mathcal{Y}^{(1)}$, then some of the 1-form $\omega_{r+1}|_{dH_c}, \ldots, \omega_k|_{dH_c}$ are closed and give rise to local first integrals of the distribution P_c. The distribution P_c can be restricted to the mutual level surface of these integrals, etc.

We continue in this fashion obtaining the sequence of subalgebras

$$\mathcal{Y} \supset \mathcal{Y}^{(1)} \supset \mathcal{Y}^{(2)} \supset \cdots,$$

where $\mathcal{Y}^{(i+1)} = [\mathcal{Y}^{(i)}, \mathcal{Y}^{(i)}]$. If the algebra $\mathcal{Y}$ is *solvable*, i.e., $\mathcal{Y}^{(l)}$ is commutative for some l (which is equivalent to $\mathcal{Y}^{(l+1)} = 0$), then after l steps we shall be able to apply Corollary 4.3. All this work yields the following theorems (which, by the way, explain the term "solvable"):

THEOREM 4.4. *If a completely integrable distribution P of codimension k possesses a solvable k-dimensional nondegenerate Lie algebra of symmetries $\mathcal{Y} \subset \operatorname{sym} P$, then P is integrable by quadratures, i.e., one can find a complete set of first integrals for P by integrating closed 1-forms and solving functional equations.*

THEOREM 4.5 (Bianchi–Lie). *If an ordinary differential equation of order k possesses a solvable k-dimensional nondegenerate Lie algebra of symmetries, then it is integrable by quadratures.*

It is these results which caused the concept of solvability, originated in the Galois theory of algebraic equations, to be extended to the theory of Lie groups and algebras. Note also that the Lie groups of finite symmetries can be used instead of the algebras of symmetries, since one can always pass to the corresponding algebra by considering infinitesimal generators of the group.

EXAMPLE 4.9. Consider the equation

$$au^2 u''' + buu'u'' + cu'^3 = 0.$$

It possesses a solvable three-dimensional nondegenerate algebra of symmetries consisting of the translation along x (i.e., the field $\partial/\partial x$) and of two independent scale transformations (i.e., the fields $x\partial/\partial x$ and $u\partial/\partial u$). It is convenient to take the fields

$$X_1 = \frac{\partial}{\partial x},$$
$$X_2 = -x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2p_1\frac{\partial}{\partial p_1} + 3p_2\frac{\partial}{\partial p_2} + 4p_3\frac{\partial}{\partial p_3},$$
$$X_3 = u\frac{\partial}{\partial u} + p_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial p_2} + p_3\frac{\partial}{\partial p_3}$$

for a basis of the symmetry algebra and follow the above-described scheme to accomplish the integration.

REMARK 4.2. Interestingly, in the well-known Kamke's reference book [**49**] on ordinary differential equations, for each of the following two differential equations (numbers 7.8 and 7.9 in Kamke's book):

$$4u^2u''' - 18uu'u'' + 15u'^3 = 0,$$
$$9u^2u''' - 45uu'u'' + 40u'^3 = 0,$$

which both are of the type considered in Example 4.9, a different solution procedure is given.

EXERCISE 4.4. Prove that any Lie algebra consisting of infinitesimal translations and infinitesimal scale transformations is solvable. To this end, check that if $\mathcal{Y}$ is a Lie algebra of the considered type, then its commutator subalgebra $\mathcal{Y}^{(1)}$ consists of translations only and, hence, is commutative.

Note that any two-dimensional Lie algebra is solvable. Indeed, if X_1 and X_2 is a basis of $\mathcal{Y}$, then the subalgebra $\mathcal{Y}^{(1)}$ is generated by $[X_1, X_2]$ and, thus, $\mathcal{Y}^{(2)} = 0$. Therefore, any completely integrable distribution of codimension two, in particular, any second-order ordinary differential equation, can be integrated by quadratures, if a two-dimensional Lie algebra of its symmetries is known.

EXAMPLE 4.10. Consider the second-order equation

$$u'' = u' + u^n - \frac{2n+2}{(n+3)^2}u, \tag{4.1}$$

where $n \neq -3$, $n \in \mathbb{R}$.

The independent variable x does not enter into the equation explicitly, hence the field

$$X_1 = \frac{\partial}{\partial x}$$

is a symmetry. It is not hard to prove that the field

$$X_2 = e^{kx}\left[\frac{\partial}{\partial x} + \frac{k+1}{2}u\frac{\partial}{\partial u} + \left(\frac{k(k+1)}{2}u + \frac{1-k}{2}p\right)\frac{\partial}{\partial p}\right],$$

where $k = (1-n)/(n+3)$, is also a symmetry of the equation under consideration[5].

The geometric image associated with the given equation is the hypersurface $\mathcal{E}$ in the four-dimensional space with coordinates x, u, p, q, where p and q stand for u' and u'' respectively, defined by the equation

$$q = p + u^n - \frac{2n+2}{(n+3)^2}u,$$

together with the one-dimensional distribution on $\mathcal{E}$ given by the forms $\omega_1 = du - p\,dx$ and $\omega_2 = dp - q\,dx$.

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[5]Contrary to the previous example, where all symmetries was seen with "the naked eye", finding X_2 is not as straightforward. To do this, one needs the method of generating functions for symmetries of the Cartan distribution. It is discussed in §5.

The functions x, u, p can be taken for coordinates on $\mathcal{E}$. Consider the matrix

$$M = \|\omega_i(X_j)\| = \begin{pmatrix} -p & e^{kx}\left(\dfrac{k+1}{2}u - p\right) \\ -q & e^{kx}\left(\dfrac{k(k+1)}{2}u + \dfrac{1-k}{2}p - q\right) \end{pmatrix}.$$

The inverse matrix has the form

$$M^{-1} = \frac{1}{\Delta}\begin{pmatrix} e^{kx}\left(\dfrac{k(k+1)}{2}u + \dfrac{1-k}{2}p - q\right) & -e^{kx}\left(\dfrac{k+1}{2}u - p\right) \\ q & -p \end{pmatrix},$$

where $\Delta = \det M = e^{kx}T$,

$$T = -\frac{k(k+1)}{2}up - \frac{1-k}{2}p^2 + \frac{k+1}{2}uq.$$

Multiplying the matrix M^{-1} by the column consisting of ω_1 and ω_2, we obtain new basis forms defining $\mathcal{C}(\mathcal{E})$:

$$\omega_1' = \frac{1}{T}\left[\left(\frac{k(k+1)}{2}u + \frac{1-k}{2}p - q\right)du - \left(\frac{k+1}{2}u - p\right)dp\right] + dx,$$

$$\omega_2' = \frac{1}{\Delta}(q\,du - p\,dp).$$

By Theorem 4.1 the form ω_1' is closed, since $[X_1, X_2] = kX_2$. To find its integral, note that

$$\frac{\partial T}{\partial u} = \frac{1-k^2}{2}p + (1-k)u^n + \frac{(k+1)^2(k-1)}{4}u,$$

$$\frac{\partial T}{\partial p} = \frac{1-k^2}{2}u - (1-k)p.$$

Therefore, if $k \neq 1$, then the form ω_1' can be written as

$$\omega_1' = \frac{1}{(k-1)T}\left(\frac{\partial T}{\partial u}du + \frac{\partial T}{\partial p}dp\right) + dx = \frac{1}{k-1}\,d\ln(Te^{(k-1)x}).$$

Thus, $f = Te^{(k-1)x}$ is a first integral of the distribution. Note that we have reduced the order of equation (4.1), because it is equivalent to the family of first-order equations of the form

$$\frac{k-1}{2}p^2 + \frac{1-k^2}{2}up + \frac{k+1}{2}u^{n+1} + \frac{(k+1)^2(k-1)}{8}u^2 - ce^{(1-k)x} = 0, \qquad (4.2)$$

where c is an arbitrary constant.

Let us now restrict the distribution $\mathcal{C}(\mathcal{E})$ to the surface H_c defined by (4.2). On this surface, p can be expressed in terms of x and u as follows:

$$p = \frac{k+1}{2}u \pm \sqrt{\frac{1+k}{1-k}u^{n+1} + \frac{2c}{k-1}e^{(1-k)x}}. \qquad (4.3)$$

Furthermore, note that on H_c we have $\Delta = ce^x$. Therefore,

$$\omega_2'|_{H_c} = -d\left(\frac{e^{-x}p^2}{2c}\right) + \frac{e^{-x}}{c}\left(p + u^n + \frac{k^2-1}{4}u\right)du - \frac{e^{-x}}{2c}p^2\,dx,$$

where p is given by (4.3). From the proof of Theorem 4.4 it follows that this form is closed; its integral is

$$g = -\frac{e^{-x}}{2c}\left(\frac{k+1}{2}u \pm \sqrt{\frac{1+k}{1-k}u^{n+1} + \frac{2c}{k-1}e^{(1-k)x}}\right)^2 + \frac{e^{-kx}}{k(k-1)}$$

$$+ \frac{e^{-x}}{c}\left(\frac{u^{n+1}}{n+1} + \frac{(k+1)^2}{8}u^2\right) \pm \frac{e^{-x}}{c}\int_0^u \sqrt{\frac{1+k}{1-k}\eta^{n+1} + \frac{2c}{k-1}e^{(1-k)x}}\, d\eta.$$

All solutions of the equation at hand are given by the implicit relation $g = c_1$, $c_1 = \mathrm{const}$. The Chebyshev theorem implies that the integral g is an elementary function whenever $2/(n+1)$ is integer.

EXERCISE 4.5. 1. Complete the integration of 4.1 in the case $n = -1$.

2. Integrate the equation $u'' = u' + ae^{bu} - 2/b$ by means of the symmetry algebra with the generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = e^{-x}\left(\frac{\partial}{\partial x} + \frac{2}{b}\frac{\partial}{\partial u} + (p - \frac{2}{b})\frac{\partial}{\partial p}\right).$$

5. Generating functions

In this section, we describe the generating function method for finding symmetries of the Cartan distribution.

Let $\mathcal{E}$ be an ordinary differential equation of order k resolved with respect to the highest-order derivative:

$$u^{(k)} = f(x, u, u^{(1)}, \ldots, u^{(k-1)}). \tag{5.1}$$

Let us view x, $u = p_0, p_1, \ldots, p_{k-1}$ as coordinates on the hypersurface $\mathcal{E} = \{p_k = f(x, u, p_1, \ldots, p_{k-1})\}$. In this coordinates, the restriction of any vector field X to $\mathcal{E}$ is written as

$$X = \alpha\frac{\partial}{\partial x} + \beta_0\frac{\partial}{\partial p_0} + \cdots + \beta_{k-1}\frac{\partial}{\partial p_{k-1}}.$$

Note that the algebra of characteristic symmetries of $\mathcal{C}(\mathcal{E})$ consists of fields of the form $\lambda\bar{D}$, where λ is an arbitrary function on $\mathcal{E}$ and

$$\bar{D} = \frac{\partial}{\partial x} + p_1\frac{\partial}{\partial p_0} + \cdots + p_{k-1}\frac{\partial}{\partial p_{k-2}} + f\frac{\partial}{\partial p_{k-1}}. \tag{5.2}$$

is the *total derivative operator* with respect to x on the equation $\mathcal{E}$.

Let us remark that $\bar{D}(p_i) = p_{i+1}$, $i < k-1$, and, by (5.1), $\bar{D}(p_{k-1}) = f$.

In the quotient space of all fields by $\mathrm{char}\,\mathcal{C}(\mathcal{E})$, one has

$$\frac{\partial}{\partial x} = -p_1\frac{\partial}{\partial p_0} - \cdots - p_{k-1}\frac{\partial}{\partial p_{k-2}} - f\frac{\partial}{\partial p_{k-1}},$$

so that it is sufficient to search for symmetries only among vector fields of the form

$$X = \beta_0\frac{\partial}{\partial p_0} + \cdots + \beta_{k-1}\frac{\partial}{\partial p_{k-1}}. \tag{5.3}$$

For further calculations, it is convenient to consider the operation of *horizontalization* on the space of 1-form on $\mathcal{E}$ which is defined by

$$\daleth dx = dx,$$
$$\daleth dp_i = p_{i+1}\, dx, \quad i < k-1,$$
$$\daleth dp_{k-1} = f\, dx$$

and extended to all of the space of 1-form as a linear map over the ring of smooth functions.

LEMMA 5.1. *The following statements holds*:

1. *The operation of horizontalization takes a 1-form ω to zero if and only if ω vanishes on the Cartan distribution.*
2. *For any function $g \in C^\infty(\mathcal{E})$, one has*

$$\daleth dg = \bar{D}(g)\, dx.$$

PROOF. 1. Let $\omega = \gamma\, dx + \sum_i \gamma_i\, dp_i$. The equality $\daleth\omega = 0$ means that $\gamma + \sum_i \gamma_i p_{i+1} + \gamma_{k-1} f = 0$. Therefore,

$$\omega = -\left(\sum_i \gamma_i p_{i+1} + \gamma_{k-1} f\right) dx + \sum_i \gamma_i\, dp_i = \sum_i \gamma_i \omega_i.$$

2. It is straightforward to compute that

$$\daleth dg = \daleth\left(\frac{\partial g}{\partial x}\, dx + \sum_i \frac{\partial g}{\partial p_i}\, dp_i\right) = \left(\frac{\partial g}{\partial x} + \sum_i p_{i+1}\frac{\partial g}{\partial p_i} + f\frac{\partial g}{\partial p_{k-1}}\right) dx = \bar{D}(g)dx. \quad \square$$

Suppose now that a vector field X of the form (5.3) is a symmetry of the Cartan distribution. By virtue of Theorem 3.1 and Lemma 5.1, this is equivalent to

$$\daleth X(dp_i - p_{i+1}\, dx) = 0, \qquad i < k-1, \tag{5.4}$$
$$\daleth X(dp_{k-1} - f dx) = 0. \tag{5.5}$$

From (5.4) it follows that

$$(\bar{D}(\beta_i) - \beta_{i+1})\, dx = 0.$$

Hence, $\beta_{i+1} = \bar{D}(\beta_i)$ for all $i < k-1$. Denoting β_0 by φ, we arrive to the following expression for X:

$$X = \sum_{i=0}^{k-1} \bar{D}^i(\varphi)\frac{\partial}{\partial p_i} \tag{5.6}$$

In particular, we see that the field X, which is a symmetry of the Cartan distribution, is uniquely defined by one function $\varphi = X(u)$. The function φ will be referred to as the *generating function* of the field X given by (5.6). The symmetry with the generating function φ will be denoted by X_φ.

Clearly, the generating function of a symmetry for a given differential equation cannot be arbitrary. From (5.5) it follows that

$$\bar{D}^k(\varphi) - \sum_{i=0}^{k-1} \frac{\partial f}{\partial p_i}\bar{D}^i(\varphi) = 0. \tag{5.7}$$

The linear differential operator at the left-hand side of (5.7) is denoted by $\ell_F^{\mathcal{E}}$ and is called the *universal linearization*[6] of the function $F = p_k - f$ restricted to the equation $\mathcal{E}$.

The above discussion can be summarized as follows.

THEOREM 5.2. *Suppose that the equation $\mathcal{E}$ has the form $F = 0$, where $F = p_k - f(x, u, \ldots, p_{k-1})$; then the correspondence $\varphi \mapsto X_\varphi$ defines the isomorphism*

$$\ker \ell_F^{\mathcal{E}} \cong \operatorname{sym} \mathcal{E}.$$

REMARK 5.1. The assumption that the equation is resolved with respect to the the highest-order derivative is not in fact essential. In Ch. 4, we shall prove that this theorem holds for "almost all" functions F. In the general case, the operator $\ell_F^{\mathcal{E}}$ is defined by

$$\ell_F^{\mathcal{E}} = \sum_i \frac{\partial F}{\partial p_i} D^i.$$

Obviously, this formula gives the above expression for $\ell_F^{\mathcal{E}}$ in the case when $F = p_k - f(x, u, \ldots, p_{k-1})$.

Since the fields of the form (5.6) are in one-to-one correspondence with the generating functions, the generating function φ of the field X_φ is recovered by means of the formula

$$\varphi = X_\varphi \lrcorner \omega_0 = \omega_0(X_\varphi), \tag{5.8}$$

where $\omega_0 = dp_0 - p_1 \, dx$. Note that the form ω_0 vanishes on the characteristic symmetries, because $\omega_0(\bar{D}) = 0$, i.e., the right-hand side of (5.8) is defined by the coset of X_φ modulo the characteristic symmetries. So, to find the generating function, it is not necessary for the symmetry to be reduced to the form (5.6).

Note also that, since no differential other than dx and du appears in the expression of ω_0, the generating function of a symmetry X is uniquely determined by the coefficients at $\partial/\partial x$ and $\partial/\partial u$ in the coordinate expression for this symmetry:

$$\omega_0 \left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial u} + \cdots \right) = \beta - \alpha p_1.$$

Consider now a symmetry X such that its coefficients α and β depend on x and u only. Geometrically, this means that the image X_0 of X under the projection $\pi \colon \mathbb{R}^{k+2} \to \mathbb{R}^2$, $\pi(x, u, p_1, \ldots, p_k) = (x, u)$, is well defined. The field X_0 has the form

$$X_0 = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial u}.$$

In the case considered, the symmetry X is uniquely determined by the field X_0 and is called the *lifting* of X_0. Vector fields on the plane $\mathbb{R}^2$, i.e., infinitesimal changes of independent and dependent variables, are said to be infinitesimal *point transformations*[7].

Thus, point transformations (or, more precisely, the liftings of point transformations) are those symmetries of the Cartan distribution whose generating functions

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[6]The general theory of such operators is discussed in Ch. 4.

[7]The meaning of this term will be clarified in Ch. 2 and 3 in studies of more general contact transformations.

depend on x, u and p_1 only, with linear dependence on p_1. If the generating function is an arbitrary function of x, u and p_1, then the corresponding vector field is called *contact*. Symmetries whose generating functions depend on all coordinates $x, u, p_1, \ldots, p_{k-1}$ are called *higher*. The general theory of higher symmetries will be developed in Ch. 4.

Here are point transformations which are mostly frequently encountered in applications:

1. Translation along x: $X_0 = \partial/\partial x$. The generating function is $-p_1$.
2. Translation along u: $X_0 = \partial/\partial u$. The generating function is 1.
3. Scale transformation: $X_0 = ax\partial/\partial x + bu\partial/\partial u$, where a and b are constant. The generating function is $bu - axp_1$.

Consider now a system of ordinary differential equations. For the sake of simplicity, restrict ourselves to first-order systems resolved with respect to the highest-order derivatives:

$$\begin{cases} \dot{u}_1 = f_1(x, u_1, \ldots, u_n), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{u}_n = f_n(x, u_1, \ldots, u_n). \end{cases} \tag{5.9}$$

As before such a system is geometrized by considering the manifold $\mathcal{E}$ with the coordinates $x, u_1, \ldots, u_n$ equipped with the one-dimensional completely integrable Cartan distribution $\mathcal{C}(\mathcal{E})$. This distribution is defined by the forms

$$\begin{cases} \omega_1 = du_1 - f_1\, dx, \\ \cdots\cdots\cdots\cdots\cdots \\ \omega_n = du_n - f_n\, dx. \end{cases}$$

Its integral curves are graphs of solutions of the system under consideration. A counterpart of Theorem 5.2 in the case of systems of ordinary equations can be formulated as follows:

THEOREM 5.3. *The space of nontrivial infinitesimal symmetries of system* (5.9) *is isomorphic to the kernel of the matrix operator*

$$\ell_F^{\mathcal{E}} = \begin{pmatrix} \bar{D} - \partial f_1/\partial u_1 & -\partial f_1/\partial u_2 & \ldots & -\partial f_1/\partial u_n \\ -\partial f_2/\partial u_1 & \bar{D} - \partial f_2/\partial u_2 & \ldots & -\partial f_2/\partial u_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -\partial f_n/\partial u_1 & -\partial f_n/\partial u_2 & \ldots & \bar{D} - \partial f_n/\partial u_n \end{pmatrix},$$

where $\bar{D} = \partial/\partial x + f_1\partial/\partial u_1 + \cdots + f_n\partial/\partial u_n$, *acting on the space of n-vector functions of variables* $x, u_1, \ldots, u_n$.

The symmetry corresponding to a vector function $\varphi = (\varphi_1, \ldots, \varphi_n) \in \ker \ell_F^{\mathcal{E}}$ *under this isomorphism has the form*

$$X_\varphi = \varphi_1 \frac{\partial}{\partial u_1} + \cdots + \varphi_n \frac{\partial}{\partial u_n}.$$

The vector function φ is called the *generating section* of the symmetry X_φ.

Let us now discuss the Lie algebra structure on the space of generating functions and sections induced by the commutator operation of symmetries. Computing the commutator of symmetries $X_\varphi = \sum_i \bar{D}^i(\varphi)\partial/\partial p_i$ and $X_\psi = \sum_i \bar{D}^i(\psi)\partial/\partial p_i$ yields:

$$[X_\varphi, X_\psi] = \sum_j \sum_i \left(\bar{D}^i(\varphi) \frac{\partial \bar{D}^j(\psi)}{\partial p_i} - \bar{D}^i(\psi) \frac{\partial \bar{D}^j(\varphi)}{\partial p_i} \right) \frac{\partial}{\partial p_j}.$$

Denote the generating function of the field obtained by $\{\varphi, \psi\}$. Contracting this field with $\omega_0 = dp_0 - p_1\, du$, we get

$$\{\varphi, \psi\} = \sum_{i=0}^{k-1} \left(\bar{D}^i(\varphi) \frac{\partial \psi}{\partial p_i} - \bar{D}^i(\psi) \frac{\partial \varphi}{\partial p_i} \right).$$

This bracket is called the *higher Jacobi bracket* (on the equation $\mathcal{E}$).

For a system of the form (5.9) the Jacobi bracket $\chi = \{\varphi, \psi\}$ of generating sections $\varphi = (\varphi_1, \ldots, \varphi_n)$ and $\psi = (\psi_1, \ldots, \psi_n)$ is given by the formula

$$\chi_i = \sum_{j=1}^{n} \left(\varphi_j \frac{\partial \psi_i}{\partial u_j} - \psi_j \frac{\partial \varphi_i}{\partial u_j} \right), \qquad j = 1, \ldots, n,$$

which immediately follows from the fact that the symmetry with the generating section φ has the form $X_\varphi = \varphi_1 \partial/\partial u_1 + \cdots + \varphi_n \partial/\partial u_n$ and that the Jacobi bracket corresponds to the commutator of symmetries.

6. How to search for equations integrable by quadratures: an example of using symmetries

Let us find all equations of the form

$$u'' = u' + f(u) \tag{6.1}$$

possessing a two-dimensional Lie algebra of point symmetries [**26, 77**]. As it was mentioned above, all two-dimensional algebras are solvable, and therefore all such equations are integrable by quadratures.

Since the independent variable x does not enter explicitly into the equation, the translation along x is a symmetry. So, the problem is to determine when this equation has one more symmetry with a generating function of the form

$$\varphi = \alpha p + \beta, \tag{6.2}$$

where $p = p_1$, α and β are functions of x, u, such that $\{p, \varphi\}$ is a linear combination of p and φ. In what follows, we exclude the trivial case when the function $f(u)$ is linear.

We shall use variables x, u, p as coordinates on the surface $\mathcal{E}$ corresponding to equation (6.1). In these coordinates, the operator of total derivative has the form

$$\bar{D} = \frac{\partial}{\partial x} + p \frac{\partial}{\partial u} + (p + f) \frac{\partial}{\partial p}. \tag{6.3}$$

The generating function of a symmetry $\varphi(x, u, p)$ must satisfy the equation

$$\bar{D}^2(\varphi) - \bar{D}(\varphi) - f'\varphi = 0.$$

Taking into account equalities (6.2) and (6.3), we can rewrite this equation as

$$\alpha_{uu} p^3 + (2\alpha_u + 2\alpha_{xu} + \beta_{uu}) p^2 + (3\alpha_u f + \alpha_x + \alpha_{xx} + 2\beta_{xu}) p \\ + (2\alpha_x f + \beta_u f + \beta_{xx} - \beta_x - \beta f') = 0, \tag{6.4}$$

where the indices x and u label the derivative with respect to these variables. The left-hand side of this expression is a polynomial in p, so that the coefficients of this

polynomial vanish. Thus, equation (6.4) is equivalent to the system

$$\begin{cases} \alpha_{uu} = 0, \\ 2\alpha_u + 2\alpha_{xu} + \beta_{uu} = 0, \\ 3f\alpha_u + \alpha_x + \alpha_{xx} + 2\beta_{xu} = 0, \\ 2f\alpha_x + f\beta_u + \beta_{xx} - \beta_x - f'\beta = 0. \end{cases} \tag{6.5}$$

The first equation yields

$$\alpha = \gamma u + \delta, \qquad \gamma, \delta \in C^\infty(x),$$

where $C^\infty(x)$ stands for the ring of smooth functions of x. Substituting this expression into the second equation, we obtain

$$\beta = -(\gamma' + \gamma)u^2 + \varepsilon u + \zeta, \qquad \varepsilon, \zeta \in C^\infty(x).$$

Then the third equation is reduced to

$$3f\gamma = 3(\gamma' + \gamma'')u - \delta' - \delta'' - 2\varepsilon'.$$

Since the function $f(u)$ was supposed to be nonlinear, it follows that $\gamma = 0$, $\varepsilon = (\varkappa - \delta - \delta')/2$, where $\varkappa = \text{const}$.

Now, the last equation of system (6.5) takes the form

$$(\varepsilon u + \zeta)f' - \eta f = \theta u + \lambda, \tag{6.6}$$

where $\eta = 2\delta' + \varepsilon$, $\theta = \varepsilon'' - \varepsilon'$, $\lambda = \zeta'' - \zeta'$ are functions belonging to $C^\infty(x)$.

Equation (6.6) is an ordinary differential equation for $f(u)$, involving the variable x as a parameter. If $\varepsilon \neq 0$, $\eta \neq 0$, and $\varepsilon \neq \eta$, its general solution has the form

$$f(u) = \mu\left(u + \frac{\zeta}{\varepsilon}\right)^{\frac{\eta}{\varepsilon}} + \frac{\theta}{\varepsilon - \eta}\left(u + \frac{\zeta}{\varepsilon}\right) + \frac{\theta\zeta - \varepsilon\lambda}{\varepsilon\eta}, \tag{6.7}$$

where $\mu \in C^\infty(x)$.

Of all the functions (6.7), we have to choose those which does not depend on x and are nonlinear in u. If $\eta/\varepsilon \neq 2$, then these requirements hold if and only if

$$\mu = a, \qquad \frac{\zeta}{\varepsilon} = b, \qquad \frac{\eta}{\varepsilon} = c,$$
$$\frac{\theta}{\varepsilon - \eta} = d, \qquad \frac{\theta\zeta - \varepsilon\lambda}{\varepsilon\eta} = e, \tag{6.8}$$

where a, b, c, d, and e are some constants.

Taking into consideration all the relations on the functions δ, ε, η, ζ, θ, λ and using (6.8), we easily obtain the following expressions:

$$\varepsilon = he^{kx}, \quad \zeta = b\varepsilon, \quad \eta = c\varepsilon, \quad \theta = (k^2 - k)\varepsilon, \quad \lambda = b\theta,$$

where $k = (1 - c)/(c + 3)$ and h is a constant. Hence,

$$f(u) = a(u + b)^c - \frac{2c + 2}{(c + 3)^2}(u + b). \tag{6.9}$$

From the relations on δ, ε, η, ζ, θ, and λ it follows that if $\eta = 2\varepsilon$ and function (6.7) does not depend on x, then it has the form (6.9) for $c = 2$.

Now consider the possibilities previously excluded. Either of the assumptions $\eta = 0$ and $\eta = \varepsilon$ results in the linearity of $f(u)$. In the case $\varepsilon = 0$, we have a new series of solutions to equation (6.6):

$$f(u) = ae^{bu} - \frac{2}{b}, \qquad a, b = \text{const.}$$

From the computation above, we obtain the following

THEOREM 6.1. *Among all nonlinear equations of the form*

$$u'' = u' + f(u)$$

only equations of the two following series:

1. $u'' = u' + a(u+b)^c - \dfrac{2c+2}{(c+3)^2}(u+b) \qquad a,b,c \in \mathbb{R}, \quad c \neq 1, -3$

2. $u'' = u' + ae^{bu} - \dfrac{2}{b}, \qquad a,b \in \mathbb{R}, \quad b \neq 0.$

possess a two-dimensional algebra of point symmetries. In the first case, this algebra spans the generating functions

$$\begin{cases} \varphi_1 = p, \\ \varphi_2 = e^{kx}\left(p - \dfrac{k+1}{2}(u+b)\right), \end{cases} \qquad k = \frac{1-c}{c+3}.$$

In the second case, the algebra is generated by

$$\begin{cases} \varphi_1 = p, \\ \varphi_2 = e^{-x}\left(p - \dfrac{2}{b}\right). \end{cases}$$

First-Order Equations

In this chapter, the geometric approach to differential equations and their solutions, discussed above in the case of ordinary differential equation, is extended to scalar differential equations of first order.

As in Ch. 1, we interpret such differential equations as manifolds endowed with the Cartan distribution and, in this language, study their symmetries, integrability, etc.

Geometric theory of first-order equations is closely related to symplectic geometry and, in particular, Hamiltonian mechanics. These questions are also the concern of the present chapter.

1. Contact transformations

1.1. Contact elements and the Cartan distribution. Let $x_1, \ldots, x_n$ denote coordinates on $\mathbb{R}^n$. In this coordinates, a first-order differential equation for one unknown function u has the form

$$F\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = 0,$$

where F is a smooth function of $2n + 1$ variables.

Denote the partial derivatives of u by $p_1 = \partial u/\partial x_1, \ldots, p_n = \partial u/\partial x_n$ and consider the $(2n+1)$-dimensional vector space $J^1(\mathbb{R}^n)$ with coordinates $x_1, \ldots, x_n$, $u, p_1, \ldots, p_n$. A differential equation gives rise to a locus

$$\mathcal{E} = \{F(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = 0\}$$

in this space.

As a rule, we shall assume that the differential

$$dF = \sum_{i=1}^n \frac{\partial F}{\partial x_i}\, dx_i + \frac{\partial F}{\partial u}\, du + \sum_{i=1}^n \frac{\partial F}{\partial p_i}\, dp_i$$

is nonzero on some everywhere dense subset of the hypersurface $\mathcal{E}$.

Under this assumption, any function G vanishing on $\mathcal{E}$ has the form $G = \mu F$ near the hypersurface $\mathcal{E}$, where μ is a smooth function of $x_1, \ldots, x_n$, u, $p_1, \ldots, p_n$.

Differential equations on manifolds other than $\mathbb{R}^n$ (e.g., on torus) are not infrequent in applications. Thus, let us define, in a coordinate-free manner, the space $J^1(M)$ for an arbitrary smooth manifold M.

Consider the $(n + 1)$-dimensional manifold $J^0(M) = M \times \mathbb{R}$; a point of $J^0(M)$ is a pair (x, u), where $x \in M$, $u \in \mathbb{R}$. A *contact element* at a point $\theta \in J^0(M)$ is a pair (θ, L), where $L \subset T_\theta(J^0(M))$ is an n-dimensional plane. A contact element (θ, L) is called *nonsingular*, if the plane L does not contain the vertical vector $\partial/\partial u$.

EXERCISE 1.1. Prove that the set of all nonsingular contact elements on $J^0(M)$ is a smooth manifold.

DEFINITION 1.1. The manifold of all nonsingular contact elements on $J^0(M)$ is called the *manifold of 1-jets* of smooth functions on M and is denoted by $J^1(M)$.

Note that the projections

$$\pi_{1,0}\colon J^1(M) \longrightarrow J^0(M), \qquad \pi_{1,0}(\theta, L) = \theta \tag{1.1}$$

$$\pi_1\colon J^1(M) \longrightarrow M, \qquad \pi_1((x, u), L) = x, \quad (x, u) \in J^0(M). \tag{1.2}$$

are well-defined and are smooth vector bundles. The $C^\infty(M)$-module of sections of bundle (1.2) will be denoted by $\mathcal{J}^1(M)$.

The manifold $J^1(M)$ is basic to geometric formulation of first-order differential equations.

Let $\Gamma(f) \subset J^0(M)$ be the graph of a function $f \in C^\infty(M)$, i.e., $\Gamma(f)$ consists of the points $b = (a, f(a))$, where $a \in M$. At every point $b = (a, f(a))$ of this graph, consider the nonsingular contact element (b, L), where $L = T_a(\Gamma(f))$ is the tangent plane to the graph. Clearly, every nonsingular contact element $\theta = (b, L_\theta) \in J^1(M)$ is tangent to the graph of an appropriate function.

Let us give a coordinate description of contact elements. Take a function $f \in C^\infty(M)$ and suppose that in local coordinates $x_1, \ldots, x_n, u$ it has the form $u = f(x_1, \ldots, x_n)$. Let $\xi_1, \ldots, \xi_n, \eta$ be coordinates on $T_b(J^0(M))$, where $b = (a, f(a))$, $a \in M$, with respect to the basis $\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial u$. Clearly, the plane tangent to the graph of f is given by the equation

$$\eta = p_1 \xi_1 + \cdots + p_n \xi_n, \tag{1.3}$$

where $p_1 = \partial f/\partial x_1(a), \ldots, p_n = \partial f/\partial x_n(a)$.

Thus, on the manifold of 1-jets there exist *special* (or *canonical*) *local coordinates* $x_1, \ldots, x_n, u, p_1, \ldots, p_n$. Their geometric meaning is as follows: if $\theta = (b, L_\theta) \in J^1(M)$, where L_θ is the plane tangent to the graph of the function $u = u(x_1, \ldots, x_n)$ at $b = (a, f(a))$, then the contact element (b, L_θ) has the coordinates

$$\left(a_1, \ldots, a_n, f(a), \frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right).$$

EXERCISE 1.2. Show that $J^1(M) = T^*(M) \times \mathbb{R}$.

By this exercise, one can define the projection $\pi\colon J^1(M) \to T^*(M)$. In the special coordinates it takes the form

$$\pi(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = (x_1, \ldots, x_n, p_1, \ldots, p_n). \tag{1.4}$$

Let us now discuss a geometric interpretation of solutions to a given differential equation. For any smooth function $f \in C^\infty(M)$, define the map

$$j_1(f)\colon M \longrightarrow J^1(M), \qquad j_1(f)\colon a \longmapsto (b, L_{f(a)}), \tag{1.5}$$

which takes each point $a \in M$ to the nonsingular contact element consisting of the point $b = (a, f(a))$ and the plane tangent to the graph of f at this point.

The map $j_1(f)$ is said to be the *1-jet* of the function f. Its graph $N_f = j_1(f)(M) \subset J^1(M)$ is a n-dimensional manifold. In the special local coordinates

FIGURE 2.1
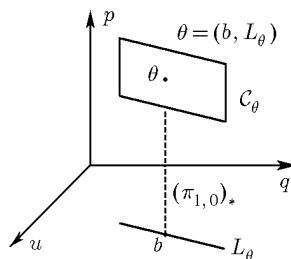
$x_1, \ldots, x_n$, u, $p_1, \ldots, p_n$ this manifold is obviously given by the equations

$$
\begin{cases}
u = f(x_1, \ldots, x_n), \\
p_1 = \dfrac{\partial f}{\partial x_1}(x_1, \ldots, x_n), \\
\quad\ldots\ldots\ldots\ldots\ldots\ldots \\
p_n = \dfrac{\partial f}{\partial x_n}(x_1, \ldots, x_n).
\end{cases}
\tag{1.6}
$$

Note that, similar to the case of one independent variable, not every n-dimensional submanifold in $J^1(M)$ has the form N_f. To describe submanifolds of the form N_f, we supply the manifold $J^1(M)$ with the *Cartan distribution*, which is defined below.

Let $\theta = (b, L_\theta)$ be a point of $J^1(M)$, $L_\theta \subset T_b(J^0(M))$. Denote by $\mathcal{C}_\theta$ the set of vectors in the tangent space $T_\theta(J^1(M))$ whose images under the projection $(\pi_{1,0})_*$ belong to the plane L_θ:

$$
\mathcal{C}_\theta = \{\, \xi \in T_\theta(J^1(M)) \mid (\pi_{1,0})_*(\xi) \in L_\theta \,\}.
$$

$\mathcal{C}_\theta$ is a hyperplane called the *Cartan plane* at $\theta \in J^1(M)$ (see Figure 2.1).

DEFINITION 1.2. The distribution

$$
\theta \longmapsto \mathcal{C}_\theta, \qquad \theta \in J^1(M),
$$

on the manifolds of 1-jets $J^1(M)$ is called the *Cartan distribution*.

PROPOSITION 1.1. *Let $f \in C^\infty(M)$ be a function, $a \in M$, $\theta = j_1(f)(a) \in N_f$. Then the manifold N_f is tangent to $\mathcal{C}_\theta$.*

PROOF. The map $\pi_{1,0}$ takes the manifold N_f to the graph of f. Therefore the tangent plane $T_\theta(N_f)$ is projected to the tangent plane to the graph of f at the point $b = (a, f(a))$, i.e., on L_θ. Hence, $T_\theta(N_f) \subset \mathcal{C}_\theta$. $\square$

This proposition can be restated as follows:

COROLLARY 1.2. *The image of the map $j_1(f)\colon M \to J^1(M)$ is an integral manifold of the Cartan distribution.*

EXERCISE 1.3. a. Show that the fibers of the projection (1.1) are integral manifolds of the Cartan distribution.

b. Let V be an arbitrary k-dimensional submanifold of $J^0(M)$, $1 \le k \le n = \dim M$. Consider the manifold $P \subset J^1(M)$ consisting of contact elements (b, L)

such that $b \in V$ and $L \supset T_b(V)$. Prove that P is an integral manifold of the Cartan distribution.

Let us now define a differential form U_1 which determines the Cartan distribution on $J^1(M)$. Take a contact element $\theta = (b, L_\theta)$, where $b \in J^0(M)$ and $L_\theta \subset T_b(J^0(M))$. The image of a tangent vector $\xi \in T_\theta(J^1(M))$ under the projection $\pi_{1,0} \colon J^1(M) \to J^0(M)$, $\pi_{1,0}(\theta) = b$, can be uniquely decomposed to the sum $\xi = \xi_0 + \xi_0'$, where $\xi_0 = (\pi_{1,0})_*(\xi) = \rho_\xi \, \partial/\partial u|_b$, $\xi_0' \in L_\theta$. Set $U_1(\xi) = \rho_\xi$. The following proposition is obvious.

PROPOSITION 1.3. $\xi \in \mathcal{C}_\theta$ if and only if $U_1(\xi) = 0$.

Thus, the 1-form U_1 defines the Cartan distribution. Let us describe this form in local coordinates.

Let $x_1, \ldots, x_n$, u, $p_1, \ldots, p_n$ be special local coordinates on $J^1(M)$ in the vicinity of a point θ. Since the plane $L_\theta \subset T_b(J^0(M))$ is given by the equation $\eta = p_1\xi_1 + \cdots + p_n\xi_n$, where $\xi_1, \ldots, \xi_n$, η are the coordinates in $T_b(J^0(M))$, we have

$$\sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} + \left(\sum_{i=1}^n p_i\xi_i \right) \frac{\partial}{\partial u} \in L_\theta. \tag{1.7}$$

Suppose that

$$\xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} + \sum_{i=1}^n \zeta_i \frac{\partial}{\partial p_i} \in T_\theta(J^1(M));$$

then

$$(\pi_{1,0})_*(\xi) = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u}.$$

From (1.7) it follows that

$$(\pi_{1,0})_*(\xi) = \left(\eta - \sum_{i=1}^n p_i\xi_i \right) \frac{\partial}{\partial u} + \chi,$$

with $\chi \in L_\theta$. Hence,

$$U_1 = du - \sum_{i=1}^n p_i \, dx_i. \tag{1.8}$$

REMARK 1.1. Contact geometry closely resembles symplectic one. This is essentially due to the isomorphism $J^1(M) = T^*M \times \mathbb{R}$. For a detailed discussion of symplectic geometry see, e.g., [**33, 3, 7**].

In view of this remark, it is not surprising that there exists a universal element on the manifold $J^1(M)$ similar to the universal form $p \, dq$ in symplectic geometry [**72, 77**].

PROPOSITION 1.4. *There exists a unique element $\rho \in \mathcal{J}^1(J^1(M))$ on the manifold $J^1(M)$ such that for any section $\theta \in \mathcal{J}^1(M)$ one has $\theta^*(\rho) = \theta$.*

PROOF. Consider an arbitrary point $x \in J^1(M)$. It is clear that there exists a function $f \in C^\infty(M)$ such that $x = j_1(f)|_a$, $a = \pi_1(x)$. By definition, put $\rho|_x = j_1(\pi_1^*(f))|_x$. Let us prove the equality $\theta^*(\rho) = \theta$. Take $x = \theta(a)$. We have

$$\theta^*(\rho)|_a = \theta^*(j_1(\pi_1^*(f))|_x) = j_1(\theta^*\pi_1^*(f))|_a = j_1(f)|_a = x = \theta(a).$$

To prove uniqueness of ρ, observe that if one has $(\theta)^*(\rho') = 0$ for some element ρ' and all $\theta \in \mathcal{J}^1(M)$, then $\rho' = 0$. $\square$

EXERCISE 1.4. Prove that in the local coordinates the element ρ has the form $\sum_{i=1}^n p_i\, dx_i$.

The decomposition $J^1(M) = T^*(M) \times \mathbb{R}$ induces the decomposition $\mathcal{J}^1(M) = \Lambda^1(M) \oplus C^\infty(M)$. Define an operator $S\colon \mathcal{J}^1(M) \to \Lambda^1(M)$ by the formula[1] $S(f, \omega) = df - \omega$.

EXERCISE 1.5. Prove that the image of the operator $j_1\colon C^\infty(M) \to \mathcal{J}^1(M)$ coincides with the kernel of the operator[2] S.

It is readily seen that $U_1 = S(\rho)$.

The following proposition states that the converse of Corollary 1.2 is also true:

PROPOSITION 1.5. *A submanifold $N \subset J^1(M)$ that projects to M without degeneration is a maximal integral manifold of the Cartan distribution if and only if it is the graph of a function $f \in C^\infty(M)$, i.e., $N = j_1(f)(M)$.*

PROOF. An n-dimensional manifold $N \subset J^1(M)$ that projects to M without degeneration has the form

$$\begin{cases} u = f(x_1, \ldots, x_n), \\ p_1 = g_1(x_1, \ldots, x_n), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ p_n = g_n(x_1, \ldots, x_n). \end{cases}$$

The manifold N is an integral manifold of the Cartan distribution if and only if $U_1|_N = 0$. In the coordinates $x_1, \ldots, x_n$ on N, the restriction of U_1 to N can be written as $\sum_{i=1}^n (\partial f/\partial x_i - g_i)\, dx_i$. Thus, N will be an integral manifold if and only if $g_i(x_1, \ldots, x_n) = \partial f/\partial x_i$, $i = 1, \ldots, n$, and this is precisely the assertion of the proposition. $\square$

Thus, the Cartan distribution on $J^1(M)$ is the geometric structure which distinguishes the graphs of 1-jets from all n-dimensional submanifolds of $J^1(M)$.

Consider a submanifold $\mathcal{E} \subset J^1(M)$. The Cartan distribution on $J^1(M)$ induces the *Cartan distribution $\mathcal{C}(\mathcal{E})$ on $\mathcal{E}$*, $\mathcal{C}(\mathcal{E})_\theta = \mathcal{C}_\theta \cap T_\theta(\mathcal{E})$.

DEFINITION 1.3.   1. A submanifold $\mathcal{E} \subset J^1(M)$ of codimension one supplied with the induced Cartan distribution is called a *first-order differential equation.*

  2. A *generalized solution* $L \subset \mathcal{E}$ to an equation $\mathcal{E}$ is an n-dimensional maximal integral manifold of the Cartan distribution on $\mathcal{E}$.

EXAMPLE 1.1. Consider the stationary Hamilton–Jacobi equation

$$\left(\frac{\partial S}{\partial q}\right)^2 + q^2 = C, \qquad C = \text{const},$$

---

[1]The operator S is the simplest operator in the series of the Spencer operators

$$S_{k,l}\colon \mathcal{J}^k \otimes \Lambda^l \to \mathcal{J}^{k-1} \otimes \Lambda^{l+1}, \, S_{k,l}(aj_k(f) \otimes \omega) = j_{k-1}(f) \otimes (da \wedge \omega).$$

For discussion of the Spencer operators see [**60**].

[2]The exact sequence

$$0 \longrightarrow C^\infty(M) \overset{j_1}{\longrightarrow} \mathcal{J}^1(M) \overset{S}{\longrightarrow} \Lambda^1(M) \longrightarrow 0$$

is the simplest Jet-Spencer complex (see [**60**]).
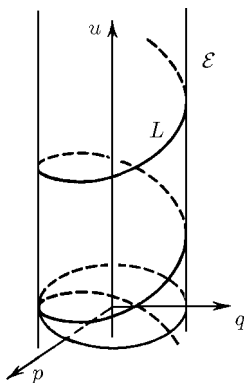
FIGURE 2.2
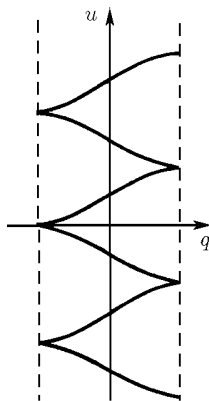
FIGURE 2.3

describing one-dimensional harmonic oscillator. The corresponding locus $p^2 + q^2 = C$ is the cylinder in the (q, u, p)-space. The Cartan distribution $\mathcal{C}(\mathcal{E})$ is a direction field on this cylinder. Integral curves of $\mathcal{C}(\mathcal{E})$ (see Figure 2.2) are generalized solutions of the equation. It is obvious that these curves project to the q-axis with degeneration. Figure 2.3 shows the projections of these curves to the (q, u)-plane.

If the tangent spaces $T_\theta(\mathcal{E})$ nowhere coincide with the Cartan planes $\mathcal{C}_\theta$, then the induced Cartan distribution $\mathcal{C}(\mathcal{E})$ is of codimension 1.

DEFINITION 1.4. A point $\theta \in \mathcal{E}$ is called a *singular point of the equation* $\mathcal{E}$ if $T_\theta(\mathcal{E}) = \mathcal{C}_\theta$.

EXERCISE 1.6. Show that the origin of coordinates is a singular point of the equations

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} = f \qquad \text{and} \qquad \left(\frac{\partial f}{\partial x_1} \right)^2 + \left(\frac{\partial f}{\partial x_2} \right)^2 = f.$$

Below, we work, as a rule, in the vicinity of nonsingular points of $\mathcal{E}$.

1.2. Contact transformations. A smooth transformation $J^1(M) \to J^1(M)$ is called a *contact transformation*, if it takes the Cartan distribution to itself.
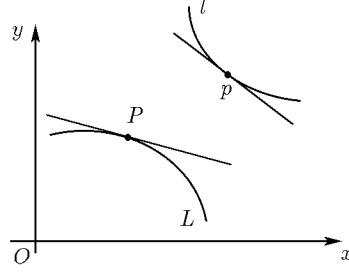
FIGURE 2.4

Thus, $F\colon J^1(M) \to J^1(M)$ is a contact transformation, if either of the following two equivalent conditions holds:

1. $F_*(\mathcal{C}_\theta) \subset \mathcal{C}_{F(\theta)}$ for all $\theta \in J^1(M)$.
2. $F^*(U_1) = \lambda U_1$, where $\lambda \in C^\infty(J^1(M))$.

In coordinates the second condition takes the form:

$$dU - \sum_{i=1}^{n} P_i \, dX_i = \lambda\left(du - \sum_{i=1}^{n} p_i \, dx_i \right), \tag{1.9}$$

where $X_i = F^*(x_i)$, $U = F^*(u)$, $P_i = F^*(p_i)$, $\lambda \in C^\infty(J^1(M))$.

Historically, contact transformations were understood as "transformations of curves on the (x, y)-plane" of the form

$$\begin{aligned} X &= f(x, y, y'), \\ Y &= \varphi(x, y, y'), \end{aligned} \tag{1.10}$$

satisfying the following condition: that the images of any two curves tangent to each other are also tangent. Consider a curve l given by an equation $y = y(x)$ and a point $p \in l$. Suppose that a transformation F of the form (1.10) takes l to a curve L given by an equation $Y = Y(X)$ and $P = F(p)$ (see Figure 2.4). It is clear that

$$\frac{dY}{dX} = \frac{\dfrac{\partial \varphi}{\partial x} + \dfrac{\partial \varphi}{\partial y}y' + \dfrac{\partial \varphi}{\partial y'}y''}{\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}y' + \dfrac{\partial f}{\partial y'}y''} = \frac{\left(\dfrac{\partial \varphi}{\partial x} + \dfrac{\partial \varphi}{\partial y}y'\right)\left(1 + \dfrac{\dfrac{\partial \varphi}{\partial y'}y''}{\dfrac{\partial \varphi}{\partial x} + \dfrac{\partial \varphi}{\partial y}y'}\right)}{\left(\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}y'\right)\left(1 + \dfrac{\dfrac{\partial f}{\partial y'}y''}{\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}y'}\right)}.$$

Obviously, dY/dX does not depend on y'' (this is just the condition that the transformation F preserves the tangency of curves), if

$$\frac{\partial \varphi}{\partial y'}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y'\right) = \frac{\partial f}{\partial y'}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}y'\right). \tag{1.11}$$

It can easily be checked that (1.11) exactly means that the transformation $J^1(\mathbb{R}^1) \to J^1(\mathbb{R}^1)$, $j_1(y(x))|_p \mapsto j_1(Y(X))|_P$ preserves the differential form $dy - p\,dx$.

Contact transformations can be obtained by lifting smooth transformations of the space $J^0(M)$. Namely, let $F\colon J^0(M) \to J^0(M)$ be such a transformation. In

the coordinates $x_1, \ldots, x_n$, u, it has the form

$$(x_1, \ldots, x_n, u) \longmapsto (a_1(x, u), \ldots, a_n(x, u), b(x, u)). \tag{1.12}$$

This transformation induces the transformation of $J^1(M)$. The coordinate description of the latter is as follows.

EXERCISE 1.7. Show that under the transformation (1.12)

$$\begin{pmatrix} p_1 \\ \ldots \\ p_n \end{pmatrix} \longmapsto \Delta^{-1} \times \begin{pmatrix} \partial b/\partial x_1 + p_1 \partial b/\partial u \\ \ldots \\ \partial b/\partial x_n + p_n \partial b/\partial u \end{pmatrix}, \tag{1.13}$$

where Δ is the matrix with the elements $\Delta_{ik} = \partial a_k/\partial x_i + p_i \partial a_k/\partial u$.

EXERCISE 1.8. Prove that the transformation given by (1.12) and (1.13) is contact.

Formulas (1.12) and (1.13) define the transformation of $J^1(M)$, which is called the *lifting* of a given transformation of $J^0(M)$ to $J^1(M)$. The lifting of F takes each contact element $\theta = (b, L_\theta)$ to the element $\bar{F}(\theta) = (F(b), (F_*)_b(L_\theta))$.

Since $\pi_{1,0} \circ \bar{F} = F \circ \pi_{1,0}$ and $(F_*)_b(L_\theta) = L_{\bar{F}(\theta)}$, we have $(\bar{F}_*)_b(\mathcal{C}_\theta) \subset \mathcal{C}_{\bar{F}(\theta)}$. Thus, the transformations of the form $\bar{F}$ preserve the Cartan distribution and, therefore, take the integral manifolds of this distribution to themselves[3].

Contact transformations that are liftings of transformations of $J^0(M)$ are called *point transformations.*.

Point transformations do not exhaust contact ones.

EXAMPLE 1.2. Consider the *Legendre transformation*

$$L: \begin{cases} x_i \longmapsto p_i, \\ u \longmapsto \sum_{k=1}^{n} x_k p_k - u, \\ p_i \longmapsto x_i, \qquad i = 1, \ldots, n, \end{cases} \tag{1.14}$$

where $x_1, \ldots, x_n$, u, $p_1, \ldots, p_n$ are special local coordinates.

EXERCISE 1.9. Show that the Legendre transformation preserves the Cartan distribution.

EXERCISE 1.10. Let $F\colon J^0(M) \to J^0(M)$ be a transformation of the form $F(x, u) = (f(x), u)$, where $f\colon M \to M$. Prove that the lifting $\bar{F}$ of F preserves the form U_1, i.e., $\bar{F}^*(U_1) = U_1$.

EXAMPLE 1.3. The following transformation is a generalization of the Legendre transformation and is called the *Euler transformation*:

$$E: \begin{cases} x_i \longmapsto p_i, \\ x_l \longmapsto x_l, \\ u \longmapsto \sum_{s=1}^{k} x_s p_s - u, \\ p_i \longmapsto x_i, \\ p_l \longmapsto -p_l, \qquad i = 1, \ldots, k, \quad l = k+1, \ldots, n. \end{cases}$$

---

[3]Strictly speaking, the lifting fails to exist at the points θ for which the element $\bar{F}(\theta)$ is singular. Analytically, this means that the matrix Δ is not invertible at these points. In what follows we do not consider such points.
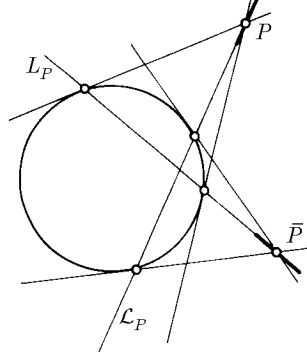
FIGURE 2.5

It can easily be checked that $E^*(U_1) = -U_1$, so that the Euler transformation is contact.

Here are more examples of contact transformations (see [**33**]).

EXAMPLE 1.4. Consider the transformation given by

$$\bar{x}_i = x_i + \frac{\theta p_i}{\sqrt{1 + \sum_{k=1}^n p_k^2}}, \quad \bar{u} = u - \frac{\theta}{\sqrt{1 + \sum_{k=1}^n p_k^2}}, \quad \bar{p}_i = p_i. \quad (1.15)$$

EXERCISE 1.11. Prove that transformation (1.15) is contact. Is it a point transformation?

EXAMPLE 1.5. The transformation of $\mathbb{R}^3$ given by the formulas

$$\bar{q} = \frac{p}{pq - u}, \quad \bar{u} = -\frac{1}{pq - u}, \quad \bar{p} = -\frac{q}{u}, \quad (1.16)$$

is contact. Its geometric meaning is as follows. Consider the unit circle

$$q^2 + u^2 = 1.$$

(see Figure 2.5). Recall that the polar line of a point $P = (a, b)$ with respect to this circle is the straight line L_P given by the equation

$$aq + bu - 1 = 0.$$

If P lies outside the circle, then its polar line passes through the intersection points of the tangents to the circles passing through P with the circle.

Consider the straight line $\mathcal{L}_P$ passing through P with a slope equal to p_0. There exists a unique point on L_P such that $\mathcal{L}_P$ is its polar line with respect to the circle.

It is not hard to prove that the coordinates of P are given by (1.16).

By taking an arbitrary second-order curve instead of the circle, one can define a contact transformation in a similar way.

EXAMPLE 1.6. Consider the contact transformation of $\mathbb{R}^3$ given by the formulas

$$\bar{x} = \frac{(xp - u)p}{1 + p^2}, \quad \bar{u} = -\frac{xp - u}{1 + p^2}, \quad \bar{p} = \frac{xp^2 - x - 2up}{up^2 - u + 2xp}. \quad (1.17)$$

and called the *pedal transformation*. Let us give its geometric interpretation (see Figure 2.6).
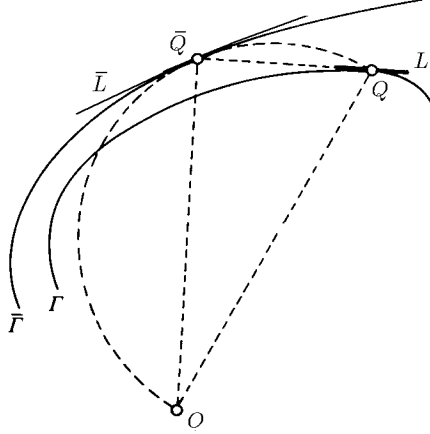
FIGURE 2.6

Let O be the origin of coordinates. Consider a point $Q = (x_0, u_0)$ and the straight line L passing through Q with a slope equal to p. Drop a perpendicular to L from O. Through the foot of the perpendicular let us pass a tangent to the circle with the diameter OQ. It can easily be checked that formulas (1.17) give the coordinates of $\bar{Q}$ and the slope of the tangent $\bar{L}$.

Let us discuss the action of contact transformations on differential equations.

A contact transformation $F\colon J^1(M) \to J^1(M)$ takes a hypersurface $\mathcal{E}$, corresponding to an equation, to a hypersurface $\mathcal{E}'$ (which may coincides with $\mathcal{E}$). Since contact transformations preserve the Cartan distribution, $\mathcal{E}'$ is also "almost everywhere" a differential equation. Note that the transformation $\bar{F}$ establishes a one-to-one correspondence between (generalized) solutions of the initial equation and the transformed one.

DEFINITION 1.5. Equations $\mathcal{E}$ and $\mathcal{E}'$ in $J^1(M)$ are called (locally) equivalent, if there exists a contact transformation of $J^1(M)$ such that the image of $\mathcal{E}$ under this transformation is $\mathcal{E}'$.

EXAMPLE 1.7. Consider the action of the lifting of the transformation $J^0(M) \to J^0(M)$, $(x_1, x_2, u) \mapsto (u, x_2, x_1)$ on the equation $\mathcal{E}_1 = \{u = \sqrt{p_2^2/p_1 - x_1^2}\}$. From formula (1.13) it follows that

$$(x_1, x_2, u, p_1, p_2) \longmapsto (u, x_2, x_1, 1/p_1, -p_2/p_1).$$

The transformed equation $\mathcal{E}_1' = \{x_1 = \sqrt{p_2^2/p_1 - u^2}\}$ coincides with the initial one for $x_1 > 0$, $u > 0$.

The same transformation takes the equation $\mathcal{E}_2 = \{4u - p_1^2 - p_2^2 = 0\}$ to $\mathcal{E}_2' = \{4x_1 p_1^2 - 1 - p_2^2 = 0\}$. The image of the graph of the solution to $\mathcal{E}_2$ given by $u = x_1^2 + x_2^2$ is the surface $u^2 = x_1 - x_2^2$, which is a multiple-valued solution to $\mathcal{E}_2'$. The points $(x_1 = x_2^2, u = 0)$ on this surface are singular, since the surface is tangent to the vector $\partial/\partial u$ at these points.

For $x_1 > x_2^2$ the solutions $u = \pm\sqrt{x_1 - x_2^2}$ are single-valued and smooth.

DEFINITION 1.6. Let $\mathcal{E} \subset J^1(M)$ be a first-order differential equation. A contact transformation $F\colon J^1(M) \to J^1(M)$ is called finite contact symmetry of $\mathcal{E}$, if $F(\mathcal{E}) = \mathcal{E}$.

Consider an equation $\mathcal{E}$ given by a hypersurface $H = 0$, where $H \in C^\infty(J^1(M))$ is a smooth function with nonzero differential. It is obvious that a contact transformation F is a symmetry of $\mathcal{E}$ if and only if $F^*(H) = \lambda H$ for some function $\lambda \in C^\infty(J^1(M))$.

Let us also stress that symmetries of $\mathcal{E}$ take (generalized) solutions of this equation to solutions of the same equations.

EXAMPLE 1.8. The Legendre transformation is a finite contact symmetry of the equation

$$\mathcal{E} = \left\{ u - \frac{1}{2} \sum_{i=1}^{n} x_i p_i = 0 \right\}. \tag{1.18}$$

Equation (1.18) has the n-parameter family of solutions $u = \sum_{i=1}^{n} b_i x_i^2$, where $b_i \in \mathbb{R}$. This family is invariant under the Legendre transformation:

$$L\left(j_1\left(\sum_{i=1}^{n} b_i x_i^2 \right) \right) = j_1\left(\sum_{i=1}^{n} \frac{x_i^2}{4 b_i} \right).$$

The graph of the constant function $u = c$ transformed by the Legendre transformation,

$$\begin{cases} u = c, \\ p_1 = 0, \\ \dots\dots \\ p_n = 0, \end{cases} \qquad \longmapsto \qquad \begin{cases} u = -c, \\ x_1 = 0, \\ \dots\dots \\ x_n = 0, \end{cases}$$

is an integral manifold whose projection to the manifold of independent variables is the point $(x_1 = \cdots = x_n = 0)$.

Let us now use the Clairaut equation to illustrate applications of such degenerate integral manifolds.

1.3. Clairaut equation and its integrals. The general Clairaut equation has the form

$$F\left(\sum_{i=1}^{n} x_i p_i - u, p_1, \dots, p_n \right) = 0, \tag{1.19}$$

where F is a smooth function of $(n+1)$ variables. The image of this equation under the Legendre transformation is the equation

$$F(u, x_1, \dots, x_n) = 0, \tag{1.20}$$

which contains no derivatives. The locus associated to (1.20) is an n-dimensional surface in the $(n+1)$-dimensional space $J^0(\mathbb{R}^n)$, the corresponding first-order differential equation being its inverse image under the projection $\pi_{1,0} \colon J^1(\mathbb{R}^n) \to J^0(\mathbb{R}^n)$.

Take a point $(a_1, \dots, a_n, b) \in J^0(\mathbb{R}^n)$ such that $F(b, a_1, \dots, a_n) = 0$. The fiber of $\pi_{1,0} \colon J^1(\mathbb{R}^n) \to J^0(\mathbb{R}^n)$ over this point is a generalized solution of (1.20).

Applying the Legendre transformation to these generalized solutions yields an n-parameter family of single-valued solutions to (1.19), which is said to be a *complete integral of the Clairaut equation*.

On the other hand, solving (1.20) for u in the vicinity of a nonsingular point (i.e., where $\partial F / \partial u \neq 0$) yields

$$u = f(x_1, \dots, x_n). \tag{1.21}$$

To find the image of this solution under the Legendre transformation observe that the graph of the 1-jet of (1.21) has the form

$$\begin{cases} p_1 = \dfrac{\partial f}{\partial x_1}(x_1, \ldots, x_n), \\ \quad\cdots\cdots\cdots\cdots\cdots \\ p_n = \dfrac{\partial f}{\partial x_n}(x_1, \ldots, x_n), \\ F(u, x_1, \ldots, x_n) = 0. \end{cases} \tag{1.22}$$

Under the Legendre transformation, (1.22) maps to the system

$$\begin{cases} x_1 = \dfrac{\partial f_l}{\partial x_1}(p_1, \ldots, p_n), \\ \quad\cdots\cdots\cdots\cdots\cdots \\ x_n = \dfrac{\partial f_l}{\partial x_n}(p_1, \ldots, p_n), \\ F\left(\displaystyle\sum_{i=1}^{n} x_i p_i - u, p_1, \ldots, p_n\right) = 0, \end{cases} \tag{1.23}$$

which is compatible.

Express $p_1, \ldots, p_n$ in terms of $x_1, \ldots, x_n$ by the first n equations. Substituting them to the last equation and solving it for u in terms of $x_1, \ldots, x_n$ yields a finite number of solution to the Clairaut equation, which are called its *exceptional integrals*.

EXAMPLE 1.9. Consider the following Clairaut equation with two independent variables

$$x_1 p_1 + x_2 p_2 - u = \frac{1}{3}(p_1^3 + p_2^3). \tag{1.24}$$

Using the Legendre transformation, we obtain the equation

$$u = \frac{1}{3}(x_1^3 + x_2^3),$$

which gives the following complete integral of the initial equation

$$u = a_1 x_1 + a_2 x_2 - \frac{1}{3}(a_1^3 + a_2^3), \qquad a_1, a_2 = \text{const.}$$

Exceptional integrals are found from the system

$$\begin{cases} u = x_1 p_1 + x_2 p_2 - \dfrac{1}{3}(p_1^3 + p_2^3), \\ x_1 = p_1^2, \\ x_2 = p_2^2. \end{cases} \tag{1.25}$$

It follows that exceptional integrals exist in the quadrant $x_1 > 0$, $x_2 > 0$ and have the form

$$p_1 = \alpha_1 \sqrt{x_1}, \qquad p_2 = \alpha_2 \sqrt{x_2},$$
$$u = \frac{2}{3}(\alpha_1 x_1^{3/2} + \alpha_2 x_2^{3/2}), \tag{1.26}$$

with $\alpha_1 = \pm 1$ and $\alpha_2 = \pm 1$. Thus, formula (1.26) defines four exceptional integrals. However, from the geometric point of view, the system (1.25) defines only one

generalized solution, which is a surface of the fourth order and degenerately projects to the (x_1, x_2)-plane. Formula (1.26) defines the four branches of this generalized solution.

1.4. Contact manifolds in mechanics.

DEFINITION 1.7. A $(2n+1)$-dimensional manifold M equipped with a 1-form θ is called a *contact manifold* if $\theta \wedge (d\theta)^n$ is a volume form.

An odd-dimensional manifold equipped with a closed 2-form of maximal rank will be said to be an *almost contact manifold*.

It immediately follows from §1.1 that the manifold of 1-jets $J^1(M)$ is a contact manifold.

Note that at any point of a contact manifold the restriction of $d\theta$ to the kernel of θ is nondegenerate. In particular, any contact manifold is almost contact (with the 2-form $d\theta$).

Given a 2-form ω on a manifold M, the set of vectors ξ such that $\xi \lrcorner \omega = 0$ is called the *characteristic distribution* of the form ω. A vector field X on an almost contact manifold (M, ω) is called *characteristic* if $X \lrcorner \omega = 0$.

Let us give some examples of contact manifolds ([**2**]).

EXAMPLE 1.10. Let (M, ω, H) be a Hamiltonian system (i.e., (M, ω) is a symplectic manifold and H is a smooth function on M) and H_e be a regular energy surface. We claim that $(H_e, \omega|_{H_e})$ is an almost contact manifold and the restriction of the Hamiltonian field X_H to H_e is a characteristic vector field on this manifold.

Indeed, $d(\omega|_{H_e}) = d\omega|_{H_e} = 0$, so that the form $\omega|_{H_e}$ is closed. Since the form ω is nondegenerate and codimension of H_e is equal to 1, ω is of maximal rank on H_e. Further, the field X_H is tangent to H_e, therefore $X_H|_{H_e} \lrcorner \omega|_{H_e} = 0$.

EXAMPLE 1.11. Let (M, ω) be a symplectic manifold and $\pi_2 \colon \mathbb{R} \times M \to M$ be the natural projection $\pi_2(t, x) = x$. We claim that $(\mathbb{R} \times M, \pi_2^*(\omega))$ is almost contact.

Indeed, $d\pi_2^*(\omega) = 0$, hence the form $\widetilde{\omega} = \pi_2^*(\omega)$ is closed. To check the maximal rank condition, it is sufficient to show that the characteristic distribution of this form is one-dimensional. If the vector $((t, r), v_p) \in T_{(t,p)}(\mathbb{R} \times M)$ lies in the characteristic space, then

$$\pi_2^*(\omega)_{(t,p)}(((t, r), v_p), ((t, s), w_p)) = 0,$$

where $w_p \in T_p M$ and $s = \partial/\partial t$. By definition of $\pi_2^*(\omega)$, we have $\omega_p(v_p, w_p) = 0$ for any w_p. Thus, $v_p = 0$ and the characteristic distribution is generated by the vector field

$$\overline{\frac{\partial}{\partial t}}\bigg|_{(t,p)} = ((t, 1), 0) \in T_{(t,p)}(\mathbb{R} \times M) = T_t\mathbb{R} \times T_pM.$$

Note that if $\omega = d\theta$ and $\widetilde{\theta} = dt + \pi_2^*(\theta)$, where $t \colon \mathbb{R} \times M \to \mathbb{R}$, $t(t, x) = t$, then $\widetilde{\omega} = d\widetilde{\theta}$ and $(\mathbb{R} \times M, \widetilde{\theta})$ is a contact manifold.

A smooth mapping $X \colon \mathbb{R} \times M \to TM$ is called a *vector field depending on time*, if for any $t \in \mathbb{R}$ the mapping $X \colon \{t\} \times M \to TM$ is a vector field on M. To any vector field depending on time, one associates the vector field $\widetilde{X}$ on $\mathbb{R} \times M$ by $\widetilde{X} = \overline{\partial/\partial t} + X$.

Take a function $H \in C^\infty(\mathbb{R} \times M)$. For each $t \in \mathbb{R}$, there is the Hamiltonian vector field X_{H_t}, where $H_t \colon M \to \mathbb{R}$, $H_t(x) = H(t,x)$. Define the vector field depending on time $X_H \colon \mathbb{R} \times M \to TM$ by the formula $X_H(t,x) = X_{H_t}(x)$. Let $\widetilde{X}_H$ be the corresponding field on $\mathbb{R} \times M$.

THEOREM 1.6 (Cartan). *Let* (M,ω) *be a symplectic manifold and* $H \in C^\infty(\mathbb{R} \times M)$ *be a function. Set*

$$\omega_H = \pi_2^*(\omega) + dH \wedge dt.$$

Then:

1. $(\mathbb{R} \times M, \omega_H)$ *is an almost contact manifold.*
2. *The field* $\widetilde{X}_H$ *generates the characteristic distribution of the form* ω_H.
3. *If* $\omega = d\theta$ *and* $\theta_H = \pi_2^*\theta + H\,dt$, *then* $\omega_H = d\theta_H$; *in this case, if* $H + \pi_2^*(\theta)(X_H)$ *is nowhere zero,* $(\mathbb{R} \times M, \theta_H)$ *is a contact manifold.*

EXERCISE 1.12. Prove Theorem 1.6.

2. Infinitesimal contact transformations and characteristic fields

In Ch. 1 we saw that passing to the infinitesimal point of view is useful in the theory of ordinary differential equation. In this section, we consider infinitesimal contact transformation.

2.1. Infinitesimal contact transformations. Let $A_t \colon J^1(M) \to J^1(M)$ be a one-parameter group of finite contact transformations. Consider the vector field $X = d/dt|_{t=0} A_t^*$. By definition of contact transformations, $A_t^*(U_1) = \lambda_t U_1$, where $\lambda_t \in C^\infty(J^1(M))$, so that

$$X(U_1) = \lambda U_1, \qquad \text{where } \lambda = \left.\frac{d\lambda_t}{dt}\right|_{t=0}. \tag{2.1}$$

DEFINITION 2.1. A vector field X on the manifold $J^1(M)$ is called an *infinitesimal contact transformation* or a *contact vector field*, if

$$X(U_1) = \lambda U_1, \qquad \text{where } \lambda \in C^\infty(J^1(M)).$$

EXERCISE 2.1. Prove that contact vector fields are closed under the commutator operation.

It can easily be checked that a one-parameter group corresponding to a contact vector field consists of finite contact transformations.

EXAMPLE 2.1. In the special local coordinates $x_1, \ldots, x_n$, u, $p_1, \ldots, p_n$ on $J^1(M)$ the vector fields $\partial/\partial x_1, \ldots, \partial/\partial x_n$ and $\partial/\partial u$ are contact, while the fields $\partial/\partial p_i$ are not.

EXAMPLE 2.2. Let $n = 2$. The vector field $X = x_1\partial/\partial x_2 - x_2\partial/\partial x_1 + p_1\partial/\partial p_2 - p_2\partial/\partial p_1$ is an infinitesimal contact transformation, since $X(du - p_1\,dx_1 - p_2\,dx_2) = p_2\,dx_1 + p_1\,dx_2 - p_1\,dx_2 - p_2\,dx_1 = 0$.

EXAMPLE 2.3. Consider the vector field $X \in \mathrm{D}(J^1(M))$ in special local coordinates given by

$$X = \sum_{i=1}^{n} \alpha_i x_i \frac{\partial}{\partial x_i} + \beta u \frac{\partial}{\partial u} + \sum_{i=1}^{n} (\beta - \alpha_i) p_i \frac{\partial}{\partial p_i}, \tag{2.2}$$

where $\alpha_1, \ldots, \alpha_n, \beta$ are constant. It is easily shown that $X(U_1) = \beta U_1$, so that X is an infinitesimal contact transformation.

In §1 we saw that any smooth transformation $A\colon J^0(M) \to J^0(M)$ gives rise to the contact transformation $\bar{A}\colon J^1(M) \to J^1(M)$. The same is true for infinitesimal transformations as well. Namely, let $X \in \mathrm{D}(J^0(M))$ and A_t be the one-parameter group of shifts along trajectories of X. Clearly, the vector field $\bar{X} = d/dt|_{t=0}\, \bar{A}_t^* \in \mathrm{D}(J^1(M))$ is contact. This vector field $\bar{X}$ is called the *lifting* of the field X to the manifold of 1-jets $J^1(M)$.

All contact fields discussed in Examples 2.1–2.3 are in fact liftings of some fields. The fields $\partial/\partial x_i$, $\partial/\partial u \in \mathrm{D}(J^1(M))$ (see Example 2.1) are liftings of $\partial/\partial x_i$ and $\partial/\partial u$ considered as vector fields on $J^0(M)$. The vector field from Example 2.2 is the lifting of the infinitesimal rotation $x_1\partial/\partial x_2 - x_2\partial/\partial x_1$, while the field from Example 2.3 is the lifting of the infinitesimal scale transformation

$$\alpha_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n x_n \frac{\partial}{\partial x_n} + \beta u \frac{\partial}{\partial u} \in \mathrm{D}(J^0(M)).$$

This is an immediate consequence of the corresponding coordinate formulas for the liftings of A_t.

EXERCISE 2.2. Check that a contact vector field $X \in \mathrm{D}(J^1(M))$ is the lifting of a field $X^{(0)} \in \mathrm{D}(J^0(M))$ if and only if in the special local coordinates $x_1, \ldots, x_n$, $u, p_1, \ldots, p_n$ on $J^1(M)$ one has $[X, \partial/\partial p_i] = \alpha_{i1}\partial/\partial p_1 + \cdots + \alpha_{in}\partial/\partial p_n$, where α_{ik} are smooth functions on $J^1(M)$, $1 \le i, k \le n$.

EXAMPLE 2.4. The vector field $X = \sum_{i=1}^n (2p_i\partial/\partial x_i + p_i^2\partial/\partial u)$ is contact (because $X(U_1) = 0$), however, $[X, \partial/\partial p_l] = -2\partial/\partial x_l - 2p_l\partial/\partial u$. Hence X cannot be the lifting of a vector field $X^{(0)} \in \mathrm{D}(J^0(M))$.

Let us now give a description of contact vector fields in terms of generating functions [**72**, **77**].

THEOREM 2.1. *A contact vector field X on $J^1(M)$ is uniquely determined by the function $f = U_1(X)$. To every function $f \in C^\infty(J^1(M))$ there corresponds a contact field X_f, i.e., $f = U_1(X_f)$. The correspondence $f \mapsto X_f$ is linear over $\mathbb{R}$ and satisfies the following properties:*

$$
\begin{aligned}
&U_1(X_f) = f, \\
&X_f(U_1) = X_1(f) \cdot U_1, \\
&X_{f+g} = X_f + X_g, \qquad g \in C^\infty(J^1(M)), \\
&X_{fg} = fX_g + gX_f - fgX_1, \\
&X_f(f) = X_1(f) \cdot f.
\end{aligned}
$$

PROOF. Consider the 2-form dU_1. We claim that for any point $\theta \in J^1(M)$ the form $(dU_1)_\theta$ is nondegenerate on the Cartan plane $\mathcal{C}_\theta$. Indeed, from the coordinate expression for dU_1 it follows that $X_1|_\theta \lrcorner (dU_1)_\theta = 0$, where $X_1 = \partial/\partial u$, and $(dU_1)_\theta$ is nondegenerate on any hyperplane that does not contain $X_1|_\theta$. Since $(U_1)_\theta(X_1) = 1 \ne 0$, one has $\mathcal{C}_\theta \not\ni X_1|_\theta$.

Thus, the map $Y \mapsto Y \lrcorner dU_1$ establishes an isomorphism between vector fields that lie in the Cartan distribution (i.e., on which the form U_1 vanishes) and 1-forms that vanish on X_1.

Let X be a contact vector field on $J^1(M)$, $X(U_1) = hU_1$, where $h \in C^\infty(J^1(M))$. Decomposing X into the sum

$$X = f \cdot X_1 + Y, \qquad \text{where } f \in C^\infty(J^1(M)), \quad U_1(Y) = 0, \tag{2.3}$$

we get

$$Y \lrcorner\, dU_1 = X \lrcorner\, dU_1 = hU_1 - d(X \lrcorner U_1).$$

Thus,

$$Y \lrcorner\, dU_1 = hU_1 - df. \tag{2.4}$$

Applying both sides of (2.4) to the field X_1, we get $h = X_1(f)$.

The equality $U_1(Y) = 0$ means that Y_θ lies in $\mathcal{C}_\theta$ for any $\theta \in J^1(M)$. Since the 2-form dU_1 is nondegenerate on $\mathcal{C}_\theta$, the vector Y_θ can be recovered by means of (2.4). Hence, the field X is uniquely determined by the function $f = U_1(X)$.

Conversely, for an arbitrary function f on $J^1(M)$ define the field Y by (2.4), with $h = X_1(f)$. Then (2.3) yields $X = X_f$.

The rest of the proof is by direct verification. $\qquad\qquad\qquad\qquad\qquad\square$

DEFINITION 2.2. Given a contact vector field X, the function $f = U_1(X)$ is called the *generating function* of X.

EXERCISE 2.3. Let $f_1, \ldots, f_k \in C^\infty(J^1(M))$ and $\varphi \in C^\infty(\mathbb{R}^k)$ be an arbitrary smooth function. Using Theorem 2.1, prove that

$$Y_{\varphi(f_1,\ldots,f_k)} = \sum_{i=1}^{k} \frac{\partial \varphi}{\partial f_i} Y_{f_i}, \tag{2.5}$$

where $Y_f = X_f - fX_1$.

In view of (2.5), to describe X_f in local coordinates it suffices to obtain the coordinate formulas for Y_{x_i}, Y_u, Y_{p_i} $(i = 1, \ldots, n)$. It immediately follows from the definitions that

$$Y_{x_i} = \frac{\partial}{\partial p_i}, \qquad i = 1, \ldots, n, \tag{2.6}$$

$$Y_u = \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i}, \tag{2.7}$$

$$Y_{p_i} = -\frac{\partial}{\partial x_i} - p_i \frac{\partial}{\partial u}, \qquad i = 1, \ldots, n. \tag{2.8}$$

Using (2.5), we get

$$Y_f = -\sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial}{\partial x_i} - \left(\sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i} \right) \frac{\partial}{\partial u} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} + p_i \frac{\partial f}{\partial u} \right) \frac{\partial}{\partial p_i}. \tag{2.9}$$

Since $X_f = Y_f + fX_1$, we obtain

$$X_f = -\sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial}{\partial x_i} + \left(f - \sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i} \right) \frac{\partial}{\partial u} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} + p_i \frac{\partial f}{\partial u} \right) \frac{\partial}{\partial p_i}. \tag{2.10}$$

REMARK 2.1. Consider the pullback $f = \pi^*(H)$ of a function $H \in C^\infty(T^*(M))$ along the projection $\pi\colon J^1(M) \to T^*(M)$. The projection of the field X_f to $T^*(M)$ is the Hamiltonian field X_H. From the equalities $X_f(U_1) = X_1(f)U_1$ and $dU_1 =$

$-\pi^*(\omega)$, where ω is the symplectic structure on $T^*(M)$, it follows that $X_H \lrcorner\, d\omega = dH$.

EXERCISE 2.4. The $2n+1$ vector fields (2.6), (2.7), and (2.8) at each point θ lie in the $2n$-dimensional plane $\mathcal{C}_\theta$ at each point θ, so that they are linearly dependent. Find the linear dependence (over $C^\infty(J^1(M))$) between these fields.

EXERCISE 2.5. Consider the trivial extension of a vector field $Y \in \mathrm{D}(M)$ to $J^0(M) = M \times \mathbb{R}$. Find the generating function for the lifting $\bar{Y} \in \mathrm{D}(J^1(M))$ of Y.

EXERCISE 2.6. Let $f \in C^\infty(J^0(M))$ be a function and $\pi^*(f) \in C^\infty(J^1(M))$ be the pullback of f along the projection π defined by (1.1). Prove that the contact field $X_{\pi^*(f)}$ is the lifting of some field $Y \in \mathrm{D}(J^0(M))$ and describe Y in coordinates.

The isomorphism between contact vector fields and functions on $J^1(M)$ makes it possible to define a number of brackets on the set of functions.

First, let X_f and X_g be the contact vector fields with the generating functions $f, g \in C^\infty(J^1(M))$. The commutator of these fields is a contact field as well.

DEFINITION 2.3. The generating function $\{f, g\} = U_1([X_f, X_g])$ of the contact field $[X_f, X_g]$ is called the *Jacobi bracket* of f and g.

PROPOSITION 2.2. *For $f, g \in C^\infty(J^1(M))$, one has*

$$\{f, g\} = X_f(g) - X_1(f)g. \tag{2.11}$$

PROOF. Since $X_f(U_1) = X_f \lrcorner\, dU_1 + d(X_f \lrcorner\, U_1) = X_f \lrcorner\, dU_1 + df$ and $X_f(U_1) = X_1(f) \cdot U_1$, it follows that

$$dU_1(X_f, X_g) = gX_1(f) - X_g(f).$$

Hence,

$$\{f, g\} = U_1([X_f, X_g]) = X_f(U_1(X_g)) - X_g(U_1(X_f)) - dU_1(X_f, X_g)$$
$$= X_f(g) - X_g(f) - gX_1(f) + X_g(f) = X_f(g) - X_1(f)g. \quad \square$$

THEOREM 2.3. *The set of function on $J^1(M)$ is a Lie algebra over $\mathbb{R}$ with respect to the Jacobi bracket.*

PROOF. It is obvious that the Jacobi bracket is skew-symmetric and bilinear. The Jacobi identity is a direct consequence of (2.11). $\qquad\square$

EXERCISE 2.7. Using (2.11), prove that in the special coordinates the Jacobi bracket takes the form

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right) + \sum_{i=1}^n p_i \left(\frac{\partial f}{\partial u} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial u} \frac{\partial f}{\partial p_i} \right) + f \frac{\partial g}{\partial u} - g \frac{\partial f}{\partial u}.$$

Another bracket is defined as follows:

DEFINITION 2.4. The *Mayer bracket* $[f, g]$ of two functions $f, g \in C^\infty(J^1(M))$ is defined by $[f, g] = dU_1(X_f, X_g) \in C^\infty(J^1(M))$.

The Mayer bracket is bilinear over $\mathbb{R}$ and skew-symmetric, but instead of the Jacobi identity, one has

$$[[f, g], h] + [[h, f], g] + [[g, h], f] = X_1(f)[g, h] + X_1(g)[h, f] + X_1(h)[f, g].$$

Using
$$[f, g] = Y_f(g) = X_f(g) - fX_1(g),$$
we obtain the following coordinate formula for the Mayer bracket:
$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right) + \sum_{i=1}^{n} p_i \left(\frac{\partial f}{\partial u} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial u} \frac{\partial f}{\partial p_i} \right).$$

Finally, let us define the Poisson bracket.

DEFINITION 2.5. The *Poisson bracket* (f, g) of two functions $f, g \in C^\infty(J^1(M))$ is defined by $(f, g) = X_f(g) \in C^\infty(J^1(M))$.

EXERCISE 2.8. Prove the following properties of the Poisson bracket:

1. Bilinearity over $\mathbb{R}$.
2. $(f, g) + (g, f) = X_1(f)g + X_1(g)f$.
3. $(f, g) = [f, g] + X_1(g)f$.
4. $(f, g) = \{f, g\} + X_1(f)g$.

In coordinates, the Poisson bracket looks as
$$(f, g) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right) + \sum_{i=1}^{n} p_i \left(\frac{\partial f}{\partial u} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial u} \frac{\partial f}{\partial p_i} \right) + f \frac{\partial g}{\partial u}.$$

REMARK 2.2. The restrictions of all the three brackets to the functions which are pullbacks of functions on $T^*(M)$ coincide with the Poisson bracket on $T^*(M)$, because $X_1(\pi^*(H)) = 0$ for all $H \in C^\infty(T^*(M))$.

2.2. Infinitesimal symmetries of equations.

DEFINITION 2.6. Let $\mathcal{E} \subset J^1(M)$ be a first-order differential equation. A contact vector field $X \in D(J^1(M))$ is called an *infinitesimal contact symmetry* of the equation $\mathcal{E}$, if at each point $\theta \in \mathcal{E}$ the vector X_θ is tangent to $\mathcal{E}$.

In the sequel, such a symmetry will be called simply a "symmetry". The generating function of a symmetry will also be referred to as "symmetry" for short.

It is clear that the set of symmetries of $\mathcal{E}$ is a Lie algebra over $\mathbb{R}$ with respect to the commutation operation. In terms of generating functions, the Lie algebra structure is given by the Jacobi bracket.

If the equation $\mathcal{E}$ is (locally) of the form $\mathcal{E} = \{F = 0\}$, $F \in C^\infty(J^1(M))$, then X is a symmetry of $\mathcal{E}$, if $X(F)|_{\mathcal{E}} \equiv 0$, or, which is equivalent, $X(F) = \mu F$ for some function $\mu \in C^\infty(J^1(M))$.

Note that the equality $X_F(F) = X_1(F) \cdot F$ implies that the equation $\mathcal{E} = \{F = 0\}$ always possesses at least one symmetry, namely, X_F.

As an example, let us discuss equations, which are symmetric with respect to the contact fields from Examples 2.1–2.3.

EXAMPLE 2.5. The vector field X given by (2.2) (see Example 2.3) is a symmetry of $\mathcal{E} = \{F = 0\}$, if the function F is quasihomogeneous in the following sense: for any real numbers $\alpha_1, \ldots, \alpha_n, \beta$ there exists $\gamma \in \mathbb{R}$ such that for each $\tau \in \mathbb{R}$ one has

$$F(\tau^{\alpha_1} x_1, \ldots, \tau^{\alpha_n} x_n, \tau^\beta u, \tau^{\beta - \alpha_1} p_1, \ldots, \tau^{\beta - \alpha_n} p_n)$$
$$\equiv \tau^\gamma F(x_1, \ldots, x_n, u, p_1, \ldots, p_n). \quad (2.12)$$

Indeed, suppose that F satisfies (2.12) and consider the trajectory of X passing through a point $\theta = (x_1^{(0)}, \ldots, x_n^{(0)}, u^{(0)}, p_1^{(0)}, \ldots, p_n^{(0)}) \in \mathcal{E}$:

$$\begin{cases} x_i = x_i^{(0)} e^{\alpha_i t}, \\ u = u^{(0)} e^{\beta t}, \\ p_i = p_i^{(0)} e^{(\beta - \alpha_i)t}, \qquad i = 1, \ldots, n. \end{cases}$$

Set $\tau = e^t$ and compute F at a point $(x_1, \ldots, x_n, u, p_1, \ldots, p_n)$ on the considered trajectory:

$$F(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = \tau^\gamma F(x_1^{(0)}, \ldots, x_n^{(0)}, u^{(0)}, p_1^{(0)}, \ldots, p_n^{(0)}) = 0.$$

Therefore, the trajectory of X passing through θ lies entirely on $\mathcal{E}$. Hence, $X(F) = 0$ on $\mathcal{E}$.

PROPOSITION 2.4. *The contact field $\partial/\partial x_i$ (resp., $\partial/\partial u$) is a symmetry of $\mathcal{E} = \{F = 0\}$, where $F \in C^\infty(J^1(M))$, if and only if F does not depend on x_i (resp., u).*

PROOF. If $\mathcal{E} = \{G = 0\}$ and $\partial G/\partial x_i = \mu G$, $\mu \in C^\infty(J^1(M))$, then $G = e^\nu F$, where ν is a function satisfying $\partial \nu/\partial x_i = \mu$ and F does not depend on x_i. Since e^ν is nowhere zero, we have $\mathcal{E} = \{G = 0\} = \{F = 0\}$. The field $\partial/\partial u$ can be handled in the same way. $\square$

EXERCISE 2.9. Prove that the vector field from Example 2.2 for $n = 2$ is a symmetry of an equation $\mathcal{E}$ if and only if $\mathcal{E}$ is of the form

$$F(u, x_1^2 + x_2^2, p_1^2 + p_2^2, x_1 p_1 + x_2 p_2) = 0.$$

2.3. Characteristic vector fields and integration of first-order equations. Recall (see Ch. 1, §3) that a vector field X is a characteristic symmetry of a distribution P, if X belongs to P and the Lie derivative with respect to X preserves P.

PROPOSITION 2.5. *The Cartan distribution on $J^1(M)$ has no nontrivial characteristic symmetries.*

PROOF. If X is a characteristic symmetry, then X is contact and $U_1(X) = 0$. Hence, by Theorem 2.1, $X = 0$. $\square$

Recall also that from the geometric viewpoint the finding of (generalized) solutions to a differential equation amounts to integration of the Cartan distribution restricted to the locus of this equation. So, the problem of finding all (generalized) solutions to a first-order differential equation $\mathcal{E}$ can be stated as follows:

> *To find all maximal integral manifolds of the induced Cartan distribution $\mathcal{C}(\mathcal{E})$ on the manifold $\mathcal{E} \subset J^1(M)$.*

Now, we claim that although the Cartan distribution on $J^1(M)$ has no characteristic symmetries, any nontrivial first-order equation has a nontrivial characteristic symmetry. Indeed, consider the 1-form $U_1|_\mathcal{E}$ that defines the Cartan distribution $\mathcal{C}(\mathcal{E})$. For any point $\theta \in J^1(M)$, the 2-form $(dU_1)_\theta$ is nondegenerate on the Cartan plane $\mathcal{C}_\theta$. If θ is a nonsingular point of the equation $\mathcal{E} = \{F = 0\}$, then

$$\mathcal{C}(\mathcal{E})_\theta = \mathcal{C}_\theta \cap T_\theta(\mathcal{E})$$

is a hyperplane in $\mathcal{C}_\theta$. On this hyperplane, the 2-form $(dU_1)_\theta$ is of rank $2n-2$ and, therefore, is degenerate. Thus, on the everywhere dense set of generic points of $\mathcal{E}$

there exists the direction field $l_\theta \subset T_\theta(\mathcal{E})$, where $l_\theta \subset \mathcal{C}(\mathcal{E})_\theta$ is the degeneration subspace of $(dU_1)_\theta$.

PROPOSITION 2.6. *The direction field l_θ is tangent to every generalized solution of $\mathcal{E}$.*

PROOF. Recall that an n-dimensional plane $R \subset \mathcal{C}_\theta$ is called *Lagrangian* with respect to the 2-form $(dU_1)_\theta$, if the restriction $(dU_1)_\theta|_R$ vanish. It is not hard to prove that if R is contained in a hyperplane $S \subset \mathcal{C}_\theta$, then R must contain the 1-dimensional degeneration subspace of $(dU_1)_\theta|_S$.

Now, let N be a generalized solution to $\mathcal{E}$, i.e., a maximal n-dimensional integral manifold of the Cartan distribution contained in $\mathcal{E}$. Since $U_1|_N = 0$, we have $(dU_1)_N = 0$, and, therefore, for any $\theta \in N$ the plane $T_\theta(N) \subset \mathcal{C}(\mathcal{E})_\theta \subset \mathcal{C}_\theta$ is Lagrangian with respect to $(dU_1)_\theta$. Hence, $l_\theta \subset T_\theta(N)$. $\square$

The field $\{l_\theta\}$ is called the *characteristic direction field*. Its integral curves are called the *characteristics* of $\mathcal{E}$.

PROPOSITION 2.7. *Let $\mathcal{E} = \{F = 0\}$ be an equation. The characteristics of $\mathcal{E}$ are integral curves of the characteristic vector field Y_F.*

PROOF. Since $U_1(Y_F) = 0$, we have $(Y_F)_\theta \in \mathcal{C}_\theta$ for any point $\theta \in J^1(M)$. From the formula

$$Y_F \lrcorner\, dU_1 = X_1(F)U_1 - dF \tag{2.13}$$

(see (2.4)) it follows that

$$Y_F(F) = dF(Y_F) = X_1(F)U_1(Y_F) - dU_1(Y_F, Y_F) = 0,$$

therefore Y_F is tangent to every level surface of F. Thus, $(Y_F)_\theta \in \mathcal{C}(\mathcal{E})_\theta$ for all $\theta \in \mathcal{E}$.

By virtue of (2.13), the form $Y_F \lrcorner\, dU_1$ vanish on the $(2n-1)$-dimensional plane $\mathcal{C}(\mathcal{E})_\theta$. Hence, $(Y_F)_\theta$ is a degeneration vector of $(dU_1)_\theta$ on this plane, i.e., for all $\theta \in \mathcal{E}$ we have $(Y_F)_\theta \in l_\theta$, which proves the theorem. $\square$

PROPOSITION 2.8. *The vector field Y_F is an infinitesimal symmetry of the distribution $\mathcal{C}(\mathcal{E})$.*

PROOF. Since $Y_F = X_F - FX_1$, the vector fields Y_F and X_F coincide on $\mathcal{E} = \{F = 0\}$. To conclude the proof, it remains to recall that X_F is an infinitesimal symmetry of the Cartan distribution. $\square$

Now let us turn to the discussion of the Cauchy problem. By Propositions 2.6–2.7, a noncharacteristic Cauchy problem for the equation $\mathcal{E} = \{F = 0\}$ reduces to the solution of the following system of characteristic equations:

$$\begin{cases} \dot{x}_i = -\dfrac{\partial F}{\partial p_i}, \\[2mm] \dot{u} = -\displaystyle\sum_{i=1}^n p_i \dfrac{\partial F}{\partial p_i}, \\[2mm] \dot{p}_i = \dfrac{\partial F}{\partial x_i} + p_i \dfrac{\partial F}{\partial u}, \qquad i = 1, \dots, n, \end{cases} \tag{2.14}$$

describing the trajectories of the field Y_F.

Further on an $(n-1)$-dimensional integral manifold of the Cartan distribution contained in $\mathcal{E}$ will be called *Cauchy data* for $\mathcal{E}$.

EXAMPLE 2.6. Let $\Gamma \subset M$ be a smooth hypersurface, u_0 be a smooth function on it, and $\widetilde{u}_0$ be an extension of u_0 such that the graph $j_1(\widetilde{u}_0)(\Gamma)$ is contained in $\mathcal{E}$. This graph is an $(n-1)$-dimensional manifold of Cauchy data.

"Almost always" the graph $j_1(\widetilde{u}_0)(\Gamma)$ is determined by the initial function u_0 and does not depend on the choice of $\widetilde{u}_0$. To prove this, observe that for any point $a \in \Gamma$ the n-dimensional plane tangent to the graph of $\widetilde{u}_0$ at a point $(a, u_0(a))$ must contain the $(n-1)$-dimensional plane tangent to the surface $\{ (x, u_0(x)) \mid x \in \Gamma \}$. Therefore, all contact elements $j_1(\widetilde{u}_0)(a)$ lie on a 1-dimensional curve, which in a general case intersects $\mathcal{E}$ at one point.

In particular, suppose that in local coordinates the hypersurface $\Gamma \subset M$ is given by $\Gamma = \{x_n = 0\}$ and the equation $\mathcal{E}$ is resolved with respect to p_n:

$$p_n = f(x_1, \ldots, x_n, u, p_1, \ldots, p_{n-1}).$$

Then, for any function $u_0 \in C^\infty(\Gamma)$ and any point $a \in \Gamma$, the contact element $j_1(\widetilde{u}_0)(a)$ is uniquely determined and in coordinates has the form

$$\left(a_1, \ldots, a_n, u_0(a), \left.\frac{\partial u_0}{\partial x_1}\right|_a, \ldots, \left.\frac{\partial u_0}{\partial x_{n-1}}\right|_a, f\left(a, u_0(a), \left.\frac{\partial u_0}{\partial x_1}\right|_a, \ldots, \left.\frac{\partial u_0}{\partial x_{n-1}}\right|_a \right) \right).$$

PROPOSITION 2.9. *Let $\mathcal{E} \subset J^1(M)$ be a first-order differential equation and $R \subset \mathcal{E}$ be a manifold of Cauchy data such that none of the characteristic directions l_θ, $\theta \in R$, is tangent to R. Then, in a neighborhood of R, there exists a generalized solution of $\mathcal{E}$ containing R.*

PROOF. For each point $\theta \in R$ there exists a segment of a characteristic passing through θ that does not intersect R in other point. Take the union $N = \bigcup_{\theta \in R} \chi_\theta \subset \mathcal{E}$ of these segments. In view of the theorem about smooth dependence of solutions on initial conditions, without loss of generality we can assume that N is a smooth manifold.

Since for $\theta \in R$ the tangent plane $T_\theta(N)$ is generated by the plane $T_\theta(R) \subset \mathcal{C}_\theta(\mathcal{E})$ and the straight line $l_\theta \subset \mathcal{C}_\theta(\mathcal{E})$, we get $T_\theta(N) \subset \mathcal{C}_\theta$.

Consider a point $\theta \in N$ that does not belong to R. There is a shift along trajectories of the characteristic field which takes this point to a point $\theta_0 \in R$. The manifold N can be considered as invariant under this shift, so that the plane $T_\theta(N)$ maps to $T_{\theta_0}(N)$. Since the characteristic field belongs the Cartan distribution, the shift along its trajectories takes the space $\mathcal{C}_\theta(\mathcal{E})$ to $\mathcal{C}_{\theta_0}(\mathcal{E})$. Hence, $T_\theta(N) \subset C_\theta$.

Thus, N is the equation we seek for. $\square$

EXAMPLE 2.7. Consider the equation with two independent variables

$$u - \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} = 0,$$

i.e., $\mathcal{E} = \{u - p_1 p_2 = 0\}$. Let us find the solution whose restriction to the hypersurface $\Gamma = \{x_2 = 0\}$ equals $u_0(x_1) = x_1^2$. The graph of u_0 uniquely determines the following 1-dimensional Cauchy data

$$x_1 = \tau, \quad x_2 = 0, \quad u = \tau^2, \quad p_1 = 2\tau, \quad p_2 = u/p_1 = \tau/2,$$

where τ is a parameter.

Using (2.9), we get

$$Y_F = p_2 \frac{\partial}{\partial x_1} + p_1 \frac{\partial}{\partial x_2} + 2p_1 p_2 \frac{\partial}{\partial u} + p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2}.$$

Parametric equations of trajectories of this field have the form

$$\begin{cases} x_1 = \tau(e^t + 1)/2, \\ x_2 = 2\tau(e^t - 1), \\ u = \tau^2 e^{2t}, \\ p_1 = 2\tau e^t, \\ p_2 = \tau e^t/2, \end{cases} \tag{2.15}$$

where t is a parameter on trajectories.

Equations (2.15) give the solution in the parametric form. Solving the first two equations for τ and t and substituting the expressions obtained into the third equation yields the solution to the Cauchy problem in the explicit form: $u = (4x_1 + x_2)^2/16$.

EXAMPLE 2.8. Consider the *Hamilton–Jacobi equation*[4]

$$H\left(q, \frac{\partial u}{\partial q}\right) = C,$$

where $q = (q_1, \ldots, q_n)$, $\partial u/\partial q = (\partial u/\partial q_1, \ldots, \partial u/\partial q_n)$, C is a constant, i.e., H does not depend on u.

Note that the surface $H(q, p) = c$, as well as the Cartan distribution on $J^1(M)$, is invariant under shifts along trajectories of the field $\partial/\partial u$. The corresponding quotient equation can be considered as a submanifold of T^*M. The quotient Cartan distribution is given by the universal form $p\,dq$, since this form coincides with the projection of the form $-U_1$. Therefore, the symplectic structure on T^*M is defined by the projection of $-dU_1$.

The system (2.14) corresponding to the characteristic field takes the form

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}, \\ \dot{u} = -\displaystyle\sum_{i=1}^{n} p_i \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}, \qquad i = 1, \ldots, n. \end{cases}$$

Note that the first and the third equations correspond to the Hamiltonian vector field on T^*M, i.e., the characteristics are described by the Hamilton equations.

The invariant theory of the Hamilton–Jacobi equations (including the solution of the Cauchy problem) is fully considered in [**143**].

2.4. Symmetries and first integrals. As discussed above, solution of a non-characteristic Cauchy problem for a first-order differential equation in the vicinity of nonsingular point amounts to integration of the characteristic direction field l_θ, i.e., to solution of the characteristic system (2.14).

Recall that a *first integral* for a system of differential equation (resp., a vector field) is a function constant on solutions to this system (resp., trajectories of this field). We shall say that $\varphi \in C^\infty(J^1(M))$ is a *first integral of equation*

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[4]Sometimes by the Hamilton–Jacobi equation is meant the one of the form $u_t = \varphi(t, x, u_x)$, where $x = (x_1, \ldots, x_n)$, $u_x = (u_{x_1}, \ldots, u_{x_n})$. Such an equation can be obtained by setting $x_i = q_i$, $t = q_{n+1}$, and $H(q, u_q) = u_t - \varphi(t, x, u_x)$.

$\mathcal{E} = \{F = 0\} \subset J^1(M)$, if φ is a first integral of the characteristic system (2.14) of $\mathcal{E}$. Symmetries and first integrals are intimately related.

PROPOSITION 2.10. *If X_{f_1} and X_{f_2} are two symmetries of $\mathcal{E} = \{F = 0\}$ linearly independent over $C^\infty(J^1(M))$, then the function $f = f_2/f_1$ is a first integral of $\mathcal{E}$.*

PROOF. We have
$$\lambda_i F = X_{f_i}(F) = Y_{f_i}(F) + f_i X_1(F) = f_i X_1(F) - Y_F(f_i)$$
for some functions $\lambda_i \in C^\infty(J^1(M))$, $i = 1, 2$. It follows that $Y_F(f_i) = f_i X_1(F) - \lambda_i F$. Hence

$$Y_F(f_2/f_1) = (Y_F(f_2)f_1 - Y_F(f_1)f_2)/f_1^2$$
$$= (f_1 f_2 X_1(F) - \lambda_2 f_1 F - f_1 f_2 X_1(F) + \lambda_1 f_2 F)/f_1^2 = \left(\frac{\lambda_1 f_2 - \lambda_2 f_1}{f_1^2}\right) F. \quad \square$$

EXERCISE 2.10. Prove that contact vector fields X_{f_1} and X_{f_2} are linearly independent over $C^\infty(J^1(M))$ if and only if they (or, which is the same, their generating functions) are linearly independent over $\mathbb{R}$.

EXAMPLE 2.9. The equation
$$\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} - x_1^a x_2^b u^c = 0$$
for $c \neq 2$ has the contact symmetries $X_{\alpha u - x_1 p_1}$, where $\alpha = (a+1)/(2-c)$ and $X_{\beta u - x_2 p_2}$, with $\beta = (b+1)/(2-c)$. Therefore, the function $f = (\beta u - x_2 p_2)/(\alpha u - x_1 p_1)$ is a first integral of this equation.

PROPOSITION 2.11. *If an equation $\mathcal{E} \subset J^1(M)$ possesses an $(l+1)$-dimensional Abelian algebra of symmetries, then it has l independent first integral which are mutually in involution with respect to the Mayer bracket.*

PROOF. It can easily be checked that for all $f, g \in C^\infty(J^1(M))$ one has
$$[f, g] = \{f, g\} + f'g - g'f, \tag{2.16}$$
where $f' = X_1(f)$, $g' = X_1(g)$.

Let $X_{f_1}, \ldots, X_{f_{l+1}}$ be mutually commuting symmetries of $\mathcal{E}$. Put
$$g_1 = \frac{f_1}{f_{l+1}}, \ldots, g_l = \frac{f_l}{f_{l+1}}.$$
We claim that the first integrals g_i are mutually in involution on $\mathcal{E}$.

Indeed, $\{f_i, f_k\} = 0$ on $\mathcal{E}$, because the symmetries f_i and f_k commute. Using (2.16), we get
$$[f_i, f_k] = f_i' f_k - f_k' f_i.$$
From (2.5) it easily follows that

$$[g_i, g_k] = \frac{1}{f_{l+1}^3}(f_{l+1}[f_i, f_k] - f_i[f_{l+1}, f_k] - f_k[f_i, f_{l+1}])$$
$$= \frac{1}{f_{l+1}^3}(f_{l+1}f_i'f_k - f_{l+1}f_i f_k' - f_i f_{l+1}' f_k + f_i f_{l+1} f_k' - f_{l+1}f_i' f_k + f_{l+1}' f_i f_k) = 0$$
on $\mathcal{E}$, which proves the proposition. $\square$

Thus, if $X_{f_0}, X_{f_1}, \ldots, X_{f_l}$ is a basis of an Abelian algebra of symmetries of $\mathcal{E}$, then the first integrals $g_i = f_i/f_0$, $i = 1, \ldots, l$, of the characteristic distribution are independent and are mutually in involution.

EXAMPLE 2.10. Any equation of the form $F(u, p_1, \ldots, p_n) = 0$ has the n-dimensional Abelian algebra of symmetries generated by

$$\frac{\partial}{\partial x_1} = -X_{p_1}, \ldots, \frac{\partial}{\partial x_n} = -X_{p_n}.$$

It immediately follows that $p_2/p_1, \ldots, p_n/p_1$ is an involutive system of first integrals.

3. Complete integrals of first-order differential equations

In §2, we saw that the Cauchy problem for a first-order equation amounts to integration of the characteristic system of the equation under consideration. This observation gives rise to another method of solving first-order equations, namely, the complete integral method. Complete integrals are dealt with in [**20, 113**]. Remark that the role of symmetries in the construction of completes integrals is quite important.

3.1. Complete integrals: a coordinate approach. It is a common knowledge that a system of n first-order ordinary differential equations for n unknown functions has, in the vicinity of a nonsingular point, an n-parameter family of solutions. By analogy, let us look for an n-parameter family

$$u = V(x_1, \ldots, x_n, a_1, \ldots, a_n) \tag{3.1}$$

of solutions to a first-order partial differential equation $\mathcal{E}$. Here $a_1, \ldots, a_n$ are parameters.

Differentiating this relation with respect to x_i yields

$$p_1 = \frac{\partial V}{\partial x_1}, \ldots, p_n = \frac{\partial V}{\partial x_n}. \tag{3.2}$$

If one can obtain the initial equation $\mathcal{E}$ through eliminating the constants $a_1, \ldots, a_n$ from (3.1) and (3.2), then $V(x_1, \ldots, x_n, a_1, \ldots, a_n)$ is said to be a complete integral. Writing this condition analytically, we arrive at the following

DEFINITION 3.1 (coordinate). A *complete integral* of a first-order partial differential equation with n independent variables is an n-parameter family

$$u = V(x_1, \ldots, x_n, a_1, \ldots, a_n). \tag{3.3}$$

of solutions to this equation. A complete integral is called *nondegenerate* if the matrix

$$\begin{pmatrix} \partial V/\partial a_1 & \ldots & \partial V/\partial a_n \\ \partial^2 V/\partial a_1 \partial x_1 & \ldots & \partial^2 V/\partial a_n \partial x_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \partial^2 V/\partial a_1 \partial x_n & \ldots & \partial^2 V/\partial a_n \partial x_n \end{pmatrix} \tag{3.4}$$

is of maximal rank n almost everywhere.

EXAMPLE 3.1. The family

$$u = 2a_1 x_1 x_2^3 + a_1^2 x_2^6 + a_2 x_2^2$$

is a complete integral for the equation

$$\left(\frac{\partial u}{\partial x_1}\right)^2 + x_1\frac{\partial u}{\partial x_1} + 2u = x_2\frac{\partial u}{\partial x_2}$$

In this case, the matrix (3.4) vanishes on the surface $\{x_2 = 0\}$. Outside this surface the complete integral is nondegenerate.

EXAMPLE 3.2. For each equation of the form $F(p_1,\dots,p_n) = 0$, the family $u = a_1x_1 + \cdots + a_{n-1}x_{n-1} + cx_n + a_n$, where c is defined by the equality $F(a_1,\dots,a_{n-1},c) = 0$, is a complete integral.

EXAMPLE 3.3. For a Clairaut equation of the form $u = x_1p_1 + \cdots + x_np_n + f(p_1,\dots,p_n)$, the family $u = a_1x_1 + \cdots + a_nx_n + f(a_1,\dots,a_n)$ is a complete integral.

3.2. The construction of complete integrals using symmetry algebras.
The integration method discussed below is based on "integration" of an algebra of symmetries, i.e., on construction of the Lie group of finite contact transformations for which the given algebra is the algebra of infinitesimal generators.

Let M be an n-dimensional smooth manifold, $\mathcal{E} \subset J^1(M)$ be an equation of codimension one, and $X_{f_1},\dots,X_{f_n}$ be a basis of a Lie algebra $\mathfrak{g}$ of infinitesimal symmetries of this equation. Suppose that the Lie group G corresponding to $\mathfrak{g}$ is realized as a group of finite contact transformations. By the latter we mean an exponential correspondence defined for sufficiently small values of $a_1,\dots,a_n$ which takes every vector field $a_1X_{f_1} + \cdots + a_nX_{f_n}$ to a finite transformation $A_{(a_1,\dots,a_n)}$, with

$$\frac{d}{dt}(A_{(ta_1,\dots,ta_n)})^*f = (A_{(ta_1,\dots,ta_n)})^*(a_1X_{f_1} + \cdots + a_nX_{f_n})f,$$

for any $f \in C^\infty(J^1(M))$.

Let now N be a "sufficiently good" generalized solution of $\mathcal{E}$. In more exact terms, assume that the span of $T_\theta(N)$ and the vectors $(X_{f_1})_\theta,\dots,(X_{f_n})_\theta$ is of dimension $2n$ almost everywhere in N. Since the finite contact symmetries $A_{(a_1,\dots,a_n)}$ take generalized solutions of $\mathcal{E}$ to the generalized solutions,

$$N_{(a_1,\dots,a_n)} = A_{(a_1,\dots,a_n)}(N)$$

is an n-parameter family of generalized solutions, i.e., a complete integral of $\mathcal{E}$.

Let us find out what particular solutions can be used to construct complete integrals.

Let $u = u(x_1,\dots,x_n)$ be a solution to $\mathcal{E}$. The plane tangent to the graph $N = j_1(u)(M)$ spans the vectors

$$T_i = \frac{\partial}{\partial x_i} + \frac{\partial u}{\partial x_i}\frac{\partial}{\partial u} + \sum_{k=1}^n \frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial}{\partial p_k}.$$

Thus, constructing complete integral requires that the matrix composed of the coefficients of the fields

$$T_1,\dots,T_n, X_{f_1},\dots,X_{f_n} \tag{3.5}$$

be of rank $2n$ almost everywhere in N.

EXAMPLE 3.4. As we have mentioned in Example 2.10, any equation of the form $F(u, p_1, \ldots, p_n) = 0$ possesses the n-dimensional algebra of contact symmetries generated by

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}.$$

The exponential correspondence for this algebra has the form

$$A_{(a_1,\ldots,a_n)} \colon (x_1, \ldots, x_n) \longmapsto (x_1 + a_1, \ldots, x_n + a_n).$$

Therefore, if $u = \widetilde{u}(x_1, \ldots, x_n)$ is a solution satisfying the above conditions, then

$$V(x_1, \ldots, x_n, a_1, \ldots, a_n) = \widetilde{u}(x_1 - a_1, \ldots, x_n - a_n)$$

is a complete integral for this equation.

Note that the matrix (3.5) is nonsingular, if the matrix

$$\begin{pmatrix} \partial\widetilde{u}/\partial x_1 & \partial^2\widetilde{u}/\partial x_1 \partial x_1 & \ldots & \partial^2\widetilde{u}/\partial x_1 \partial x_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \partial\widetilde{u}/\partial x_n & \partial^2\widetilde{u}/\partial x_n \partial x_1 & \ldots & \partial^2\widetilde{u}/\partial x_n \partial x_n \end{pmatrix}$$

is of rank n almost everywhere. Consider several particular cases.

a. The equation

$$\frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} = u^a$$

has the solution $\widetilde{u} = ((2 - a)^2 x_1 x_2)^{1/(2-a)}$ for $a \neq 2$ and the solution $\widetilde{u} = e^{2\sqrt{x_1 x_2}}$ for $a = 2$. Using this, we obtain the complete integral $u = [(2 - a)^2 (x_1 - a_1)(x_2 - a_2)]^{1/(2-a)}$ in the first case and $u = e^{2\sqrt{(x_1 - a_1)(x_2 - a_2)}}$ in the second case.

b. The equation

$$u = \sum_{i=1}^{n} b_i \left(\frac{\partial u}{\partial x_i} \right)^2$$

has the solution $\widetilde{u} = \sum_{i=1}^{n} x_i^2 / 4b_i$, which yields the complete integral

$$u = \sum_{i=1}^{n} \frac{(x_i - a_i)^2}{4b_i}.$$

EXAMPLE 3.5. It is not hard to construct the exponential correspondence for the Abelian algebra consisting of infinitesimal translations and scale transformations (see Examples 2.1 and 2.3). In this case, each infinitesimal transformation can be integrated separately.

The equation

$$2x_1 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} - u \frac{\partial u}{\partial x_2} = a$$

possesses the Abelian symmetry algebra with the basis

$$x_1 \frac{\partial}{\partial x_1} - p_1 \frac{\partial}{\partial p_1}, \quad \frac{\partial}{\partial x_2}.$$

The particular solution $\widetilde{u}^2 = 2x_2(x_1 - a)$ gives the following complete integral

$$V^2(x_1, x_2, a_1, a_2) = 2(x_2 - a_2)(a_1 x_1 - a).$$

Let us now discuss an example of how to construct a complete integral using a non-Abelian algebra of contact symmetries.

EXAMPLE 3.6. Suppose that an equation $F(x_3, u, p_1, p_2, p_3) = 0$, in addition to obvious symmetries $\partial/\partial x_1$ and $\partial/\partial x_2$, has a scale symmetry of the form

$$X = \sum_{i=1}^{3} \alpha_i x_i \frac{\partial}{\partial x_i} + \beta u \frac{\partial}{\partial u} + \sum_{i=1}^{3} (\beta - \alpha_i) p_i \frac{\partial}{\partial p_i}. \tag{3.6}$$

Computing shifts along trajectories of this field, we easily obtain the exponential correspondence

$$A_{(a_1, a_2, a_3)} \colon (x_1, x_2, x_3, u, p_1, p_2, p_3) \longmapsto \left[\left(x_1 + \frac{a_1}{\alpha_1 a_3} \right) e^{\alpha_1 a_3} - \frac{a_1}{\alpha_1 a_3}, \right.$$

$$\left. \left(x_2 + \frac{a_2}{\alpha_2 a_3} \right) e^{\alpha_2 a_3} - \frac{a_2}{\alpha_2 a_3}, x_3 e^{\alpha_3 a_3}, u e^{\beta a_3}, p_1 e^{(\beta - \alpha_1) a_3}, p_2 e^{(\beta - \alpha_2) a_3}, p_3 e^{(\beta - \alpha_3) a_3} \right].$$

Take as an example the equation

$$4u = \left(\frac{p_1}{p_3^{\alpha_1}} \right)^2 + \left(\frac{p_2}{p_3^{\alpha_2}} \right)^2 + 4 x_3 p_3,$$

which has the symmetry of the form (3.6) with $\alpha_3 = 1$ and $\beta = 0$. Using the particular solution $\widetilde{u} = x_1^2 + x_2^2 + x_3$ and the above exponential correspondence, we get the following complete integral

$$V = \left[\left(x_1 + \frac{a_1}{\alpha_1 a_3} \right) e^{-\alpha_1 a_3} - \frac{a_1}{\alpha_1 a_3} \right]^2 + \left[\left(x_2 + \frac{a_2}{\alpha_2 a_3} \right) e^{-\alpha_2 a_3} - \frac{a_2}{\alpha_2 a_3} \right]^2 + x_3 e^{-a_3}.$$

3.3. Complete integrals: an invariant approach.

DEFINITION 3.2 (geometric). Let $\mathcal{E} \subset J^1(M)$ be a differential equation with one dependent variable, $n = \dim M$. Suppose that a domain $\mathcal{O} \subset \mathcal{E}$ is foliated by an n-parameter family of generalized solutions to the equation (n-dimensional integral manifolds of the distribution $\mathcal{C}(\mathcal{E})$), i.e., there are an n-dimensional smooth manifold N and a regular smooth map $\pi_N \colon \mathcal{O} \to N$ such that inverse images $N_a = \pi_N^{-1}(a)$ of points $a \in N$ are n-dimensional integral manifolds of $\mathcal{C}(\mathcal{E})$; such a foliation is called a *complete integral* of $\mathcal{E}$.

We claim that Definitions 3.1 and 3.2 are equivalent. Indeed, let us prove that from the coordinate definition it follows existence of a domain $\mathcal{O} \subset \mathcal{E}$ foliated by solutions of $\mathcal{E}$.

To this end, define a map $\bar{\alpha} \colon M \times \bar{N} \to \mathcal{E}$, where $\bar{N} = \{a = (a_1, \dots, a_n)\}$ is a parameter domain, by the formula $(b, a) \mapsto j_1(V(x, a))(b)$. Since the matrix (3.4) is nonsingular, this map has rank $2n$ at each point and, therefore, is a local diffeomorphism. Hence, we can choose open domains $U \subset M$, $N \subset \bar{N}$, $\mathcal{O} \subset \mathcal{E}$ such that the restriction

$$\alpha \colon U \times N \to \mathcal{O} \tag{3.7}$$

of $\bar{\alpha}$ is a diffeomorphism and for any $a \in N$ the surface $W_a = \alpha(U \times \{a\})$ is a solution to $\mathcal{E}$ over the domain U. We construct the desired map π_N by putting $\pi_N(\theta) = \beta_2(\theta)$ if $\alpha^{-1}(\theta) = (\beta_1(\theta), \beta_2(\theta))$. Clearly, $W_a = \pi_N^{-1}(a)$.

Using the above procedure yields a system of local coordinates

$$(x_1, \dots, x_n, a_1, \dots, a_n), \tag{3.8}$$

in $\mathcal{O} \subset \mathcal{E}$ such that the integral surfaces of the complete integral are given by

$$\begin{cases} a_1 = C_1, \\ \dots\dots \\ a_n = C_n, \end{cases}$$

where $C_1, \dots, C_n$ are constants.

EXERCISE 3.1. Show that the geometric definition of a complete integral implies the coordinate one.

Given a complete integral, we need not explicitly solve system (2.14) to find the characteristics. Indeed, observe that the map α (see (3.7)) is given by

$$\begin{cases} u = V(x, a), \\ p_1 = V_{x_1}(x, a), \\ \dots\dots\dots \\ p_n = V_{x_n}(x, a), \end{cases} \tag{3.9}$$

which describe the relation between the special local coordinates x, u, p and the coordinates (3.8). It follows that in the coordinates (3.8) the form U_1 is written as

$$U_1 = V_{a_1} \, da_1 + \cdots + V_{a_n} \, da_n = V_{a_1} \left(da_1 + \frac{V_{a_2}}{V_{a_1}} \, da_2 + \cdots + \frac{V_{a_n}}{V_{a_1}} \, da_n \right).$$

We can assume without loss of generality that locally $V_{a_1} \neq 0$, so that $\mathcal{C}(\mathcal{E})$ is given by the 1-form

$$\omega = da_1 + \frac{V_{a_2}}{V_{a_1}} \, da_2 + \cdots + \frac{V_{a_n}}{V_{a_1}} \, da_n.$$

Taking the functions

$$a_1, \dots, a_n, \; y_1, \; y_2 = \frac{V_{a_2}}{V_{a_1}}, \dots, y_n = \frac{V_{a_n}}{V_{a_1}},$$

where y_1 is an arbitrary independent function, for coordinates on $\mathcal{E}$, we get

$$\omega = da_1 + y_2 \, da_2 + \cdots + y_n \, da_n.$$

Therefore, $\partial/\partial y_1$ defines the degeneration direction of $d\omega$ on $\mathcal{C}(\mathcal{E})$ (i.e., $\partial/\partial y_1 \lrcorner \omega = 0$ and $\partial/\partial y_1 \lrcorner d\omega = 0$), while the functions $a_1, \dots, a_n, y_2, \dots, y_n$ are first integrals of the characteristic distribution.

EXAMPLE 3.7. Consider the equation

$$\frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} - x_1 \frac{\partial u}{\partial x_1} - x_2 \frac{\partial u}{\partial x_2} = 0.$$

It can easily be checked that the function $V(x_1, x_2, a_1, a_2) = a_1 + (x_1 + x_2 a_2)^2 / 2a_2$ is a complete integral for this equation.

We have $V_{a_1} = 1$, $p_1 = (x_1 + x_2 a_2)/a_2$, $p_2 = x_1 + x_2 a_2$, whence $a_2 = p_2/p_1$, $a_1 = u - p_1 p_2/2$. Substituting a_1 and a_2 to the expression for V_{a_2} yields $y_2 = p_1(p_2 x_2 - p_1 x_1)/2p_2$. The functions a_1, a_2, and y_2 are first integrals of the characteristic distribution.

EXAMPLE 3.8. Consider the equation

$$2x_1 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} - u \frac{\partial u}{\partial x_2} - a = 0.$$

A complete integral for this equation can be obtained by means of the relation

$$u^2 = V^2 = 2(a_1 x_1 - a)(x_2 - a_2).$$

It is easily shown that

$$y_2 = \frac{V_{a_2}}{V_{a_1}} = -\frac{a_1 p_2}{x_1 p_1}, \quad a_2 = x_2 - \frac{p_1(a_1 x_1 - a)}{a_1 p_2},$$

where a_1 is the integral defined by $2p_1(a_1 x_2 - a)^2 = u^2 p_2 a_1$. The functions a_1, a_2, y_2 constitute a complete set of first integrals of the characteristic distribution on the equation at hand.

3.4. The Lagrange–Charpit method. To construct a complete integral in the case of two independent variables it suffices to know one nontrivial first integral of the characteristic distribution. The method allowing one to do this is called the *Lagrange–Charpit method* [**113, 36**].

Let $\mathcal{E} = \{F = 0\}$ be a first-order equation with two independent variables, $F \in C^\infty(J^1(\mathbb{R}^2))$, and $f \in C^\infty(J^1(\mathbb{R}^2))$ be an integral of the characteristic vector field Y_F, i.e.,

$$Y_F(f) = 0. \tag{3.10}$$

Let us look for generalized solutions of $\mathcal{E}$ satisfying

$$\begin{cases} F(x_1, x_2, u, p_1, p_2) = 0, \\ f(x_1, x_2, u, p_1, p_2) = a_1, \end{cases} \tag{3.11}$$

where a_1 is a constant.

Solving (3.11) for p_1 and p_2 yields

$$\begin{cases} p_1 = g_1(x_1, x_2, u, a_1), \\ p_2 = g_2(x_1, x_2, u, a_1). \end{cases}$$

It is readily seen that under conditions (3.10) and (3.11) the form $U_1 = du - p_1 \, dx_1 - p_2 \, dx_2$ is closed, so that the system

$$\begin{cases} \dfrac{\partial u}{\partial x_1} = g_1(x_1, x_2, u, a_1), \\ \dfrac{\partial u}{\partial x_2} = g_2(x_1, x_2, u, a_1), \end{cases}$$

is compatible and has a solution $u = V(x_1, x_2, a_1, a_2)$, which is the desired complete integral of $\mathcal{E}$.

EXAMPLE 3.9. Consider the equation

$$x_2 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} - u \frac{\partial u}{\partial x_1} + c \frac{\partial u}{\partial x_2} = 0,$$

where a is a nonzero constant. It is easy to prove that p_2 is a first integral of the characteristic distribution of the equation under consideration. Setting $p_2 = a_2$ and solving the equation for p_1, we get $p_1 = ca_2/(u - x_2 a_2)$. Thus, the system of equations $\partial u/\partial x_2 = a_2$ and $\partial u/\partial x_1 = ca_2/(u - x_2 a_2)$ is compatible. Integrating the first equation with respect to x_1, we get $u = a_2 x_2 + v(x_1)$. The second equation yields $v_{x_1} = ca_2/v$, whence $v = \pm\sqrt{2ca_2 x_1 + a_1}$. Therefore, $u = V_\pm(x_1, x_2, a_1, a_2) = a_2 x_2 \pm \sqrt{2ca_2 x_1 + a_1}$ is a complete integral of the equation in hand for either of the two signs.

The Lagrange–Charpit method can be generalized to the case of arbitrary number of independent variables. To describe this generalization, let us first discuss the problem of the compatibility for an overdetermined system

$$\begin{cases} f_1 = c_1, \\ \dots\dots \\ f_r = c_r, \end{cases} \tag{3.12}$$

with $f_i \in C^\infty(J^1(M))$, $i = 1, \dots, r$, and $c_1, \dots, c_r$ constants.

PROPOSITION 3.1. *A sufficient condition for system* (3.12) *to be compatible is that for all* $1 \le i, k \le r$ *the Mayer brackets* $[f_i, f_k]$ *vanish on the surface* (3.12).

PROOF. First of all, note that the characteristic vector fields $Y_{f_1}, \dots, Y_{f_r}$ are tangent to the $(2n - r + 1)$-dimensional surface in $J^1(M)$ given by (3.12), since $Y_{f_i}(f_k) = [f_i, f_k] = 0$ on this surface.
Further, we have

$$[Y_f, Y_g] = Y_{[f,g]} + g'Y_f - f'Y_g - [f, g]X_1, \tag{3.13}$$

where $f' = U_1([X_1, X_f])$, $g' = U_1([X_1, X_g])$.

EXERCISE 3.2. Prove (3.13).

Using (3.13), we obtain $[Y_{f_i}, Y_{f_k}] = f'_k Y_{f_i} - f'_i Y_{f_k}$ on the surface (3.12), so that, by the Frobenius theorem, the vector fields $Y_{f_1}, \dots, Y_{f_r}$ generate a completely integrable distribution Y on this surface.
Next, observe that the manifold of intersection of an n-dimensional integral manifold of the Cartan distribution with the surface (3.12) is at least of dimension $n - r$. Take an $(n - r)$-dimensional integral manifold of the Cartan distribution belonging to the surface (3.12) such that for any $\theta \in R$ the tangent plane $T_\theta(R)$ and the span of $(Y_{f_1})_\theta, \dots, (Y_{f_r})_\theta$ intersect at zero. Let N be the union of all integral manifolds of the distribution Y passing through points of R. We claim that the n-dimensional manifold N is a generalized solution of (3.12) near R.
Indeed, for each point $\theta \in R$ the space $T_\theta(N)$ is the sum of $T_\theta(R) \subset \mathcal{C}_\theta$ and $Y_\theta \subset C_\theta$. Thus, $T_\theta(N) \subset \mathcal{C}_\theta$ for all $\theta \in N$ near R. $\qquad\square$

Recall that we have been working under the assumption that the form $dF|_\mathcal{E}$, where F is the function that defines the equation at hand $\mathcal{E} = \{F = 0\}$, is almost everywhere nonzero. In this case, any function $f \in C^\infty(J^1(M))$ vanishing on $\mathcal{E}$ has the form $f = \nu F$ for some $\nu \in C^\infty(J^1(M))$.
Under this condition, $f \in C^\infty(J^1(M))$ is a first integral of the characteristic distribution of $\mathcal{E}$ if and only if

$$[F, f] = \lambda F \qquad \text{for some } \lambda \in C^\infty(J^1(M)). \tag{3.14}$$

EXAMPLE 3.10. Consider the differential equation

$$h(x_1^2 + x_2^2)\left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] - \left(x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} - u\right)^2 = 0.$$

In this case $F = h(x_1^2 + x_2^2)(p_1^2 + p_2^2) - (x_1 p_1 + x_2 p_2 - u)^2$. Taking $f = (x_2 p_1 - x_1 p_2)/u$, we get

$$[F, f] = Y_F(f) = \frac{2(x_2 p_1 - x_1 p_2)}{u^2} F.$$

Therefore condition (3.14) is satisfied, so that the function f can be used in construction of a complete integral by means of the Lagrange–Charpit method.

System (3.12) is said to be *in involution* if $[f_i, f_k] = 0$, $1 \leq i, k \leq r$, by virtue of this system.

Let us now state the Lagrange–Charpit method in the general case.

THEOREM 3.2. *Let $\mathcal{E} = \{F = 0\} \subset J^1(M)$ be a first order differential equation with n independent variables and $f_2, \ldots, f_n$ be smooth functions on $J^1(M)$ such that the following conditions hold*:
1. $[F, f_k] = \lambda_k F$, $k = 2, \ldots, n$, $\lambda_k \in C^\infty(J^1(M))$.
2. $[f_i, f_k] = \lambda_{i,k} F$, $2 \leq i, k \leq n$, $\lambda_{i,k} \in C^\infty(J^1(M))$.
3. *The functions F, $f_2, \ldots, f_n$ are functionally independent.*

Then a complete integral of $\mathcal{E}$ can be computed by the following system

$$\begin{cases} F = 0, \\ f_2 = a_2, \\ \ldots\ldots\ldots \\ f_n = a_n. \end{cases} \tag{3.15}$$

PROOF. System (3.15) defines an $(n+1)$-dimensional surface P_a, $a = (a_2, \ldots, a_n)$, in $\mathcal{E}$. Let $\mathcal{C}(P_a)$ denote the induced Cartan distribution on this surface, i.e., the distribution given by the 1-form $\omega_a = U_1|_{P_a}$ (equivalently, $\mathcal{C}(P_a)_\theta = \mathcal{C}_\theta \cap T_\theta(P_a)$ for all $\theta \in P_a$).

The distribution $\mathcal{C}(P_a)$ is completely integrable. Indeed, since $\mathcal{C}(P_a)$ is generated by $Y_F, Y_{f_2}, \ldots, Y_{f_n}$, this fact is a part of the proof of Proposition 3.1.

Let f_1 be a first integral of this distribution. The system

$$\begin{cases} F = 0, \\ f_1 = b_1, \\ \ldots\ldots\ldots \\ f_n = b_n \end{cases}$$

defines a generalized solution to $\mathcal{E}$ for arbitrary constants $b_1, \ldots, b_n$. Thus, we have constructed a complete integral for $\mathcal{E}$ (see Definition 3.2). Regularity of the parameter map can be proved using the Frobenius theorem. $\square$

EXAMPLE 3.11. Consider the following equation with n independent variables:

$$u \frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_2} \cdot \ldots \cdot \frac{\partial u}{\partial x_n} - x_1 x_2 \cdots x_n = 0.$$

The functions

$$f_1 = \frac{p_1 u^{1/n}}{x_1}, \quad f_2 = \frac{p_2 u^{1/n}}{x_2}, \ldots, f_n = \frac{p_n u^{1/n}}{x_n}$$

are first integrals of the characteristic distribution for the considered equation $\mathcal{E} = \{u p_1 p_2 \cdots p_n - x_1 \cdots x_n = 0\}$.

It can easily be checked that the integrals $f_1, \ldots, f_n$ are in involution, so that the system

$$\begin{cases} p_1 u^{1/n} = A_1 x_1, \\ \ldots\ldots\ldots\ldots\ldots \\ p_n u^{1/n} = A_n x_n, \end{cases}$$

is compatible for all $A_1, \ldots, A_n = \text{const}$, being compatible with the initial equation $\mathcal{E}$ for $A_1 \cdot \ldots \cdot A_n = 1$. Solutions of this system are written as

$$u = \left(\frac{n+1}{2n} \sum_{i=1}^{n} A_i x_i^2 + A_n \right)^{n/(n+1)}, \qquad A_n = \text{const}.$$

In particular, putting $A_n = (A_1 \cdots A_{n-1})^{-1}$, we get a complete integral of $\mathcal{E}$.

CHAPTER 3

The Theory of Classical Symmetries

The methods of Ch. 2 were based on representation of a first order differential equation as a submanifold in the space $J^1(M)$ endowed with the Cartan distribution. In this chapter we extend the approach to arbitrary systems of nonlinear partial differential equations. We consider here the space of k-jets and the most important geometric structures related to it. We define classical symmetries of differential equations and, using some examples of equations in mathematical physics, illustrate methods of computation of these symmetries. Besides, we discuss applications of classical symmetries to construction of exact solutions.

1. Equations and the Cartan distribution

Consider a system of nonlinear differential equations of order k:

$$\begin{cases} F_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = 0, \\ \dots\dots\dots\dots \\ F_r(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = 0, \end{cases} \tag{1.1}$$

where F_l are some smooth functions, $\boldsymbol{x} = (x_1, \dots, x_n)$ are independent variables, $\boldsymbol{u} = (u^1, \dots, u^m)$ are unknown functions, while $\boldsymbol{p}$ denotes the set of partial derivatives $p^j_\sigma = \partial^{|\sigma|} u^j / \partial x_1^{i_1} \cdots \partial x_n^{i_n}$, $\sigma = (i_1, \dots, i_n)$ being a multi-index, $|\sigma| = i_1 + \cdots + i_n \le k$.

Geometric study of system (1.1) consists in treating the equations $F_i(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = 0$, $1 \le i \le r$, not as conditions on functions $\boldsymbol{u}$ themselves, but on k-th order Taylor series of these functions. This approach allows to introduce a finite-dimensional space whose coordinates correspond to the values of the functions $\boldsymbol{u}$ and of their derivatives.

Thus we shall consider the variables

$$x_1, \dots, x_n,\ u^1, \dots, u^m,\ p^i_\sigma, \qquad |\sigma| \le k,$$

as coordinates in some space $J^k(n, m)$ whose dimension is

$$\dim J^k(n, m) = n + m \sum_{i=0}^{k} \binom{n + i - 1}{n - 1} = n + m \binom{n + k}{k}.$$

Relations (1.1) determine in $J^k(n, m)$ a surface $\mathcal{E}$ of codimension r which is the geometric image corresponding to the given system of nonlinear differential equations. This surface in $J^k(n, m)$ will be called a *differential equation of order k* with n independent and m dependent variables. (We shall make this definition more precise below.) The surface $\mathcal{E}$ is a coordinate-free object unlike its representation as a system of the form (1.1), since the same equation can possess different analytical representations.

The fact that the variable p_σ^j, $\sigma = (i_1, \ldots, i_n)$, corresponds to the partial derivative of u^j with respect to $x_1, \ldots, x_n$ is expressed geometrically in the following way. Consider on $J^k(n, m)$ the *Cartan distribution* $\mathcal{C} = \mathcal{C}^k(n, m)$ determined by the basic 1-forms

$$\omega_\sigma^j = dp_\sigma^j - \sum_{i=1}^n p_{\sigma+1_i}^j dx_i, \qquad 1 \le j \le m, \quad |\sigma| \le k - 1, \tag{1.2}$$

where $1_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ and 1 stays at the i-th place[1]. A straightforward computation shows the number of the basic 1-forms equals

$$N = m \sum_{i=0}^{k-1} \binom{n+i-1}{n-1} = m \binom{n+k-1}{k-1}.$$

It is easily checked that the forms ω_σ^j are linear independent at every point and consequently the number N coincides with codimension of the Cartan distribution $\mathcal{C}$. Hence, dimension of the Cartan distribution is

$$\dim \mathcal{C}^k(n, m) = \dim J^k(n, m) - \operatorname{codim} \mathcal{C}^k(n, m) = n + m \binom{n+k-1}{n-1}.$$

Consider the projection

$$\pi_k \colon J^k(n, m) \to \mathbb{R}^n, \qquad \pi_k(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = \boldsymbol{x}. \tag{1.3}$$

In the preceding chapters (see §2 of Ch. 1 and §1 of Ch. 2) we showed that the Cartan distribution on the manifolds $J^k(1, 1)$ and $J^1(n, 1)$ allows one to distinguish between all sections of the mapping (1.3) and those ones which correspond to jets of smooth functions.

In general, the following theorem is valid:

THEOREM 1.1. *Let $Q \subset J^k(n, m)$ be an n-dimensional surface nondegenerately projecting to the space of independent variables $\mathbb{R}^n$ in a neighborhood of a point $y \in Q$. The surface Q is an integral manifold of the Cartan distribution $\mathcal{C}$ in the neighborhood under consideration if and only if it can be given by relations of the form*

$$\begin{cases} u^1 = f^1(x_1, \ldots, x_n), \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ u^m = f^m(x_1, \ldots, x_n), \\ p_\sigma^j = \dfrac{\partial^{|\sigma|} f^j}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}(x_1, \ldots, x_n). \end{cases} \tag{1.4}$$

for some smooth functions $f^1, \ldots, f^m$, where $\sigma = (i_1, \ldots, i_n)$ and $|\sigma| \le k$.

PROOF. In fact, nondegeneracy of the projection $\pi_k|_Q$ in a neighborhood of the point y (see Figure 3.1) means that locally the equation of Q can be represented as

$$\begin{cases} u^j = f^j(x_1, \ldots, x_n), \\ p_\sigma^j = f_\sigma^j(x_1, \ldots, x_n), \end{cases}$$

---

[1] Here and below we formally set $p_{(0, \ldots, 0)}^j = u^j$.
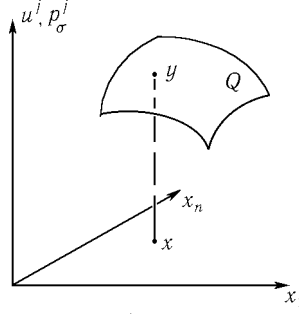
FIGURE 3.1. Nondegenerate projection of Q to $\mathbb{R}^n$

$j = 1, \ldots, m$, $|\sigma| \leq k$, where f^j, f^j_σ are some smooth functions. The surface Q is an integral manifold for the distribution $\mathcal{C}$, if the basis 1-forms vanish on this surface, i.e.,

$$\omega^j_\sigma\big|_Q = df^j_\sigma - \sum f^j_{\sigma+1_i} dx_i = \sum \left(\frac{\partial f^j_\sigma}{\partial x_i} - f^j_{\sigma+1_i} \right) dx_i = 0,$$

which is equivalent to the equalities

$$f^j_\sigma = \frac{\partial^{|\sigma|} f^j}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}$$

for all $j = 1, \ldots, m$ and $|\sigma| \leq k$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

DEFINITION 1.1. The surface Q given by the relations (1.4) will be denoted by $\Gamma^k_{\boldsymbol{f}}$ and called the *graph of the k-jet* of the vector function $\boldsymbol{f} = (f^1, \ldots, f^m)$.

From Theorem 1.1 it follows that a solution of a given equation $\mathcal{E} \subset J^k(n, m)$ is exactly an n-dimensional integral manifold of the Cartan distribution $\mathcal{C}^k$ nondegenerately projecting to the space $\mathbb{R}^n$ and lying on the surface $\mathcal{E}$:

$$\omega^j_\sigma\big|_Q = 0, \quad \ker d\pi_k\big|_y = 0, \qquad y \in Q, \quad Q \subset \mathcal{E}.$$

Solutions of the equation $\mathcal{E}$ can be defined without using the fact that $\mathcal{E}$ is a surface in $J^k(n, m)$. Namely, consider the manifold $\mathcal{E}$ together with the distribution $\mathcal{C}(\mathcal{E})$ induced on $\mathcal{E}$ by the distribution $\mathcal{C}$. The plane of the distribution $\mathcal{C}(\mathcal{E})$ at every point $y \in \mathcal{E}$ is the intersection of the plane $\mathcal{C}_y$ with the tangent plane $T_y\mathcal{E}$ to the surface $\mathcal{E}$. Obviously, integral manifolds of the distribution $\mathcal{C}(\mathcal{E})$ nondegenerately projecting to the space of independent variables $\mathbb{R}^n$ coincide with integral manifolds of the distribution $\mathcal{C}$ lying in $\mathcal{E}$ and possessing the same property. Therefore, one can say that solutions are n-dimensional manifolds of the manifold $\mathcal{C}(\mathcal{E})$ diffeomorphically projecting to the space $\mathbb{R}^n(x_1, \ldots, x_n)$. Recall that this is the approach to solutions of equations from which we started Ch. 1. Considering arbitrary n-dimensional maximal integral manifolds, we arrive to the concept of generalized solutions.

EXAMPLE 1.1 (the Burgers equation). Consider the Burgers equation

$$u_t - uu_x - u_{xx} = 0.$$

In the 8-dimensional space of jets $J^2(2,1)$, the surface $\mathcal{E}$ corresponds to this equation and is described in standard coordinates

$$x = x_1, \; t = x_2, \; u, \; p_{(1,0)}, \; p_{(0,1)}, \; p_{(1,1)}, \; p_{(0,2)}, \; p_{(2,0)}$$

by the relation

$$p_{(0,1)} - up_{(1,0)} - p_{(2,0)} = 0.$$

The Cartan distribution on $J^2(2,1)$ is determined by the basis 1-forms

$$\begin{aligned}
\omega_{(0,0)} &= du - p_{(1,0)}dx_1 - p_{(0,1)}dx_2, \\
\omega_{(1,0)} &= dp_{(1,0)} - p_{(2,0)}dx_1 - p_{(1,1)}dx_2, \\
\omega_{(0,1)} &= dp_{(0,1)} - p_{(1,1)}dx_1 - p_{(0,2)}dx_2.
\end{aligned}$$

Solutions of this equation are two-dimensional integral manifolds of this distribution lying in $\mathcal{E}$.

Choosing the functions $x = x_1$, $t = x_2$, u, $p_{(1,0)}$, $p_{(0,1)}$, $p_{(1,1)}$, $p_{(0,2)}$ for coordinates on the surface $\mathcal{E}$ and changing $p_{(2,0)}$ by $p_{(0,1)} - up_{(1,0)}$ everywhere, we shall obtain the distribution on the 7-dimensional space given by the basis forms

$$\begin{aligned}
\widetilde{\omega}_{(0,0)} &= du - p_{(1,0)}dx_1 - p_{(0,1)}dx_2, \\
\widetilde{\omega}_{(1,0)} &= dp_{(1,0)} - (p_{(0,1)} - up_{(1,0)})dx_1 - p_{(1,1)}dx_2, \qquad (1.5) \\
\widetilde{\omega}_{(0,1)} &= dp_{(0,1)} - p_{(1,1)}dx_1 - p_{(0,2)}dx_2.
\end{aligned}$$

Two-dimensional integral manifolds of this distribution, where the functions $u, p_{i,j}$ are expressed via x_1, x_2 (which is equivalent to nondegeneracy of the differential of the projection to $\mathbb{R}^2(x,t)$), correspond to solutions of the equation. For example, the two-dimensional surface

$$u = -\frac{x}{t}, \; p_{(1,0)} = -\frac{1}{t}, \; p_{(0,1)} = \frac{x}{t^2}, \; p_{(1,1)} = \frac{1}{t^2}, \; p_{(0,2)} = -\frac{2x}{t^3}.$$

corresponds to the solution $u = -x/t$.

Note that contrary to the case of systems of ordinary differential equations ($n = 1$), the Cartan distribution $\mathcal{C}$ on the manifolds $J^k(n,m)$, as well as the distributions $\mathcal{C}(\mathcal{E})$ obtained by restricting $\mathcal{C}$ to a partial differential equation $\mathcal{E}$, is not, in general, completely integrable. Nevertheless, for some overdetermined systems the distribution $\mathcal{C}(\mathcal{E})$ may be completely integrable.

EXERCISE 1.1. Let $m = k = 1$, $n = r = 2$. Consider the system of equations

$$\begin{cases} u_x = f(x, y, u), \\ u_y = g(x, y, u). \end{cases}$$

Prove that if this system is compatible, then the Cartan distribution restricted to the corresponding surface is completely integrable.

2. Jet manifolds and the Cartan distribution

In this section, we consider the basic objects important for the geometrical theory of differential equations. These are the jet manifolds and the Cartan distributions on them. We already met these objects when studying ordinary differential equations and first-order partial differential equations. Here we deal with the most general definitions and study basic properties.
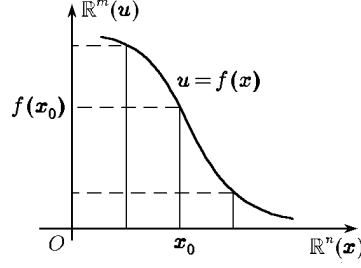
FIGURE 3.2

2.1. Geometric definition of the jet spaces. Up to now we considered equations on a vector function $\boldsymbol{u} = (u^1, \ldots, u^m)$ depending on the variables $\boldsymbol{x} = (x_1, \ldots, x_n)$. From geometric point of view, the function $\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x})$ is an analytical representation for a section of the projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$. In fact, to choose a function $\boldsymbol{f}(\boldsymbol{x})$ means to put into correspondence to every point $\boldsymbol{x}_0 \in \mathbb{R}^n$ another point $\boldsymbol{f}(\boldsymbol{x}_0) \in \mathbb{R}^m$, which can be considered as a point of the fiber $\mathbb{R}^m$ over the point $\boldsymbol{x}_0$ (see Figure 3.2).

To deal with equations on an arbitrary manifold, this construction is to be generalized. Consider an m-dimensional locally trivial bundle $\pi \colon E \to M$ over an n-dimensional manifold M. Recall that a section of the bundle π is a mapping $s \colon M \to E$ such that $\pi \circ s$ is the identical mapping of the base M. In other words, the mapping s takes a point $\boldsymbol{x} \in M$ to some point of the fiber $E_{\boldsymbol{x}}$. In a particular case of the trivial bundle $M \times N \to M$, sections are the mappings $M \to N$. In what follows, for the sake of simplicity we shall always assume that all bundles under consideration are vector bundles, i.e., their fibers are vector spaces and the gluing functions are linear transformations. Nevertheless, almost all constructions considered below are valid for arbitrary locally trivial bundles [**101, 60**].

Let $\mathcal{U} \subset M$ be a neighborhood over which the bundle π becomes trivial, i.e., such that $\pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times \mathbb{R}^m$. If $e_1, \ldots, e_m$ denotes a basis in the fiber of the bundle π, i.e., in the space $\mathbb{R}^m$, then any section is represented over $\mathcal{U}$ in the form $s = f^1 e_1 + \cdots + f^m e_m$, where f^i are smooth functions on $\mathcal{U}$. If $\mathcal{U}$ is a coordinate neighborhood on the manifold M with local coordinates $x_1, \ldots, x_n$, then any point of the fiber is determined by its projection to $\mathcal{U}$ and by its coordinates $u^1, \ldots, u^m$ with respect to the chosen basis. The functions $x_1, \ldots, x_n, u^1, \ldots, u^m$ are coordinates in $\pi^{-1}(\mathcal{U})$ and are called *adapted* coordinates for the bundle under consideration. Thus, any section is represented in adapted coordinates by a vector function $\boldsymbol{f} = (f^1, \ldots, f^m)$ in the variables $x_1, \ldots, x_n$.

DEFINITION 2.1. Two sections φ_1 and φ_2 of the bundle π will be called *tangent* with order k over the point $\boldsymbol{x}_0 \in M$, if the vector functions $\boldsymbol{u} = \boldsymbol{f}_1(\boldsymbol{x})$, $\boldsymbol{u} = \boldsymbol{f}_2(\boldsymbol{x})$ corresponding to these sections have the same partial derivatives up to order k at the point $\boldsymbol{x}_0$.

Obviously, this condition is equivalent to the fact that the k-th order Taylor series of the sections coincide. Since the functions themselves may be considered as their partial derivatives of zero order, the tangency condition for $k = 0$ reduces to coincidence of $\boldsymbol{f}_1(\boldsymbol{x}_0)$ and $\boldsymbol{f}_2(\boldsymbol{x}_0)$, i.e., the graphs of the sections s_1 and s_2 must intersect the fiber $E_{\boldsymbol{x}_0}$ at the same point (Figure 3.3a). On Figures 3.3b and 3.3c
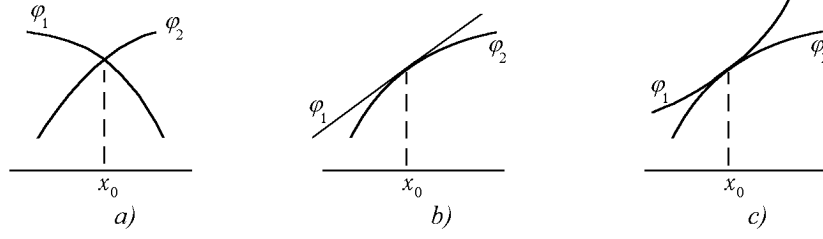
FIGURE 3.3

the cases of tangency with orders 1 and 2 are shown: a straight line tangent to a curve and an arc of the osculating circle.

EXERCISE 2.1. 1. Prove that Definition 2.1 is invariant, i.e., is does not depend on choice of adapted coordinates in the bundle π.

2. Prove that two sections have tangency of order k, if their graphs (see Definition 1.1) are tangent with the same order.

Tangency of sections with order k at a point $\mathbf{x}$ is an equivalence relation which will be denoted by $s_1 \overset{k,\boldsymbol{x}}{\sim} s_2$. The set of equivalence classes of sections, i.e., the set of k-th order Taylor series, will be denoted by $J_{\boldsymbol{x}}^k$ and called the *space of k-jets* of the bundle π at the point $\boldsymbol{x}$. The point of this space (the equivalence class of a section s) will be denoted by $[s]_{\boldsymbol{x}}^k$. Thus, if $s_1 \overset{k,\boldsymbol{x}}{\sim} s_2$, then $[s_1]_{\boldsymbol{x}}^k = [s_2]_{\boldsymbol{x}}^k$. The *space of k-jets of the bundle π* is the union of $J_{\boldsymbol{x}}^k$ over all points $\boldsymbol{x} \in M$:

$$J^k(\pi) = \bigcup_{\boldsymbol{x} \in M} J_x^k.$$

For any point $\theta = [s]_{\boldsymbol{x}}^k \in J^k(\pi)$ let us set $\pi_k(\theta) = \boldsymbol{x}$. Thus we have the projection $\pi_k \colon J^k(\pi) \to M$, and $\pi_k^{-1}(\boldsymbol{x}) = J_{\boldsymbol{x}}^k$.

For the case $k = 0$ we have $J^0(\pi) = \bigcup_{\boldsymbol{x} \in M} E_x = E$, i.e., the space $J^0(\pi)$ coincides with the total space of the bundle π.

Let us define local coordinates on the space of k-jets of the bundle π. To this end, take the functions x_i, u^j, and p_σ^j, corresponding to dependent and independent variables and to partial derivatives of latter with respect to former. In fact, let $x_1, \ldots, x_n, u^1, \ldots, u^m$ be an adapted coordinate system in the bundle π over a neighborhood $\mathcal{U}$ of the point $\boldsymbol{x} \in M$. Consider the set $\pi_k^{-1}(\mathcal{U}) \subset J^k(\pi)$. Let us complete local coordinates $x_1, \ldots, x_n, u^1, \ldots, u^m$ by the functions p_σ^j defined by the formula

$$p_\sigma^j([s]_{\boldsymbol{x}}^k) = \frac{\partial^{|\sigma|} s^j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \qquad j = 1, \ldots, m, \quad |\sigma| \le k,$$

In what follows, we shall call the coordinates p_σ^j *canonical* (or *special*) *coordinates* associated to the adapted coordinate system (x_i, u^j).

For a given bundle π one can consider all jet manifolds $J^k(\pi)$, $k = 0, 1, \ldots$, arranging them one over another as the tower

Jets and projections	Coordinates in jets
$J^{k+1}(\pi)$	$x_i, u^j, p_\sigma^j, \ |\sigma| \le k + 1$
$\downarrow \pi_{k+1,k}$	
$J^k(\pi)$	$x_i, u^j, p_\sigma^j, \ |\sigma| \le k$
$\downarrow \pi_{k,k-1}$	
$\cdots$	$\cdots$
$\downarrow \pi_{1,0}$	
$J^0(\pi) = E$	x_i, u^j
$\downarrow \pi$	
M	x_i

The projections $\pi_{t+1,t}$ are defined by the formula

$$\pi_{t+1,t}\colon J^{t+1}(\pi) \longrightarrow J^t(\pi), \qquad \pi_{t+1,t}([s]_{\boldsymbol{x}}^{t+1}) = [s]_{\boldsymbol{x}}^t,$$

where $t = 0, \ldots, k$. Since the equivalence class $[s]_{\boldsymbol{x}}^{t+1} \in J^{t+1}(\pi)$ determines the class $[s]_{\boldsymbol{x}}^t \in J^t(\pi)$ uniquely, the projections $\pi_{t+1,t}$ are well defined.

EXERCISE 2.2. 1. Prove that the family of neighborhoods $\pi_k^{-1}(\mathcal{U})$ together with the coordinate functions x_i, u^j, p_σ^j determines a smooth manifold structure in $J^k(\pi)$.

2. Prove that the projection $\pi_k\colon J^k(\pi) \to M$ is a smooth locally trivial vector bundle.

EXERCISE 2.3. Prove that the projections $\pi_{t+1,t}$ are smooth locally trivial bundles. Prove also that $\pi_t \circ \pi_{t+1,t} = \pi_{t+1}$.

2.2. The Cartan distribution. Let us introduce now the basic geometric structure on the manifold $J^k(\pi)$. This structure is the *Cartan distribution*. First of all, let us note that if s is a section of the bundle π, then for any point $x \in M$ one can define the element $j_k(s)(x) = [s]_{\boldsymbol{x}}^k \in J_{\boldsymbol{x}}^k(\pi)$. Obviously, the mapping $j_k(s)\colon M \to J^k(\pi)$ is a smooth section of the bundle $\pi_k\colon J^k(\pi) \to M$. It is called the *k-jet* of the section s. The graph of the k-jet in the space $J^k(\pi)$ is denoted by Γ_s^k.

Let us say that an n-dimensional plane in the tangent space $T_\theta(J^k(\pi))$, $\theta \in J^k(\pi)$, is an *R-plane* , if it is tangent to the graph of the k-jet of some section of the bundle π. Obviously, any R-plane is horizontal with respect to the projection $\pi_k\colon J^k(M) \to M$.

Note that a point $\theta' \in J^{k+1}(\pi)$ can be considered as the pair consisting of the point $\theta = \pi_{k+1,k}(\theta') \in J^k(\pi)$ and an R-plane $L_{\theta'} \subset T_\theta(J^k(\pi))$ defined as the plane tangent to the graph of the k-jet of some section s such that $[s]_{\boldsymbol{x}}^{k+1} = \theta'$. (It is easily seen that this plane is uniquely determined by the jet $[s]_{\boldsymbol{x}}^{k+1}$.) In other words, θ' is the set of values of derivatives up to order $k + 1$ while the plane $L_{\theta'} \subset T_\theta J^k(\pi)$ is determined by the values of first derivatives of k-th derivatives.
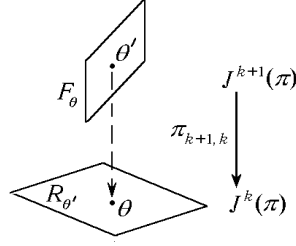
FIGURE 3.4. The fiber of the projection $\pi_{k+1,k}$
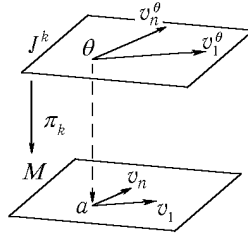
FIGURE 3.5. A basis of the plane $L_{\theta'}$

EXERCISE 2.4. Write down the conditions that an n-dimensional plane in the space $T_\theta(J^k(\pi))$ is an R-plane.

Let us fix a point $\theta \in J^k(\pi)$ and consider various points $\theta' \in J^{k+1}(\pi)$ projecting to θ under the mapping $\pi_{k+1,k}$. These points form the fiber of the bundle $\pi_{k+1,k} \colon J^{k+1}(\pi) \to J^k(\pi)$ over the point θ. We shall denote this fiber by $F_{k+1,k}(\theta)$ or by F_θ (see Figure 3.4). In other words, we fix the value of some vector function and of its derivatives up to order k and let other derivatives vary in an arbitrary way. When a point θ' moves along the fiber F_θ, the corresponding n-dimensional plane $L_{\theta'} \subset T_\theta(J^k(\pi))$ rotates somehow around the point θ, but always remains horizontal with respect to the projection $\pi_k \colon J^k(\pi) \to M$.

DEFINITION 2.2. The *Cartan plane* $\mathcal{C}_\theta = \mathcal{C}_\theta^k$ at the point $\theta \in J^k(\pi)$ is the span of all planes $L_{\theta'}$ for $\theta' \in F_\theta$, i.e., the span of all planes tangent to the graphs Γ_s^k of all k-jets of sections for which $[s]_{\boldsymbol{x}}^k = \theta$. The correspondence

$$\mathcal{C} \colon \theta \longmapsto \mathcal{C}_\theta^k$$

is called the *Cartan distribution* on $J^k(\pi)$.

Let us give a coordinate description of the Cartan distribution. To do this, we shall explicitly write down a basis of the plane $L_{\theta'}$ corresponding to the point $\theta' \in J^{k+1}(\pi)$ in an adapted coordinate system $x_1, \ldots, x_n, u^1, \ldots, u^m$. The coordinates of the point θ' will be denoted by $x_i(\theta')$, $u^j(\theta')$, $p_\sigma^j(\theta')$ (see Figure 3.5).

Let s be a section such that $[s]_a^{k+1} = \theta'$, $\Gamma_s^k \subset J^k(\pi)$ be its graph, and the R-plane $L_{\theta'}$ be tangent to Γ_s^k. Let us choose a basis in L_θ consisting of the vectors $v_1^\theta, \ldots, v_n^\theta$ whose projections to M coincide with $v_1 = \partial/\partial x_1|_a, \ldots, v_n = \partial/\partial x_n|_a$. From equations (1.4) determining the surface Γ_s^k it is easy to deduce formulas for
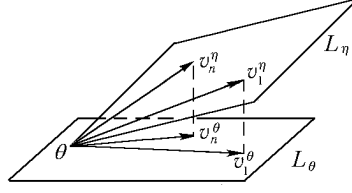
FIGURE 3.6. R-planes at the point θ
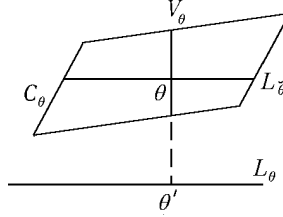
FIGURE 3.7. Structure of the Cartan plane

the basis vectors v_i^θ:

$$v_i^\theta = \frac{\partial}{\partial x_i} + \sum_{|\sigma| \leq k} \sum_{j=1}^{m} p_{\sigma+1_i}^j (\theta') \frac{\partial}{\partial p_\sigma^j}. \tag{2.1}$$

It is easily seen that the above summation over σ, $|\sigma| < k$, is determined by the point θ and is independent of the choice of the point θ' projecting to θ. Thus any plane L_η corresponding to $\eta \in F_\theta$ can be obtained from the given plane L_θ by rotating in the "vertical direction". More exactly, the position of L_η with respect to L_θ can be determined by the set of n shift vectors $\delta_i = v_i^\eta - v_i^\theta$ (see Figure 3.6) vertical with respect to the projection $\pi' = \pi_{k,k-1}$. Denote the space of vertical vectors at the point θ by $V_\theta = T_\theta(F_{\theta''})$, where $\theta'' = \pi_{k,k-1}(\theta)$; the vectors $\partial/\partial p_\sigma^j$, $|\sigma| = k$, form a basis in this space. Note that any vertical tangent vector $v \in V_\theta$ can be considered as a shift vector: for a basis vector $\partial/\partial p_\sigma^j$, $|\sigma| = k$, it suffices to take the point $\eta \in F_\theta$, such that all its coordinates, except for one, coincide with the corresponding coordinates of the point θ' while $p_{\sigma+1_i}^j(\eta) = p_{\sigma+1_i}^j(\theta') + 1$. Then $\delta_i = v_i^\eta - v_i^\theta = \partial/\partial p_\sigma^j$.

From equations (2.1) it follows that the Cartan distribution on $J^k(\pi)$ is determined by the set of 1-forms $\omega_\sigma^j = dp_\sigma^j - \sum_{i=1}^{n} p_{\sigma+1_i}^j dx_i$, $|\sigma| \leq k-1$, which are called the *Cartan forms*.

Geometric structure of Cartan planes (Figure 3.7) is described by the following theorem:

THEOREM 2.1. *Let $\mathcal{C}$ be the Cartan distribution on the manifold $J^k(\pi)$. Then the following statements are valid*:

1. *The Cartan plane $\mathcal{C}_\theta \subset T_\theta(J^k(\pi))$ is the direct sum of the vertical and horizontal subspaces*

$$\mathcal{C}_\theta = V_\theta \oplus L_{\widetilde{\theta}}, \tag{2.2}$$

where V_θ is the tangent space to the fiber of the projection $\pi_{k,k-1}$ and $L_{\widetilde{\theta}}$ is the R-plane corresponding to some point $\widetilde{\theta}$ in the fiber F_θ over the point θ.

2. The Cartan plane $\mathcal{C}_\theta$ consists of tangent vectors at the point θ, that project to L_θ under the mapping $\pi_{k,k-1}$, i.e.,

$$\mathcal{C}_\theta = (\pi_{k,k-1})_*^{-1}(L_\theta). \tag{2.3}$$

PROOF. The first statement follows from the above considerations. Let us note only that in the decomposition (2.2) the first summand, unlike the second one, is determined by the point θ uniquely.

To prove the second statement, note that the projection $(\pi_{k,k-1})_*$ takes the subspace V_θ to zero while the plane $L_{\widetilde{\theta}}$ is taken to L_θ bijectively. Therefore, $\mathcal{C}_\theta \subset (\pi_{k,k-1})_*^{-1}(L_\theta)$. The inverse embedding follows from the fact that the inverse image of the point $(\pi_{k,k-1})_*^{-1}(\theta'')$ coincides with $V_\theta \subset \mathcal{C}_\theta$. $\qquad\square$

COROLLARY 2.2. *Any horizontal (with respect to the projection $\pi_{k,k-1}$) subspace of the Cartan plane $\mathcal{C}_\theta$ cannot be of dimension greater than $n = \dim M$.*

COROLLARY 2.3. *A plane $P \subset \mathcal{C}_\theta$ is horizontal with respect to the projection $\pi_{k,k-1}$ if and only if it is horizontal with respect to the projection $\pi_k \colon J^k(\pi) \to M$ (i.e., degeneracy of the Cartan plane under the mappings*

$$J^k(\pi) \longrightarrow J^{k-1}(\pi) \longrightarrow \cdots \longrightarrow J^0(\pi) \longrightarrow M$$

may occur at the first step only).

Let us find out coordinate representation for a vector field X lying in the Cartan distribution $\mathcal{C}^k$ on the manifold $J^k(\pi)$. To do this, we shall use the splitting (2.2). The set of vertical vector fields $\partial/\partial p_\sigma^j$, $j = 1, \ldots, m$, $|\sigma| = k$, forms a basis of the vertical subspace V_θ at the point θ. A basis of the horizontal subspace can be chosen as the set of truncated operators of total derivatives

$$D_i^{(k)} = \frac{\partial}{\partial x_i} + \sum_{j=1}^m \sum_{|\sigma|<k} p_{\sigma+1_i}^j \frac{\partial}{\partial p_\sigma^j}. \tag{2.4}$$

Thus any vector field X lying in the Cartan distribution can be decomposed in the chosen basis:

$$X = \sum_{i=1}^m a^i D_i^{(k)} + \sum_{|\sigma|=k} b_\sigma^j \frac{\partial}{\partial p_\sigma^j}. \tag{2.5}$$

The distribution $\mathcal{C}^k$ is not completely integrable, since, for example, for $|\sigma| = k - 1$ the commutator

$$\left[\frac{\partial}{\partial p_{\sigma+1_i}^j}, D_i^{(k)}\right] = \frac{\partial}{\partial p_\sigma^j}$$

is not of the form (2.5). Consequently, maximal integral manifolds of the distribution $\mathcal{C}^k$ are of dimension less than $\dim \mathcal{C}^k$.

We can finally give a definition of a differential equation of order k similar to that given for differential equations of first order.

DEFINITION 2.3. A *differential equation of order k* in the bundle $\pi \colon E \to M$ is a submanifold $\mathcal{E} \subset J^k(\pi)$ endowed with the Cartan distribution $\mathcal{C}(\mathcal{E}) \colon \theta \mapsto \mathcal{C}_\theta(\mathcal{E}) = \mathcal{C}_\theta^k \cap T_\theta(\mathcal{E})$, $\theta \in \mathcal{E}$. A maximal integral manifold (of dimension $\dim M$) of the Cartan distribution is called a *(generalized) solution* of the equation $\mathcal{E}$.
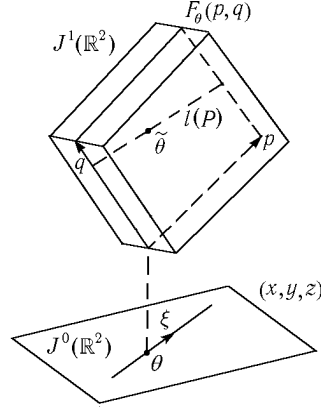
FIGURE 3.8. The ray manifold

EXERCISE 2.5. Let $\pi\colon E \to M$ be a fiber bundle and $\pi_{1,0}\colon J^1(\pi) \to E$. Show that sections of the bundle $\pi_{1,0}$ are connections in the bundle π while the condition of zero curvature determines a first order equation in the bundle $\pi_{1,0}$.

2.3. Integral manifolds of the Cartan distribution. In Theorem 1.1, we showed that the graphs Γ_f^k of k-jets of sections are integral manifolds of the Cartan distribution. Here we shall describe the structure of arbitrary maximal integral manifolds.

DEFINITION 2.4. Let $P \subset \mathcal{C}_\theta^{k-1}$ be a plane of dimension s, $s \leq \dim M$. The subset

$$l(P) = \{\, \widetilde{\theta} \in F_\theta \mid L_{\widetilde{\theta}} \supset P \,\}$$

of the fiber over the point θ is called the *ray submanifold* (or simply *ray*) corresponding to the plane P, Figure 3.8. If $N \subset J^{k-1}(\pi)$ is a submanifold, then the set

$$L(N) = \bigcup_{q \in N} l\left(T_q N\right)$$

is called the *prolongation* (or *lifting*) of the submanifold N.

EXERCISE 2.6. Show that if $\Gamma_f^{k-1} \subset N$, then $\Gamma_f^k \subset L(N)$.

EXAMPLE 2.1. Consider the manifold of 1-jets $J^1(2,1)$ of scalar functions in two variables with the coordinates x, y, z, p, q, where z is a function in x, y and p and q are its derivatives with respect to x and y.

Let us fix a point $\theta \in J^0(\mathbb{R}^2)$ with the coordinates x, y, z. The fiber F_θ of the bundle $J^1(\mathbb{R}^2) \to J^0(\mathbb{R}^2)$ over this point is the plane with coordinates p, q. Let $\xi = X\partial/\partial x + Y\partial/\partial y + Z\partial/\partial z$ be a nonzero vector at the point θ and P be the straight line determined by this vector. Let us deduce the equations describing the corresponding ray manifold $l(P)$.

Let p_0, q_0 be the coordinates of the point $\widetilde{\theta} \in F_\theta$. Then the corresponding plane $L_{\widetilde{\theta}}$ is described by the equation

$$dz - p_0\, dx - q_0\, dy = 0$$
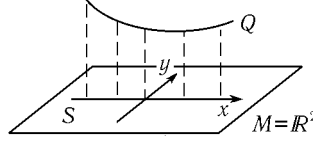
FIGURE 3.9

while the condition $L_{\widetilde{\theta}} \supset P$, i.e., $\xi \in L_{\widetilde{\theta}}$, is of the form $Xp_0 + Yq_0 = Z$. Consequently, if the values of X, Y, and Z are fixed while p and q vary, the submanifold $l(P) \subset F_\theta$ is described by the equation $Xp + Yq = Z$.

In generic position, $l(P)$ is a straight line in the plane F_θ. Note that the direction of this line is determined by the projection (X, Y) of the vector ξ to the plane $\mathbb{R}^2_{x,y}$. The position of its initial point also depends on Z. In the exceptional case, when the vector ξ is vertical (i.e., when $X = Y = 0$ and $Z \neq 0$), the set $l(P)$ is empty, since an R-plane cannot contain vertical vectors.

Consider now a submanifold $N \subset J^0(\mathbb{R}^2)$ nondegenerately projecting to $\mathbb{R}^2_{x,y}$. Depending on the dimension of N, its prolongation $L(N)$ is described in the following way:

1. The submanifold N consists of one point θ only. Then $L(N) = F_\theta$, the fiber over the point θ.
2. The submanifold N is a curve parametrized as $(\alpha(t), \beta(t), \gamma(t))$. Then $L(N)$ is a two-dimensional surface in the five-dimensional space $J^1(\mathbb{R}^2)$ given by the equations

$$\begin{cases} x = \alpha(t), \\ y = \beta(t), \\ z = \gamma(t), \\ \alpha'(t)p + \beta'(t)q = \gamma'(t). \end{cases} \tag{2.6}$$

The curve $N \subset J^0(\mathbb{R}^2)$ can be considered as the graph of some function $z(x, y)$ defined on a curve $x = \alpha(t)$, $y = \beta(t)$ lying in the plane $\mathbb{R}^2_{x,y}$. The last equation of system (2.6) shows how partial derivatives $p = z_x$, $q = z_y$ of this function are related to each other on the curve.

3. The submanifold N is a surface of the form $z = f(x, y)$. Then every point $\theta \in N$ uniquely determines a point $\widetilde{\theta} \in F_\theta$ such that $L_{\widetilde{\theta}} = T_\theta(N)$. In fact, the coordinates p, q of the point $\widetilde{\theta}$ must coincide with the slopes of the tangent plane $T_\theta(N)$. Hence, $\widetilde{\theta} = [f]^1_a$ while $L(N) = \Gamma^1_f$ is the graph of 1-jet of the function f.

EXAMPLE 2.2. Consider an integral curve Q of the Cartan distribution on $J^1(2, 1)$ and its lifting to $J^2(2, 1)$. From Corollaries 2.2 and 2.3 it follows that the lifting of Q is not empty if and only if the curve Q is horizontal with respect to the projection $\pi_1 \colon J^1(\mathbb{R}^2) \to \mathbb{R}^2$. In this case, one can choose coordinates x, y on $\mathbb{R}^2$ such that the projection S of the curve Q in a neighborhood of some point is of the form $y = 0$ (see Figure 3.9). The curve itself in this case can be described

by the relations of the form

$$\begin{cases} y = 0, \\ z = \alpha(x), \\ p = \beta(x), \\ q = \gamma(x). \end{cases} \tag{2.7}$$

Since this curve is an integral manifold of the Cartan distribution given by the equation $(dz - p\,dx - q\,dy)|_Q = 0$, one has $\beta = \alpha'$. Conditions (2.7) mean that we know the values of the function $z(x,y)$ together with the values of its derivative $z_y(x,y)$ for $y = 0$. Thus we obtained the standard formulation for the Cauchy problem for second order equations. Hence, in this situation the problem of lifting is equivalent to describing the graphs of the jets for functions satisfying Cauchy conditions.

Let $\theta(x, y, z, p, q)$ be a point of the curve Q and $\xi = (X, Y, Z, P, Q)$ be a tangent vector. Let us describe the set $l(\xi)$ in the fiber F_θ. Denote by r, s, t coordinates in the fiber of the projection $J^2(\mathbb{R}^2) \to J^1(\mathbb{R}^2)$ (they correspond to the derivatives z_{xx}, z_{xy}, z_{yy} respectively). Then the equations

$$\begin{cases} Xr + Ys = P, \\ Xs + Yt = Q \end{cases} \tag{2.8}$$

describe the line $l(\xi)$.

Note that the classical concept of a characteristic for second order equations is based on consideration of system (2.8) together with the initial equation $\mathcal{E} \subset J^2(\mathbb{R}^2)$.

Note now that for an integral manifold $N \subset J^{k-1}(\pi)$ nondegenerate with respect to the projection to the base (in what follows, such manifolds will be called *horizontal*) the corresponding manifold $L(N) \subset J^k(\pi)$ is also integral one. To see this, it suffices, applying Theorem 2.1, to check that at any point $\widetilde{\theta} \in L(N)$ the image of the tangent space $T_{\widetilde{\theta}}(L(N))$ under the projection $\pi_{k,k-1}$ lies in $L_{\widetilde{\theta}}$. By definitions and by the construction of $L(N)$, we have $(\pi_{k,k-1})_{*,\widetilde{\theta}}(L(N)) = T_{\widetilde{\theta}}(L(N)) = T_\theta(N) \subset L_{\widetilde{\theta}}$. Thus $L(N)$ is an integral manifold. Therefore, we obtained a method to construct integral manifolds in $J^k(\pi)$ starting with the ones in $J^{k-1}(\pi)$.

As it will be proved in Theorem 2.7, the above construction is of a universal nature. We shall now describe a local structure of horizontal integral manifolds N and deduce a formula to compute dimension of $L(N)$ via dimension of N.

PROPOSITION 2.4. *Any horizontal integral manifold N of the Cartan distribution on $J^k(\pi)$ locally lies on the graph $\Gamma_{\boldsymbol{f}}^k$ of the k-jet of some section $\boldsymbol{f}$.*

PROOF. Let $r = \dim N \leq n$. Let us introduce in a neighborhood of the point $a = \pi_k(\theta)$, $\theta \in N$, coordinates $x_1, \ldots, x_n$ such that $\widetilde{N} = \pi_k(N)$ is given by the equations $x_{r+1} = \cdots = x_n = 0$. Since the manifold N is horizontal, it is nondegenerate with respect to the projection $\pi_k \colon J^k(\pi) \to M$. Consequently, N is described by the equations

$$x_i = 0, \quad i > r, \qquad p_\sigma^j = \tilde{f}_\sigma^j(x_1, \ldots, x_r), \quad j = 1, \ldots, m, \ |\sigma| \leq k.$$

Restricting the Cartan forms to N (cf. the proof of Theorem 1.1), we obtain that

$$\tilde{f}_{\sigma+1_i}^j = \frac{\partial \tilde{f}_\sigma^j(x_1, \ldots, x_r)}{\partial x_i}, \qquad |\sigma| \leq k-1, \quad i = 1, \ldots, r.$$

Thus, the functions $\tilde{f}^1 \ldots, \tilde{f}^m$ are defined on $\tilde{N}$. They satisfy the above conditions and the values of all their normal derivatives are given along $\tilde{N}$. Hence, there exist functions $f^1, \ldots, f^m$ such that $f^j\big|_{\tilde{N}} = \tilde{f}^j$ and with the given values of normal derivatives on $\tilde{N}$. Consequently, $\boldsymbol{f}(\boldsymbol{x}) = (f^1, \ldots, f^m)$ is the section we need. $\qquad\square$

PROPOSITION 2.5. *Let* $\pi\colon E \to M$ *be an* m-*dimensional bundle over an* n-*dimensional manifold* M *and* $N \subset J^k(\pi)$ *be a horizontal integral manifold of the Cartan distribution of dimension* $r \leq n$. *Then*[2]

$$\dim L(N) = r + m \binom{n - r + k}{n - r - 1}.$$

PROOF. Let us fix a point $\theta \in N$ and denote by P_θ the tangent space $T_\theta(N)$. We introduce a coordinate system $x_1, \ldots, x_n$ in a neighborhood of the point $\pi_k(\theta)$ such that $\pi_k(N)$ is described by the equations $x_{r+1} = \cdots = x_n = 0$. Then the vectors

$$\xi_i = \left(\frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \sum_{|\sigma| \leq k} a_{\sigma,i}^j \frac{\partial}{\partial p_\sigma^j} \right)\Bigg|_\theta, \qquad 1 \leq i \leq r, \qquad (2.9)$$

form a basis of the plane P_θ, where $a_{\sigma,i}^j = \partial^{|\sigma|+1} \boldsymbol{f}^j / \partial x_\sigma \partial x_i$ for the section $\boldsymbol{f} = (f^1, \ldots, f^m)$ constructed in Proposition 2.4.

The condition that a point $\tilde{\theta}$ with the coordinates $\tilde{p}_\tau^j$ and lying in the fiber over θ belongs to $l(P_\theta)$ can be written in the form

$$\tilde{p}_{\sigma+1_i}^j = a_{\sigma,i}^j, \qquad 1 \leq i \leq r, \quad 1 \leq j \leq m.$$

This system of linear inhomogeneous equations on the unknowns $\tilde{p}_\tau^j$ is compatible, i.e., for all i, l, j, μ the equalities $a_{\mu+1_i,l}^j = a_{\mu+1_l,i}^j$ are valid. Therefore, relations (2.9) make it possible to find unique coordinates $\tilde{p}_\tau^j$ such that the multi-index τ, $|\tau| = k + 1$, is not equal to zero in the first r components.

There are no relations for other coordinates; the number of these coordinates for any fixed $j = 1, \ldots, m$ equals the number of all indices τ, $|\tau| = k + 1$, whose components $r + 1, \ldots, n$, do not vanish. Hence, this number is $\binom{n-r+k}{n-r-1}$.

Thus, $\dim l(P_\theta) = m \binom{n-r+k}{n-r-1}$, and since the manifold $L(N)$ is fibered over N with the fiber $l(P_\theta)$ over the point $\theta \in N$, we have $\dim L(N) = r + m \binom{n-r+k}{n-r-1}$. $\quad\square$

COROLLARY 2.6. *If* $r_1 = \dim N_1 < r_2 = \dim N_2$, *then* $\dim L(N_1) \geq \dim L(N_2)$ *and equality is achieved in the following cases only*:
  (a) $m = n = 1$.
  (b) $k = 0$, $m = 1$.
  (c) $m = 1$, $r_1 = n - 1$, $r_2 = n$.

All maximal integral manifolds of the Cartan distribution are described in the following theorem:

THEOREM 2.7. *An integral manifold* Q *of the Cartan distribution on* $J^k(\pi)$ *is maximal if and only if everywhere, possibly except for a manifold of lesser dimension, it is locally of the form* $L(N)$ *for some horizontal integral manifold* $N \subset J^{k-1}(\pi)$.

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[2]For $\beta < 0$ we set $\binom{\alpha}{\beta} = 0$.
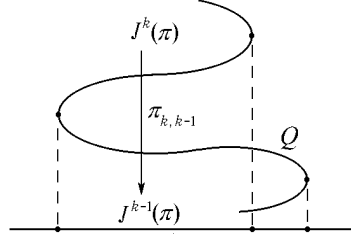
FIGURE 3.10. Regular and singular points of the projection $\pi_{k,k-1}$

PROOF. Let Q be a maximal integral manifold in $J^k(\pi)$. We shall prove the statement of the theorem for any point $\theta \in Q$ such that in its neighborhood the rank of the mapping $\pi' = \pi_{k,k-1} \colon J^k(\pi) \to J^{k-1}(\pi)$ is constant (clearly, such points form an open everywhere dense set in Q, see Figure 3.10).

In what follows, we shall need a lemma which is a consequence of the implicit function theorem and is given here without proof.

LEMMA 2.8. *Let A, B be smooth manifolds and $f \colon A \to B$ be a smooth mapping of constant rank p in a neighborhood of a point $a \in A$. Then there exists a neighborhood $\mathcal{U} \ni a$ such that the set $f(\mathcal{U})$ is a submanifold in a neighborhood of $b = f(a)$. Moreover, its tangent space at b is the image of the tangent space of A at a: $T_b(f(\mathcal{U})) = f_*(T_a(A))$.*

Let us come back to the proof of the theorem. Let $\theta \in Q$ and $N = \pi'(U)$ be the projection of the corresponding neighborhood $U \subset Q$ to $J^{k-1}(\pi)$. By Lemma 2.8, N is a submanifold in $J^{k-1}(\pi)$. Let $\theta' = \pi'(\theta)$. By the same lemma and by Theorem 2.1, one has $T_{\theta'}(N) = T_{\theta'}(\pi'(Q)) = \pi'_*(T_\theta(Q) \subset \pi'_*(\mathcal{C}_\theta) = L_\theta$. It follows that N is a horizontal manifold and $\dim N \leq n$, i.e., we can apply Proposition 2.4. The embedding $L_\theta \supset T_{\theta'}(N)$, by definition, means that $\theta \in L(N)$. Hence, $U \subset L(N)$, i.e., $Q \subset L(N)$ locally. Therefore, by maximality condition, $Q = L(N)$.

Conversely, consider a manifold of the form $L(N)$. As it was just proved, it contains in the maximal integral manifold $L(N_1)$ and $\dim L(N) \leq \dim L(N_1)$. But $N = \pi'(L(N))$, $N_1 = \pi'(L(N_1))$. Therefore, $N \subset N_1$ and $\dim N \leq \dim N_1$. By Corollary 2.6, one has $\dim L(N_1) \leq \dim L(N)$, i.e., $L(N) = L(N_1)$. $\square$

Note that if N is the graph of the $(k-1)$-jet of some section of the bundle π, then the manifold $L(N)$ is the graph of the k-jet of the same section. Hence, the graphs of jets are maximal integral manifolds of the Cartan distribution.

From the theorem proved and from Corollary 2.6 we obtain the following statement:

COROLLARY 2.9. *Except for the cases $m = n = 1$ and $k = m = 1$, the manifolds $L(N)$, where N is zero-dimensional, are of the maximal dimension among all integral manifolds of the Cartan distribution on $J^k(\pi)$. In other words, the fibers of the projection $\pi_{k,k-1}$ are integral manifolds of maximal dimension.*

This fact is of a fundamental role in description of transformations preserving the Cartan distribution. The next section deals with these transformations.

3. Lie transformations

Informally speaking, Lie transformations are transformations of independent and dependent variables, together with derivatives of the latter with respect to the former, which preserve differential relations between them. From geometric viewpoint, Lie transformations are diffeomorphisms of the jet manifolds $J^k(\pi)$ preserving the Cartan distribution, i.e., preserving the structure which contains information on differential relations between variables. Thus, Lie transformations are symmetries of the Cartan distribution. The symmetries on $J^1(M)$, i.e., contact transformations, were studied in Ch. 2. By analogy, Lie transformations of the manifold $J^k(\pi)$ are natural to be called contact transformations of order k. We shall show below that in the case of one dependent variable such transformations reduce to the usual contact transformations, while in the case of several dependent variables they reduce to point transformations, i.e., to changes of independent and dependent variables.

3.1. Finite Lie transformations. Assume that we have n independent variables $x_1, \ldots, x_n$ and m dependent variables $u^1, \ldots, u^m$. Suppose that change of variables is given by the formulas

$$\begin{cases} \overline{x}_i = f^i(\boldsymbol{x}, \boldsymbol{u}), \\ \overline{u}^j = g^j(\boldsymbol{x}, \boldsymbol{u}). \end{cases} \tag{3.1}$$

Then we can express partial derivatives $\overline{p}^j_\sigma = \partial^{|\sigma|}\overline{u}^j/\partial\overline{x}^\sigma$ via x_i, u^j, and $p^j_\tau = \partial^{|\tau|}u^j/\partial x^\tau$.

EXAMPLE 3.1. Consider the scale transformation

$$\overline{x}_i = \alpha_i x_i, \qquad \overline{u}^j = \beta_j u^j.$$

The derivatives $\overline{p}^j_i = \partial\overline{u}^j/\partial\overline{x}_i$ are transformed as

$$\overline{p}^j_i = \frac{\beta_j}{\alpha_i} p^j_i$$

in this case. One can also write down prolongations of this transformation to derivatives of arbitrary order:

$$\overline{p}^j_\sigma = \frac{\beta_j}{\alpha^\sigma} p^j_\sigma,$$

where $\alpha^{(i_1, \ldots, i_n)} = \alpha_1^{i_1} \ldots \alpha_n^{i_n}$

EXAMPLE 3.2. The translation along a constant vector in the space of dependent and independent variables is described by the formulas

$$\overline{x}_i = x_i + \xi_i, \qquad \overline{u}^j = u^j + \eta^j.$$

It acts as the identity for the variables p^j_σ,

$$\overline{p}^j_\sigma = p^j_\sigma.$$

EXAMPLE 3.3. The Galilean transformation (pass to a new frame of reference moving with a constant velocity) is given by the formulas

$$\overline{t} = t, \qquad \overline{x} = x - vt, \qquad \overline{u} = u.$$

Prolongation of this transformation to first-order derivatives is of the form

$$\overline{u}_t = u_t + vu_x, \qquad \overline{u}_x = u_x.$$

EXAMPLE 3.4. An arbitrary diffeomorphism of the plane (x, u) (i.e., change of one dependent and one independent variable)

$$\begin{cases} \overline{x} = \lambda(x, u), \\ \overline{u} = \mu(x, u) \end{cases} \tag{3.2}$$

induces a linear-fractional in p transformation of the three-dimensional space with coordinates x, u, p. It is easily checked that $\overline{p} = d\overline{u}/d\overline{x}$ and thus

$$\overline{p} = \frac{d\overline{u}}{d\overline{x}} = \frac{\mu_x dx + \mu_u du}{\lambda_x dx + \lambda_u du} = \frac{\mu_x + \mu_u p}{\lambda_x + \lambda_u p}. \tag{3.3}$$

EXERCISE 3.1. Let $m = n$, i.e., the number of independent variables coincides with that of dependent ones. Consider the change of variables

$$\overline{x}_i = u^i, \quad \overline{u}^j = x_j, \qquad i, j = 1, \dots, n = m.$$

This transformation is called the *hodograph transformation*. Derive formulas for the action of this transformation on first derivatives.

Consider now a general geometric construction corresponding to changes of variables and to prolongation of these changes to derivatives over independent variables.

Let $\pi\colon E \to M$ be a fiber bundle. Then formulas (3.1) can be interpreted as coordinate representation of a diffeomorphism in the space E. Recall that such transformations are called *point transformations*.

For any $k \geq 1$, a diffeomorphism $A\colon E \to E$ generates the diffeomorphism $A^{(k)}\colon J^k(\pi) \to J^k(\pi)$ defined in the following way. Let θ be a point of the manifold $J^k(\pi)$, b and a be its projections to E and M respectively. Choose a section φ such that $\theta = [\varphi]_a^k$ and consider its graph $\Gamma_\varphi \subset E$. Under action of the transformation A, the point b is taken to some point $b' \in E$, while the graph Γ_φ is taken to the submanifold $A(\Gamma_\varphi) \subset E$. If, in a neighborhood of the point b', this submanifold is the graph of some section φ', we shall set $\theta' = [\varphi']_{a'}^k$, where $a' = \pi(b')$ and $A^{(k)}(\theta) = \theta'$.

EXERCISE 3.2. Show that the point θ' is independent on choice of the section φ: it is uniquely determined by the k-jet of the section φ at the point a, i.e., by the point θ.

The transformation $A^{(k)}$ is called the *k-th lifting* of the point transformation A. In general, this lifting is not defined on the whole manifold $J^k(\pi)$, but (as it will be seen below) on an open everywhere dense domain in $J^k(\pi)$. This geometric fact corresponds to the following analytic observation: when one transforms derivatives, the denominator may vanish, and these are exactly the points, where the transformation is undefined (cf. equation (3.3)).

EXAMPLE 3.5. Consider the hodograph transformation of one dependent and one independent variable

$$\overline{x} = u, \qquad \overline{u} = x.$$

Let $p = du/dx$, $\overline{p} = d\overline{u}/d\overline{x}$. Then, as it follows from Exercise 3.1, $\overline{p} = 1/p$. Note that though the initial transformation A was an everywhere defined diffeomorphism of the space $\mathbb{R}^3 = J^1(\mathbb{R})$, the transformation $A^{(1)}$ is defined in the space $\mathbb{R}^3 = J^1(\mathbb{R})$ everywhere except for the plane $p = 0$.

From the above construction of the transformation $A^{(k)}$ it follows that it is a symmetry of the Cartan $\mathcal{C}^k$ distribution in its domain. In fact, the subspaces $\mathcal{C}^k_\theta$ are spans of R-planes $L_{\widetilde{\theta}}$, $\widetilde{\theta} \in J^{k+1}$, while the differential $A^{(k)}_*$ takes such planes to each other. In fact, if $A(\Gamma_\varphi) = \Gamma_{\varphi'}$, $\theta = [\varphi]^k_a$, $\widetilde{\theta} = [\varphi]^{k+1}_a$, $\widetilde{\theta}' = [\varphi']^{k+1}_{a'}$, then $A^{(k)}_{*,\theta}(L_{\widetilde{\theta}}) = L_{\widetilde{\theta}'}$.

DEFINITION 3.1. A diffeomorphism $F\colon J^k(\pi) \to J^k(\pi)$ is called a *Lie transformation*, if $F_{*,\theta}(\mathcal{C}^k_\theta) = \mathcal{C}^k_{F(\theta)}$ for any point $\theta \in J^k(\pi)$.

Since locally the Cartan distribution is described by formulas (1.2), a diffeomorphism F is a Lie transformation if and only if the equalities

$$F^*(\omega^j_\sigma) = \sum_{l=1}^m \sum_{|\tau| \le k-1} \lambda^{j,\tau}_{\sigma,l} \omega^l_\tau,$$

are locally valid for all $j = 1, \ldots, m$ and $|\sigma| \le k-1$, where $\lambda^{j,\tau}_{\sigma,l}$ are smooth functions on $J^k(\pi)$.

EXAMPLE 3.6. Let us once again obtain the formulas for lifting to $J^1(\pi)$ of the point transformation using the above formulated property of Lie transformations. We shall look for the function $\bar{p}(x, u, p)$ using the condition that the image of the distribution $du - p\,dx = 0$ under the mapping (3.2) coincides with the distribution $d\bar{u} - \bar{p}\,d\bar{x} = 0$. This condition means that when one changes all variables in the expression $d\bar{u} - \bar{p}\,d\bar{x}$ by the corresponding functions in x, u, p, one must obtain a form proportional to $du - p\,dx$. After computations, we obtain

$$d\bar{u} - \bar{p}d\,\bar{x} = \bar{u}_x\,dx + \bar{u}_u\,du - \bar{p}(\bar{x}_x\,dx + \bar{x}_u\,du)$$
$$= (\bar{u}_u - \bar{p}\bar{x}_u)\,du + (\bar{u}_x - \bar{p}\bar{x}_x)\,dx = \lambda(du - p\,dx),$$

which yields $\bar{u}_x - \bar{p}\bar{x}_x = -p(\bar{u}_u - \bar{p}\bar{x}_u)$ and consequently

$$\bar{p} = \frac{\bar{u}_x + \bar{u}_u p}{\bar{x}_x + \bar{x}_u p},$$

as it is to be by (3.3).

In a similar manner, we shall obtain the formulas for lifting to $J^1(\pi)$ of an arbitrary point transformation (3.1). Namely, the functions[3] $\bar{p}^j_i$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, are determined by m systems of linear equations (for $j = 1, \ldots, m$)

$$\begin{pmatrix} D_1(\bar{x}_1) & \ldots & D_1(\bar{x}_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ D_n(\bar{x}_1) & \ldots & D_n(\bar{x}_n) \end{pmatrix} \begin{pmatrix} \bar{p}^j_1 \\ \ldots \\ \bar{p}^j_n \end{pmatrix} = \begin{pmatrix} D_1(\bar{u}^j) \\ \ldots\ldots \\ D_n(\bar{u}^j) \end{pmatrix}, \tag{3.4}$$

where D_i denotes the *total derivative operator* along x_i, which acts on the functions $f(x_1, \ldots, x_n, u^1, \ldots, u^m)$ by the formula

$$D_i(f) = \frac{\partial f}{\partial x_i} + \sum_{j=1}^m p^j_i \frac{\partial f}{\partial u^j}.$$

The determinant of the matrix at the left-hand side of equation (3.4) is reasonable to be called the "total Jacobian" of the system of functions $\bar{x}_1, \ldots, \bar{x}_n$. Note that vanishing of this Jacobian on some open set implies functional dependence of the

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[3]In the next formula, as well as in all similar cases, we shall write p_i instead of p_{1_i}.

functions $\overline{x}_i$. Since the transformation (3.1) is a local diffeomorphism, the functions $\overline{x}_i$ are independent and thus equations (3.4) uniquely determine the values of $\overline{p}_i^j$ almost everywhere.

EXERCISE 3.3. Let $A\colon J^0(\pi) \to J^0(\pi)$ be a point transformation. Prove that if the lifting $A^{(1)}$ is defined in a neighborhood of a point $\theta \in J^1(\pi)$, then the lifting $A^{(k)}$ is defined in a neighborhood of any point θ' projecting to θ under the mapping $\pi_{k,1}\colon J^k(\pi) \to J^1(\pi)$.

The lifting procedure can be applied not only to diffeomorphisms of the manifold of dependent and independent variables $E = J^0(\pi)$, but to Lie transformations of arbitrary jet manifolds $J^k(\pi)$. In fact, let a Lie transformation $A\colon J^k(\pi) \to J^k(\pi)$ be given. As we know, the manifold $J^{k+1}(\pi)$ of contact elements of order $k+1$ can be understood as the set of all horizontal n-dimensional integral planes (R-planes) in $J^k(\pi)$. The planes which are not overturned by the transformation A (i.e., the planes L such that $A_*(L)$ is horizontal again) form an open everywhere dense set in this manifold. The lifted transformation $A^{(1)}$ acts on the set of these planes taking a plane L to $A_*(L)$.

EXAMPLE 3.7. Consider the Legendre transformation (see also §1 of Ch. 2) in the five-dimensional space $J^1(2,1)$

$$\begin{cases} \overline{x} = -p, \\ \overline{y} = -q, \\ \overline{u} = u - xp - yq, \\ \overline{p} = x, \\ \overline{q} = y. \end{cases} \tag{3.5}$$

As it was already shown in Ch. 2, this is a contact transformation and, consequently, is a Lie transformation. To describe the lifting $A^{(1)}$ of this transformation to the 8-dimensional space $J^2(2,1)$, it is necessary to express the second derivatives

$$\overline{r} = \frac{\partial^2 \overline{u}}{\partial \overline{x}^2}, \quad \overline{s} = \frac{\partial^2 \overline{u}}{\partial \overline{x}\partial \overline{y}}, \quad \overline{t} = \frac{\partial^2 \overline{u}}{\partial \overline{y}^2}$$

via x, y, u, p, q, r, s, t. Since the Cartan distribution is invariant with respect to Lie transformations, the 1-forms

$$\begin{aligned} d\overline{p} - \overline{r}\,d\overline{x} - \overline{s}\,d\overline{y} &= dx + \overline{r}\,dp + \overline{s}\,dq, \\ d\overline{q} - \overline{s}\,d\overline{x} - \overline{t}\,d\overline{y} &= dy + \overline{s}\,dp + \overline{t}\,dq \end{aligned} \tag{3.6}$$

are to be represented as linear combinations of the 1-forms

$$du - p\,dx - q\,dy, \quad dp - r\,dx - s\,dy, \quad dq - s\,dx - t\,dy.$$

Therefore, expression (3.6) identically vanishes, when one changes the form dp by $dx + s\,dy$ and dq by $s\,dx + t\,dy$. Thus, we obtain the system of equations

$$\begin{pmatrix} r & s \\ s & t \end{pmatrix} \begin{pmatrix} \overline{r} & \overline{s} \\ \overline{s} & \overline{t} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which implies

$$\overline{r} = -\frac{t}{rt - s^2}, \quad \overline{s} = \frac{s}{rt - s^2}, \quad \overline{t} = -\frac{r}{rt - s^2}. \tag{3.7}$$

EXERCISE 3.4. Let $A\colon J^k(\pi) \to J^k(\pi)$ be a Lie transformation. Prove that at the points where the lifting $A^{(l+s)}\colon J^{k+l+s}(\pi) \to J^{k+l+s}(\pi)$, $l, s \in \mathbb{N}$, is defined one has $(A^{(l)})^{(s)} = A^{(l+s)}$.

We finish this section by a complete description of Lie transformations.

THEOREM 3.1. *Any Lie transformation X of the jet manifold $J^k(\pi)$, $k \geq 1$, is described as follows*:

1. *For $\dim \pi = 1$, the transformation X is the $(k-1)$-st lifting of some (arbitrary) contact transformation of the space $J^1(\pi)$.*
2. *For $m = \dim \pi > 1$, the transformation X is the k-th lifting of some (arbitrary) diffeomorphism of the space $J^0(\pi)$ of dependent and independent variables (i.e., all Lie transformations are point transformations in this case).*

REMARK 3.1. From the theorem it follows that in the case $m = 1$ contact geometry of order k is richer, since contact transformations of $J^1(\pi)$ include both arbitrary diffeomorphisms of $J^0(\pi)$ and transformations of a more general nature. The Legendre transformation (3.5) is an example of a transformation, which is contact, but not a point one.

PROOF. The proof of this theorem is based on the above established fact: the fibers of the projection $\pi_{k,k-1}$ are integral manifolds of the Cartan distribution on $J^k(\pi)$ of maximal dimension (Corollary 2.9). The case $m = n = 1$ needs a special treatment.

Assume that $mn > 1$. Let $A\colon J^k(\pi) \to J^k(\pi)$ be a Lie transformation and $k \geq 2$, if $m = 1$, and $k \geq 1$, if $m > 1$. Since the transformation A preserves the Cartan distribution, it must take its maximal integral manifolds to maximal integral manifolds. Since A is a diffeomorphism, dimension of these manifolds is also preserved. Consequently, A takes a fiber of the projection $\pi_{k,k-1}$ to another fiber. Hence, A induces a transformation A_1 of the manifold $J^{k-1}(\pi)$: if $A(\pi_{k,k-1}^{-1}(\theta)) = \pi_{k,k-1}^{-1}(\eta)$, then $A_1(\theta) = \eta$.

Since A_1 is the projection of the Lie transformation A, the transformation A_1 itself is also a Lie transformation. Let us consider the transformation A and the lifting $(A_1)^{(1)}$ of the transformation A_1. They both are symmetries of the Cartan distribution $\mathcal{C}^k$, preserve the fibers of the projection $\pi_{k,k-1}\colon J^k(\pi) \to J^{k-1}(\pi)$, and induce the same transformation of $J^{k-1}(\pi)$. Therefore, the diffeomorphism $A' = A^{(1)} \circ A^{-1}$ is a Lie transformation of $J^k(\pi)$ inducing the identical mapping of $J^{k-1}(\pi)$.

EXERCISE 3.5. Prove that Lie transformation of the space $J^k(\pi)$ projecting to the identical mapping of $J^{k-1}(\pi)$ is identical itself.

Consequently $A = A_1^{(1)}$. One can apply the same reasoning to the transformation A_1 and continue in the same way till we obtain some transformation of the space $J^1(\pi)$ (in the case $m = 1$) or of $J^0(\pi)$ (for $m > 1$). To conclude the proof, it suffices to use Exercise 3.4.

In the case $m = n = 1$, the proof of the theorem is implied by the following considerations. Let x, p_0, $p_1, \ldots, p_k$ be canonical coordinates on $J^k(1,1)$. The Cartan distribution $\mathcal{C}$ on $J^k(1,1)$ is two-dimensional: it is generated by the vector fields $Z = \partial/\partial p_k$ and $D = \partial/\partial x + \sum_{i=0}^{k-1} p_{i+1}\partial/\partial p_i$. It suffices to prove that any Lie transformation A takes the field Z to itself, i.e., is a symmetry of the

one-dimensional distribution V spanned by the field Z; all further reasoning will be similar to the above used.

LEMMA 3.2. *Let $\mathcal{C}' = [\mathcal{C}, \mathcal{C}]$ be the distribution on the manifold $J^k(1,1)$, $k \geq 2$, spanned by the commutators of the vector fields lying in the Cartan distribution $\mathcal{C}$. Then, if $\mathrm{D}(\mathcal{P})$ denotes the set of all vector fields lying in some distribution $\mathcal{P}$, and $\mathcal{P}_D$ denotes the set of fields X, such that $[X, \mathrm{D}(\mathcal{P})] \subset \mathrm{D}(\mathcal{P})$, one has*

$$\mathrm{D}(V) = \mathrm{D}(\mathcal{C}) \cap \mathcal{C}'_{\mathrm{D}}.$$

PROOF OF THE LEMMA. Obviously, the distribution $\mathcal{C}'$ is three-dimensional. It is spanned by the fields Z, D, and $[Z, D] = \partial/\partial p_{k-1} = Y$. It is easily seen that $[Z, Y] = 0$, $[Y, D] = X = \partial/\partial p_{k-2}$. Let the field $S = \alpha Z + \beta D$ be a symmetry of the distribution $\mathcal{C}'$. Then $[Y, S] \in \mathrm{D}(\mathcal{C}')$, i.e. $[Y, S]$ is represented as a linear combination of the fields Z, D, Y. But since

$$[Y, S] = [Y, \alpha Z + \beta D] = Y(\alpha)Z + Y(\beta)D + \beta X$$

and the fields X, D, Z are linear independent at all points of the manifold J^k, we obtain $\beta = 0$. Consequently, $S \in \mathrm{D}(V)$ and this completes the proof of the lemma. $\square$

Let now $A \colon J^k(1,1) \to J^k(1,1)$. Then, obviously, $A_* \mathrm{D}(\mathcal{C}) = \mathrm{D}(\mathcal{C})$, $A_* \mathcal{C}' = \mathcal{C}'$, and $A_* \mathcal{C}'_{\mathrm{D}} = \mathcal{C}'_{\mathrm{D}}$. Then, by the lemma proved, one has $A_* \mathrm{D}(V) = \mathrm{D}(V)$. The theorem is proved. $\square$

3.2. Lie fields.

DEFINITION 3.2. A vector field X on the manifold $J^k(\pi)$ is called a *Lie field*, if shifts along its trajectories are Lie transformations.

Let X be a Lie field on $J^k(\pi)$ and $\{A_t\}$ be the corresponding one-parameter group of transformations of the manifold $J^k(\pi)$. By definition, A_t is a Lie transformation and consequently its liftings $A_t^{(l)}$ to the manifolds $J^{k+l}(\pi)$ are defined and are also Lie transformations. The field $X^{(l)}$ corresponding to the one-parameter group $\{A_t^{(l)}\}$ is called the *lifting* of the field X. Similar to the case of liftings of Lie transformations, for any natural numbers l and s the equality $(X^{(l)})^{(s)} = X^{(l+s)}$ holds.

THEOREM 3.3. *Any Lie field $J^k(\pi)$, $k \geq 1$, is of the form*:
(a) $X^{(k)}$ *for $\dim \pi > 1$, where X is a vector field on the space $J^0(\pi)$.*
(b) $X^{(k-1)}$ *for $\dim \pi = 1$, where X is a contact vector field on the space $J^1(\pi)$.*

Note that by the already discussed properties of Lie transformations, any vector field on $J^0(\pi)$ and any contact vector field on $J^1(\pi)$ can be lifted to a Lie field on $J^k(\pi)$.

Infinitesimal point of view has two important advantages. The first is that the lifting of a Lie field, unlike that of finite Lie transformation, is always defined on the whole manifold. In fact, to compute the vector of the field $X^{(1)}$ at some point $\theta \in J^{k+1}(\pi)$, it needs to know an arbitrary small part of the trajectory passing through this point. The R-plane L_θ in $J^k(\pi)$ corresponds to the point θ and this plane projects to the base M nondegenerately. It remains to note that for sufficiently small transformations the image of this plane will still project to the base nondegenerately and thus define a point in $J^{k+1}(\pi)$.

The second advantage is that there exist explicit computational formulas expressing the components of the vector field $X^{(k)}$ via the components of the field X. In addition, one obtains a possibility to use the techniques of generating functions (see below).

Let us consider a bundle $\pi\colon E \to M$ and canonical coordinates in a neighborhood of a point $\theta \in J^k(\pi)$. Note that a field X on $J^k(\pi)$ is a Lie field if and only if for any Cartan form $\omega_\sigma^j = dp_\sigma^j - \sum_{i=1}^n p_{\sigma+1_i}^j \, dx_i$ on $J^k(\pi)$ one locally has

$$X(\omega_\sigma^j) = \sum_{l=1}^m \sum_{|\tau| \le k-1} \lambda_{\sigma,l}^{j,\tau} \omega_\tau^l.$$

THEOREM 3.4. *If*

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^m \sum_{0 \le |\sigma| \le k} b_\sigma^j \frac{\partial}{\partial p_\sigma^j}, \tag{3.8}$$

is a Lie field, then the coefficients b_σ^j are computed by the recurrent formulas

$$b_{\sigma+1_i}^j = D_i(b_\sigma^j) - \sum_{s=1}^n p_{\sigma+1_s}^j D_i(a_s), \tag{3.9}$$

where $0 \le |\sigma| \le k-1$ and

$$D_i = \frac{\partial}{\partial x_i} + \sum_\sigma p_{\sigma+1_i}^j \frac{\partial}{\partial p_\sigma^j}$$

is the total derivative operator with respect to x_i.

PROOF. Let X be a Lie field of the form (3.8). Then, by invariance of the Cartan distribution, the forms

$$X(\omega_\sigma^j) = db_\sigma^j - \sum_{i=1}^n (b_{\sigma+1_i}^j \, dx_i + p_{\sigma+1_i}^j \, da_i) \tag{3.10}$$

are linear combinations of the Cartan forms.

EXERCISE 3.6. Show that any 1-form on $J^k(\pi)$ is uniquely represented as

$$\omega = \sum_{i=1}^n \varphi_i \, dx_i + \omega_{\mathcal{C}}, \tag{3.11}$$

where φ_i are functions on $J^{k+1}(\pi)$ while $\omega_{\mathcal{C}}$ is a linear combination of Cartan forms.

In particular, for any smooth function φ on $J^k(\pi)$ the equality

$$d\varphi = \sum_i D_i(\varphi) \, dx_i + \sum_{j,\sigma} \frac{\partial \varphi}{\partial p_\sigma^j} \omega_\sigma^j \tag{3.12}$$

holds.

Note that the form ω represented by equality (3.11) vanishes on the Cartan distribution if and only if the coefficients φ_i are trivial. Applying this observation to equality (3.10), we obtain

$$X(\omega_\sigma^j) = \sum_i (D_i(b_\sigma^j) - b_{\sigma+1_i}^j - \sum_s p_{\sigma+1_i}^j D_i(a_s)) \, dx_i + \omega_{\mathcal{C}},$$

which implies (3.9). □

EXERCISE 3.7. Check that equalities (3.9) comply with the representation of contact fields on $J^1(M)$ in the form

$$X_f = -\sum_i \frac{\partial f}{\partial p_i} \frac{\partial}{\partial x_i} + \left(f - \sum_i p_i \frac{\partial f}{\partial p_i} \right) \frac{\partial}{\partial u} + \sum_i \left(\frac{\partial f}{\partial x_i} + p_i \frac{\partial f}{\partial u} \right) \frac{\partial}{\partial p_i}.$$

Let us now introduce the concept of the *generating section* of a Lie field. To do this, consider evolution of the graphs of sections of the bundle π under the action of the corresponding one-parameter group of transformations.

Let $X = \sum_i a_i \partial/\partial x_i + \sum_j b^j \partial/\partial u^j + \cdots$ be a Lie field on $J^k(\pi)$ and $\{A_t\}$ be the corresponding one-parameter group of local transformations of the manifold $J^k(\pi)$. Consider a section $\boldsymbol{f}$ of the bundle π. The graph of its k-jet $\Gamma^k_{\boldsymbol{f}} \subset J^k(\pi)$ is an n-dimensional integral manifold of the Cartan distribution on $J^k(\pi)$ nondegenerately projecting to the base M. As it was already pointed out, for sufficiently small t the manifold $A_t(\Gamma^k_{\boldsymbol{f}})$ is also the graph of the k-jet for some other section $\boldsymbol{f}_t$: $A_t(\Gamma^k_{\boldsymbol{f}}) = \Gamma^k_{\boldsymbol{f}_t}$.

Thus the transformations A_t generate evolution $\boldsymbol{f}_t$ of the section $\boldsymbol{f}$. Let us find the velocity $d/dt|_{t=0} \boldsymbol{f}_t$ of this evolution at the starting moment. To compute this derivative, let us represent the field X as the sum of two fields, one of them being vertical and the other one tangent to the manifold $\Gamma^k_{\boldsymbol{f}}$.

If the section $\boldsymbol{f}$ is locally presented by the functions $u^j = f^j(x_1, \ldots, x_n)$, $j = 1, \ldots, m$, then the vector fields tangent to the manifold $\Gamma^k_{\boldsymbol{f}}$ can be written in the following form (note that on the surface $\Gamma^k_{\boldsymbol{f}}$ the equalities $p^j_\sigma = \partial^{|\sigma|} f^j / \partial x^\sigma$, $|\sigma| \leq k$, hold):

$$D_i^{(\boldsymbol{f})} = \frac{\partial}{\partial x_i} + \sum_{j=1}^m \left(\sum_{|\sigma|<k} p^j_{\sigma+1_i} \frac{\partial}{\partial p^j_\sigma} + \sum_{|\sigma|=k} \frac{\partial^{|\sigma|+1} f^j}{\partial x^\sigma \partial x_i} \frac{\partial}{\partial p^j_\sigma} \right).$$

Therefore, a field tangent to the graph $\Gamma^k_{\boldsymbol{f}}$ is of the form

$$X_2 = \sum_{i=1}^n a_i D_i^{(\boldsymbol{f})},$$

while the vertical component of the Lie field X equals

$$X_1 = X - \sum_{i=1}^n a_i D_i^{(\boldsymbol{f})} = \sum_j \left(b^j - \sum_i a_i p_i^j \right) \frac{\partial}{\partial u^j} + \cdots$$

Since the field X_2 shifts the manifold $\Gamma^k_{\boldsymbol{f}}$ along itself, it does not influence the evolution. On the other hand, the coefficients of the field X_1 at $\partial/\partial u^j$ exactly coincide with the velocities of the components $\boldsymbol{f}_t = (f_t^1, \ldots, f_t^m)$:

$$\left. \frac{d}{dt} \right|_{t=0} f_t^j = \left. \left(b^j - \sum_{i=1}^n a_i p_i^j \right) \right|_{\Gamma^k_{\boldsymbol{f}}}. \tag{3.13}$$

The vector function $\varphi = (\varphi^1, \ldots, \varphi^m)$, where

$$\varphi^j = b^j - \sum_{i=1}^n a_i p_i^j, \tag{3.14}$$

is called the *generating section* (or, in the case $m = 1$, the *generating function*) of the Lie field

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} b^j \frac{\partial}{\partial u^j} + \dots \qquad (3.15)$$

EXERCISE 3.8. Show that in the case of the trivial bundle $\pi \colon \mathbb{R} \times M \to M$, the above definition of generating function coincides with that for a contact field given in Ch. 2.

From this exercise and from Theorem 3.3 we obtain the following

PROPOSITION 3.5. *Any Lie field X on $J^k(\pi)$ is uniquely determined by its generating section $\varphi = (\varphi^1, \dots, \varphi^m)$. The components of the generating section can be computed using the formulas*

$$\varphi^j = X \lrcorner \, \omega^j_{(0,\dots,0)},$$

where $\omega^j_{(0,\dots,0)} = du^j - \sum_{i=1}^{n} p_i^j \, dx_i$.

Note that in the case $m > 1$ the generating section $\varphi = (\varphi^1, \dots, \varphi^m)$ linearly depends on the variables p_i^j and for any i, j, l the equalities

$$\frac{\partial \varphi^j}{\partial p_i^j} = \frac{\partial \varphi^l}{\partial p_i^l}, \qquad \frac{\partial \varphi^j}{\partial p_i^l} = 0, \qquad l \neq j,$$

hold.

Arbitrary generating sections lead to the theory of higher symmetries which will be considered in Ch. 4.

4. Classical symmetries of equations

4.1. Defining equations. Let $\mathcal{E} \subset J^k(\pi)$ be an equation of order k.

DEFINITION 4.1. A Lie transformation $A \colon J^k(\pi) \to J^k(\pi)$ such that $A(\mathcal{E}) \subset \mathcal{E}$ is called a *classical (finite) symmetry* of the equation $\mathcal{E} \subset J^k(\pi)$.

DEFINITION 4.2. A Lie field is called a *classical infinitesimal symmetry* of the equation $\mathcal{E} \subset J^k(\pi)$, if it is tangent to $\mathcal{E}$.

Obviously, these definitions are natural generalizations of the basic constructions considered in Ch. 1 and 2.

A direct consequence of the definitions and considerations of §§1–3 is the following[4]

PROPOSITION 4.1.   1. *Let $A \colon J^k(\pi) \to J^k(\pi)$ be a symmetry of the equation $\mathcal{E} \subset J^k(\pi)$ and f be a solution of this equation, i.e., a section of the bundle π such that its graph Γ_f^k lies in $\mathcal{E}$. Then $A(\Gamma_f^k)$ is a generalized solution of the equation $\mathcal{E}$. In particular, if the manifold $A(\Gamma_f^k)$ is of the form $\Gamma_{f'}^k$, for some section $f' = A^* f$, then f' is a solution of $\mathcal{E}$ as well.*

---

[4]Everywhere below in this chapter we use the term "symmetry" in the sense of a classical symmetry.

2. *If X is an infinitesimal symmetry of the equation $\mathcal{E}$ and f is its solution,
then for any point $\theta \in \Gamma_f^k$ there exists a neighborhood $\mathcal{U} \ni \theta$ and $\varepsilon > 0$ such
that for any $t \in [-\varepsilon, \varepsilon]$ the manifold $A_t(\mathcal{U})$, where A_t is the one-parameter
group of transformations corresponding to X, is locally of the $\Gamma_{A_t^*(f)}^k$, i.e.,
X determines a flow on the set of solutions of the equation $\mathcal{E}$.*

From practical viewpoint, finite symmetries are by all means preferable, but it
is not simple to find them. There are no general algorithms to search for them,
and they can be found accidentally only or using a sort of "physical reasons".
On the contrary, a search for infinitesimal symmetries is governed by a certain
algorithm leading, from technical point of view, to solving (usually, overdetermined)
systems of linear differential equations. The reason is that Lie fields are efficiently
described by the corresponding generating functions. That is why we work below
with infinitesimal symmetries only and often use the word "symmetry" meaning the
generating function of an infinitesimal symmetry. The latter identification does not
cause ambiguities, since the correspondence between Lie fields and their generating
functions is one-to-one.

Let φ and ψ be symmetries of the equation $\mathcal{E} \subset J^k(\pi)$, i.e., the Lie fields $X_\varphi^{(s)}$
and $X_\psi^{(s)}$ are tangent to the manifold $\mathcal{E}$ ($s = k$ for $m > 1$ and $s = k-1$ for $m = 1$).
Then obviously the commutator $[X_\varphi^{(s)}, X_\psi^{(s)}]$ is also a symmetry of the equation $\mathcal{E}$
and consequently is of the form $X_\rho^{(s)}$ for some generating section ρ. This section
is called the *Jacobi bracket* of the generating sections φ and ψ and is denoted by
$\{\varphi, \psi\}$.

Evidently, the set of symmetries forms a Lie $\mathbb{R}$-algebra with respect to the
Jacobi bracket.

If $x_1, \ldots, x_n$, $u^1, \ldots, u^m, \ldots, p_\sigma^j, \ldots$ are canonical coordinates in $J^k(\pi)$, then
from the definition of the commutator of vector fields and from equations (3.9) it
follows that the j-component of the Jacobi bracket is of the form

$$
\begin{aligned}
\{\varphi, \psi\}^j = {} & \sum_{\alpha=1}^m \left(\varphi^\alpha \frac{\partial \psi^j}{\partial u^\alpha} - \psi^\alpha \frac{\partial \varphi^j}{\partial u^\alpha} \right) \\
& + \sum_{i=1}^n \sum_{\alpha=1}^m \left(\left(\frac{\partial \varphi^\alpha}{\partial x_i} + \sum_{\beta=1}^m p_i^\beta \frac{\partial \varphi^\alpha}{\partial u^\beta} \right) \frac{\partial \psi^j}{\partial p_i^\alpha} - \left(\frac{\partial \psi^\alpha}{\partial x_i} + \sum_{\beta=1}^m p_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta} \right) \frac{\partial \varphi^j}{\partial p_i^\alpha} \right).
\end{aligned}
\tag{4.1}
$$

Let us deduce now local conditions for a Lie field X to be a symmetry of the
equation $\mathcal{E} \subset J^k(\pi)$. To do this, let us choose canonical coordinates in $J^k(\pi)$ and
assume that the submanifold $\mathcal{E}$ in these coordinates is described by the relations

$$
F^\alpha = 0, \qquad \alpha = 1, \ldots, r,
\tag{4.2}
$$

where F^α are smooth functions on $J^k(\pi)$. Suppose that the system (4.2) is locally
of maximal rank. Then the condition that a field X is tangent to the manifold
$\mathcal{E} = \{\boldsymbol{F} = 0\}$, where $\boldsymbol{F} = (F^1, \ldots, F^r)$, is of the form

$$
X(F^\alpha) = \sum_{\beta=1}^r \lambda_\beta^\alpha F^\beta, \qquad \alpha = 1, \ldots, r,
\tag{4.3}
$$

for some smooth functions λ_β^α, or equivalently

$$
X(F^\alpha)|_{\mathcal{E}} = 0, \qquad \alpha = 1, \ldots, r.
\tag{4.4}
$$

Let us represent the field X in the form $X_\varphi^{(s)}$, where $s = k$ or $s = k-1$, depending on dimension of the bundle π, and express the coefficients of the lifting of the field X_φ via generating functions using equations (3.9). Then from the relations (4.3) or from (4.4) we shall obtain a system of equations on φ called *defining equations*.

EXAMPLE 4.1. Consider in $J^2(2,1)$ a general second order equation in two independent and one dependent variable:

$$F(x_1, x_2, u, p_1, p_2, p_{(2,0)}, p_{(1,1)}, p_{(0,2)}) = 0.$$

Let

$$X_\varphi = -\sum_{i=1}^n \varphi_{p_i} \frac{\partial}{\partial x_i} + \left(\varphi - \sum_{i=1}^n p_i \varphi_{p_i}\right)\frac{\partial}{\partial u} + \sum_{i=1}^n (\varphi_{x_i} + p_i \varphi_u)\frac{\partial}{\partial p_i}, \qquad (4.5)$$

be a contact vector field in $J^1(2,1)$. Then the coefficients $b_{(i,j)}$ at $\partial/\partial p_{(i,j)}$, $i+j=2$, of the lifting of the field to $J^2(2,1)$ are of the following form:

$$b_{(2,0)} = p_{(2,0)}\varphi_u + \varphi_{x_1 x_1} + 2p_1\varphi_{x_1 u} + 2p_{(2,0)}\varphi_{x_1 p_1} + 2p_{(1,1)}\varphi_{x_1 p_2} + p_1^2 \varphi_{uu}$$
$$+ 2p_1 p_{(2,0)}\varphi_{up_1} + 2p_1 p_{(1,1)}\varphi_{up_2} + p_{(2,0)}^2\varphi_{p_1 p_1} + 2p_{(2,0)}p_{(1,1)}\varphi_{p_1 p_2} + p_{(1,1)}^2\varphi_{p_2 p_2},$$

$$b_{(1,1)} = p_{(1,1)}\varphi_u + \varphi_{x_1 x_2} + p_2\varphi_{x_1 u} + p_1\varphi_{x_2 u} + p_{(1,1)}\varphi_{x_1 p_1} + p_{(0,2)}\varphi_{x_1 p_2} + p_{(2,0)}\varphi_{x_2 p_1}$$
$$+ p_{(1,1)}\varphi_{x_2 p_2} + p_1 p_2\varphi_{uu} + (p_1 p_{(1,1)} + p_2 p_{(2,0)})\varphi_{up_1} + (p_1 p_{(0,2)} + p_2 p_{(1,1)})\varphi_{up_2}$$
$$+ p_{(2,0)}p_{(1,1)}\varphi_{p_1 p_1} + (p_{(2,0)}p_{(0,2)} + p_{(1,1)}^2)\varphi_{p_1 p_2} + p_{(1,1)}p_{(0,2)}\varphi_{p_2 p_2},$$

$$b_{(0,2)} = p_{(0,2)}\varphi_u + \varphi_{x_2 x_2} + 2p_2\varphi_{x_2 u} + 2p_{(1,1)}\varphi_{x_2 p_1} + 2p_{(0,2)}\varphi_{x_2 p_2} + p_2^2\varphi_{uu}$$
$$+ 2p_2 p_{(1,1)}\varphi_{up_1} + 2p_2 p_{(0,2)}\varphi_{up_2} + p_{(1,1)}^2\varphi_{p_1 p_1} + 2p_{(1,1)}p_{(0,2)}\varphi_{p_1 p_2} + p_{(0,2)}^2\varphi_{p_2 p_2}.$$

Then the equation $X_\varphi^{(1)}F = \lambda F$, where the coefficient of the field $X_\varphi^{(1)}$ are computed by the above formulas, is the defining equation in the situation under consideration.

EXERCISE 4.1. Deduce similar formulas for a system of two second order equations in two dependent variables.

4.2. Invariant solutions and reproduction of solutions. Let X be an infinitesimal symmetry of the equation $\mathcal{E} \subset J^k(\pi)$, $\boldsymbol{f}_0$ be its solution, $\Gamma_{\boldsymbol{f}_0}^k = j_k(\boldsymbol{f}_0)(M) \subset \mathcal{E}$. Let, further, $\{A_t\}$ be a one-parameter group of diffeomorphisms corresponding to the field X. Then the submanifold $A_t(\Gamma_{\boldsymbol{f}_0}^k) \subset J^k(\pi)$ is an integral manifold of the Cartan distribution. In the case when $A_t(\Gamma_{\boldsymbol{f}_0}^k)$ and $\Gamma_{\boldsymbol{f}_0}^k$ are horizontal with respect to the projection to the base M, then $A_t(\Gamma_{\boldsymbol{f}_0}^k) = \Gamma_{\boldsymbol{f}_t}^k$ for some section $\boldsymbol{f}_t$. In addition, by the definition of a symmetry, $A_t(\Gamma_{\boldsymbol{f}_0}^k) \subset \mathcal{E}$. Thus, at least locally, $\boldsymbol{f}_t$ is a one-parameter family of solutions of the equation $\mathcal{E}$: $j_k(\boldsymbol{f}_t)(M) = \Gamma_{\boldsymbol{f}_t}^k \subset \mathcal{E}$. A passage from the initial solution $\boldsymbol{f}$ to the family $\boldsymbol{f}_t$ is called *reproduction* of the solution $\boldsymbol{f}$ by means of the symmetry X.

If $\boldsymbol{\varphi}$ is the generating function of the field $X = X_\varphi$, then search for the family $\boldsymbol{f}_t$ reduces to solving the system of equations

$$\frac{\partial \boldsymbol{f}}{\partial t}(\boldsymbol{x}, t) = \varphi|_{\Gamma_{\boldsymbol{f}}^k}(\boldsymbol{x}, t), \qquad \boldsymbol{f}(\boldsymbol{x}, 0) = \boldsymbol{f}_0(\boldsymbol{x}), \qquad (4.6)$$

or, component-wise,

$$\frac{\partial f^j}{\partial t}(\boldsymbol{x}, t) = \varphi^j|_{\Gamma_{\boldsymbol{f}}^k}(\boldsymbol{x}, t), \quad f^j(\boldsymbol{x}, 0) = f_0^j(\boldsymbol{x}), \qquad j = 1, \ldots, m,$$

as it follows from (3.13).

If $\varphi|_{\Gamma_{\boldsymbol{f}}^k}(\boldsymbol{x}, t) = 0$, then $\boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}, 0)$ for all admissible values of t, i.e., $\boldsymbol{f}_0$ is a fixed point for the one-parameter group of transformations $\{A_t\}$ when it acts on sections. In other words, the manifold $\Gamma_{\boldsymbol{f}_0}^k$ is invariant under the field X_φ.

DEFINITION 4.3. If $\Gamma_{\boldsymbol{f}_0}^k$ is invariant with respect to the field X_φ, then $\boldsymbol{f}_0$ is called an φ-*invariant solution* of the equation $\mathcal{E}$.

The above introduced concepts can be generalized as follows. Let G be a Lie group whose Lie algebra $\mathfrak{g}$ is realized as a subalgebra in the Lie algebra of infinitesimal symmetries (or, equivalently, in the Lie algebra of generating functions of symmetries) of the equation $\mathcal{E}$. Then, starting from an arbitrary solution $\boldsymbol{f}_0$, we shall obtain a $(\dim \mathfrak{g})$-parameter family of solutions $\{\,\boldsymbol{f}_g \mid g(\Gamma_{\boldsymbol{f}_0}^k) = \Gamma_{\boldsymbol{f}_g}^k, g \in G\,\}$ of the equation $\mathcal{E}$ (provided the manifolds $g(\Gamma_{\boldsymbol{f}_0}^k)$ are horizontal with respect to the projection to the base M).

DEFINITION 4.4. If $\boldsymbol{f}_g = \boldsymbol{f}_0$ for all $g \in G$, then $\boldsymbol{f}_0$ is called a G-*invariant solution* of the equation $\mathcal{E}$ (or $\mathfrak{g}$-*invariant*, if we consider the corresponding Lie algebra).

Let $\varphi_1, \ldots, \varphi_s$ be generating sections and assume that they are generators of the Lie algebra $\mathfrak{g}$. Then to find $\mathfrak{g}$-solutions means to solve the overdetermined system of differential equations

$$\begin{cases} F_1(\theta) = 0, \\ \cdots\cdots\cdots \\ F_r(\theta) = 0, \\ \varphi_1(\theta) = 0, \\ \cdots\cdots\cdots \\ \varphi_s(\theta) = 0, \end{cases} \tag{4.7}$$

$\theta \in J^k(\pi)$, where $F_1, \ldots, F_r$ are the functions determining the equation $\mathcal{E}$. In particular, X_φ-invariant solutions can be found from the system (4.7), where the only additional equation to $\mathcal{E}$ is $\varphi = 0$.

Let us stress one important moment. If an equation possesses a classical symmetry X, one can diminish the number of independent variables by 1. In fact, consider, for simplicity, the case $\dim \pi = 1$. Then the equations $\varphi_1 = 0, \ldots, \varphi_s = 0$ can be considered as a system of equations on unknowns $p_1, \ldots, p_n$. It is natural to consider the case when this system is of maximal rank. Then, without loss of generality, one may assume that it is of the form

$$\begin{cases} p_{n-s+1} = \widetilde{\varphi}_1(\boldsymbol{x}, u, p_1, \ldots, p_{n-s}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_n = \widetilde{\varphi}_s(\boldsymbol{x}, u, p_1, \ldots, p_{n-s}). \end{cases}$$

Substituting these equalities to the initial equation, we shall obtain an equation, which does not contain the derivatives with respect to the last s variables. Obviously, in the case of several dependent variables the number of dependent ones is diminished in the same manner.

When an equation possesses two symmetries, the number of independent variables diminishes by 2, etc. In particular, if $\dim \mathfrak{g} = n - 1$, where n is the number

of independent variables, invariant solutions can be found by solving ordinary differential equations and for $\dim \mathfrak{g} = n$ the equations become "algebraic", i.e. do not contain derivatives. Some examples will be discussed below.

We shall finish this section with some remarks.

1. In general, even for small t, one can not expect the manifolds $A_t(\Gamma_{\boldsymbol{f}_0}^k)$ to be horizontal. Therefore, when reproducing a regular solution, we shall obtain general integral $\dim M$-dimensional manifolds of the Cartan distribution, i.e., "generalized" solutions, or solutions with "singularities". At some points, a tangent plane to such a solution degenerately projects to the base (and this corresponds to the fact that some derivatives of $\boldsymbol{f}_t(\boldsymbol{x})$ become infinite); therefore a solution may become multivalued. Such generalized solutions are usual in analysis of discontinuity propagation, shock waves, in catastrophe theory, etc. Physical meaning of these solutions is determined by the appropriate context.

2. For many nonlinear equations it is often convenient to find an invariant solution with respect to some symmetry and then to reproduce it by means of other symmetries. Sometimes this is the only way to find explicit formulas.

3. The concept of invariant solutions includes, as a particular case, the notion of a *self-similar solution* : a solution is called self-similar, if it is invariant with respect to the scale symmetries, i.e., symmetries of the form

$$X = \sum_i \alpha_i x_i \frac{\partial}{\partial x_i} + \sum_j \beta_j u^j \frac{\partial}{\partial u^j} \ .$$

5. Examples of computations

Many examples of symmetry computations, of constructing invariant solutions, and of other applications can be found in [**91, 114, 137, 44, 45, 46**].

5.1. The Burgers equation. The Burgers equation is of the form

$$u_t = u u_x + u_{xx} \ . \tag{5.1}$$

This equation describes the motion of weakly nonlinear waves in gases, when dissipative effects are sufficiently small to be considered in the first approximation only. When dissipation tends to zero, this equation gives an adequate description of waves in nonviscous medium. Initially proposed by Burgers to describe one-dimensional turbulence, this equation was later used to study other wave phenomena. The Burgers equation is linearized by the substitution $u = y_x/y$. This fact indicates existence of a large symmetry algebra.

Rewriting equation (5.1) in canonical coordinates

$$x_1 = x, \ x_2 = t, \ p_0 = u, \ p_1 = u_x, \ p_2 = u_t, \ p_{(2,0)} = u_{xx}$$

on $J^2(2,1)$, we obtain

$$p_2 = p_0 p_1 + p_{(2,0)}.$$

Then the defining equations for symmetries obtained in Example 4.1 acquire the form

$$(\varphi - p_1 \varphi_{p_1} - p_2 \varphi_{p_2}) p_1 + p_0 (\varphi_{x_1} + p_1 \varphi_{p_0} + p_{(2,0)} \varphi_{p_0} + \varphi_{x_1 x_1} + 2 p_1 \varphi_{x_1 p_0} + 2 p_{(2,0)} \varphi_{x_1 p_1}$$

$$+ 2 p_{(1,1)} \varphi_{x_1 p_2} + p_1^2 \varphi_{p_0 p_0} + 2 p_1 p_{(2,0)} \varphi_{p_0 p_1} + 2 p_1 p_{(1,1)} \varphi_{p_0 p_2} + p_{(2,0)}^2 \varphi_{p_1 p_1}$$

$$+ 2 p_{(2,0)} p_{(1,1)} \varphi_{p_1 p_2} + p_{(1,1)}^2 \varphi_{p_2 p_2} - \varphi_{x_2} - p_2 \varphi_{p_0} = \lambda (p_0 p_1 + p_{(2,0)} - p_2), \quad (5.2)$$

where λ is a smooth function on $J^2(2,1)$. Analysis of this equation shows that the function φ is to be of the form

$$\varphi = A(x_1, x_2, p_0)p_1 + B(x_1, x_2, p_0)p_2 + C(x_1, x_2, p_0).$$

From here, using (5.2), we see that the functions A, B, and C satisfy the equation

$$p_1 p_2 A_{p_0} + p_1 A_{x_2} + B_{x_2} p_2 + C_{x_2} + C_{p_0} p_2 - p_0 p_1 (A_{x_1} + p_1 A_{p_0}) - p_0 (C_{x_1} + p_1 C_{p_0})$$
$$- p_1 C - 2 D_x(A) p_{(2,0)} - D_x(A_{x_1} + A_{p_0}) p_1 - D_x(C_{x_1} + C_{p_0}) = 0, \quad (5.3)$$

where D_x is the total derivative over $x = x_1$.

The last equation is polynomial in p_1 and $p_{(2,0)}$. Therefore, the coefficient at $p_{(2,0)} = -4 A_{p_0} p_1 + B_{x_2} - 2 A_{x_1}$, has to vanish. In turn, this coefficient is polynomial in p_1; hence, the coefficients of this polynomial are also trivial[5].

Thus, $A_{p_0} = 0$ and $B_{x_2} - 2 A_{x_1} = 0$. Substituting these equalities to (5.3), we obtain

$$p_1 A_{x_2} + p_0 p_1 A_{x_1} + C_{x_2} - p_0 C_{x_1} - p_1 C - p_1 A_{x_1 x_1}$$
$$- C_{x_1 x_1} - 2 p_1 C_{x_1 p_0} - p_1^2 C_{p_0 p_0} = 0 \quad (5.4)$$

The last equality is quadratic in p_1, from where it follows that

$$C_{p_0 p_0} = 0, \quad A_{x_2} + p_0 A_{x_1} - C - A_{x_1 x_1} - 2 C_{x_1 p_0} = 0,$$
$$C_{x_2} - p_0 C_{x_1} - C_{x_1 x_1} = 0. \quad (5.5)$$

Differentiating the second equation of (5.5) with respect to p_0, we obtain[6]

$$A_{x_1} = C_{p_0}. \quad (5.6)$$

From the first equation in system (5.5) it follows that C is linear in p_0:

$$C = r(x_1, x_2) p_0 + s(x_1, x_2).$$

Substituting this expression to the last equation of (5.5), we shall obtain the equality

$$r_{x_2} p_0 + s_{x_2} - p_0^2 r_{x_1} - p_0 s_{x_1} - p_0 r_{x_1 x_1} - s_{x_1 x_1} = 0.$$

Equating the coefficients at the powers of p_0 to zero, we obtain the relations

$$r_{x_1} = 0, \quad r_{x_2} - s_{x_1} = 0, \quad s_{x_2} - s_{x_1 x_1} = 0. \quad (5.7)$$

The first of them means that $C_{p_0 x_1} = 0$. Taking into account (5.6), we get

$$A_{x_1 x_1} = 0.$$

Differentiating the second equation in (5.7) with respect to x_1 and using the first equation, we have $s_{x_1 x_1} = 0$, i.e.,

$$s = w(x_2) x_1 + v(x_2).$$

Taking into account the third equation of (5.7), we have the relation $s_{x_2} = 0$, i.e.,

$$s = w x_1 + v, \quad w, v \in \mathbb{R}. \quad (5.8)$$

---

[5]This recursive way of reasoning, using reduction in higher derivatives order, is typical for the first stage of solving defining equations.

[6]In general, at the second step, it is typical to obtain differential consequences and to use compatibility conditions.

Taking derivative of the second equation (5.7) with respect to x_2 and using (5.8), we obtain $r_{x_2 x_2} = 0$ and consequently

$$r = mx_2 + n, \qquad m, n \in \mathbb{R},$$

(recall that r is independent of x_1 by the first equation of (5.7)). Hence, from the second equation of (5.7) we obtain $m = w$.

As a result, we get the relation

$$A = (wx_2 + v)x_1 + nx_2 + k, \quad B = wx_2^2 + 2wx_2 + l, \quad C = (wx_2 + v)p_0 + vx_1 + n,$$

where w, v, n, k, $l \in \mathbb{R}$.

From here it follows that the functions

$$x_1 x_2 p_1 + x_2^2 p_2 + x_2 p_0 + x_1, \ x_1 p_1 + 2x_2 p_2 + p_0, \ x_2 p_1 + 1, \ p_1, \ p_2 \qquad (5.9)$$

form an $\mathbb{R}$-basis in the space of symmetries of the Burgers equation.

Coming back to "physical" notation, these functions can be represented in the form

$$xtu_x + t^2 u_t + tu + x, \ xu_x + 2tu_t + u, \ tu_x + 1, \ u_x, \ u_t. \qquad (5.10)$$

Let us finally write down the Lie fields on $J^0(2,1)$ corresponding to the above listed generating functions φ:

$$(tu + x)\frac{\partial}{\partial u} - tx\frac{\partial}{\partial x} - t^2\frac{\partial}{\partial t},$$

$$u\frac{\partial}{\partial u} - x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} \qquad \text{(scale symmetry)},$$

$$\frac{\partial}{\partial u} - t\frac{\partial}{\partial x}, \qquad \text{(Galilean symmetry)}, \qquad (5.11)$$

$$\frac{\partial}{\partial x} \qquad \text{(translation along } x),$$

$$\frac{\partial}{\partial t} \qquad \text{(translation along } t).$$

5.2. The Korteweg–de Vries equation. Another well-known equation widely used in studying nonlinear phenomena is the Korteweg–de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

The summands uu_x and u_{xxx} correspond to dissipative and dispersive phenomena in nonlinear wave processes. This equation was suggested by Korteweg and de Vries in 1895 to describe waves of small (but finite) amplitude in long time periods. Other applications of this famous equation include description of rotating fluid flow in a pipe, the theory of ion-sound or magnetohydrodynamic waves in low-temperature plasma, longitudinal waves in bars, etc.

From the mathematical point of view, the interest to the Korteweg–de Vries equation was caused by special properties both of the equation itself, and of its solutions. In particular, it possesses solitary wave solutions (solitons), as well as infinite number of commuting conservation laws (see Ch. 5) and an infinite-dimensional algebra of (higher) symmetries (Ch. 4).

Skipping technical computations, which differ from those for the Burgers equation in details only, we shall give the final answer here. An $\mathbb{R}$-basis of the symmetry Lie algebra is formed by the following generating functions:

$$xu_x + 3tu_t + 2u, \quad 6tu_x + 1, \quad u_x, \quad u_t. \qquad (5.12)$$

The Lie fields corresponding to these functions are

$$2u\frac{\partial}{\partial u} - x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t} \qquad \text{(scale symmetry)},$$

$$\frac{\partial}{\partial u} - 6t\frac{\partial}{\partial x} \qquad \text{(Galilean symmetry)},$$

$$\frac{\partial}{\partial x} \qquad \text{(translation along } x\text{)},$$

$$\frac{\partial}{\partial t} \qquad \text{(translation along } t\text{)}.$$

(5.13)

5.3. The Khokhlov–Zabolotskaya equation. Propagation of a bounded three-dimensional acoustic beam in nonlinear media is described by the equation with the following dimensionless form:

$$-\frac{\partial^2 u}{\partial q_1 \partial q_2} + \frac{1}{2}\frac{\partial^2 (u^2)}{\partial q_1^2} + \frac{\partial^2 u}{\partial q_3^2} + \frac{\partial^2 u}{\partial q_4^2} = 0, \qquad (5.14)$$

where $q_1, q_2 > 0$ and $-\infty < q_3, q_4 < +\infty$. In this equation, the value of u is proportional to the deviation of the media density from the balanced density, while the dimensionless variables q_1, q_2, q_3, q_4 are expressed via the time t and spatial variables x, y, z in the following way

$$q_1 = \frac{t - x/c_0}{\sqrt{\gamma + 1}}\sqrt{\rho_0 c_0}, \quad q_2 = \mu x, \quad q_3 = \sqrt{\frac{2\mu}{c_0}}y, \quad q_4 = \sqrt{\frac{2\mu}{c_0}}z,$$

where c_0 is the sound velocity in the media, γ is the isentropic exponent, x is the coordinate in the direction of beam propagation, μ is a small parameter, and ρ_0 is the balanced density. In the coordinates q_i, p_σ, the equation (5.14) acquires the form

$$-p_{(1,1,0,0)} + up_{(2,0,0,0)} + p_1^2 + p_{(0,0,2,0)} + p_{(0,0,0,2)} = 0. \qquad (5.15)$$

This equation is called the Khokhlov–Zabolotskaya equation.

THEOREM 5.1. *The algebra of all classical symmetries for the Khokhlov–Zabolotskaya equation* (5.15) *is generated by the following symmetries*[7]:

$$\begin{cases} f(A) = A'q_3 p_1 + 2Ap_3 + A''q_3, \\ g(B) = B'q_4 p_1 + 2Bp_4 + B''q_4, \\ h(C) = Cp_1 + C', \\ T_2 = p_2, \\ L = q_1 p_1 + q_2 p_2 + q_3 p_3 + q_4 p_4, \\ M = 2q_1 p_1 + 4q_2 p_2 + 3q_3 p_3 + 3q_4 p_4 + 2u, \\ M_{34} = q_4 p_3 - q_3 p_4, \end{cases} \qquad (5.16)$$

where A, B, C are arbitrary smooth functions in q_2.

---

[7]Note that computation needed to prove the theorem are rather laborious. This is the reason to use symbolic computation programs in this field. A detailed analysis of computations can be found in [**109, 73, 75**]

5.3.1. *"Physically meaningful" symmetries.* Let us single out of all classical symmetries of equation (5.15) the "physically meaningful" ones, i.e., those preserving the condition of solution decay at infinity.

Consider the symmetry $f(A)$. Then

$$X_{f(A)} = A'q_3 \frac{\partial}{\partial q_1} - 2A \frac{\partial}{\partial q_3} + A''q_3 \frac{\partial}{\partial u}.$$

It is easily seen that the flow of this field is described by the system of equations

$$\begin{cases} q_1 = AA'\tau^2 + A'q_3^0\tau + q_1^0, \\ q_2 = q_2^0, \\ q_3 = 2A\tau + q_3^0, \\ q_4 = q_4^0, \\ u = -AA''\tau^2 - A''q_3^0\tau + u^0, \end{cases} \qquad (5.17)$$

where u^0, q_1^0, q_2^0, q_3^0, q_4^0 are initial values.

Consider an arbitrary solution $u^0 = u^0(q_1^0, q_2^0, q_3^0, q_4^0)$ of equation (5.15) decaying at infinity, i.e., satisfying the condition $u^0 \to 0$ as $\|(q_1^0, q_2^0, q_3^0, q_4^0)\| \to \infty$. Then the flow (5.17) takes this function to the solution

$$u = -AA''\tau^2 - A''q_3^0\tau + u^0(q_1^0, q_2^0, q_3^0, q_4^0)$$
$$= u(AA'\tau^2 + A'q_3^0\tau + q_1^0, q_2^0, 2A\tau + q_3^0, q_4^0).$$

Let $\|(AA'\tau^2 + A'q_3^0\tau + q_1^0, q_2^0, 2A\tau + q_3^0, q_4^0)\| \to \infty$ for a fixed value of the parameter τ. If $A'' \neq 0$, then the condition of decay of u at infinity will not fulfil, since we shall have $u \to \infty$ as $q_3^0 \to \infty$ in this case. Hence, $A'' = 0$. Obviously, this is also sufficient for the decay of solution at infinity.

From the condition $A'' = 0$ it follows that $A = C_1 q_2 + C_2$, where C_1, C_2 are constants. Thus, among infinite number of symmetries $f(A)$ physically meaningful are the following two only:

$$T_3 = p_3, \qquad R_3 = q_3 p_1 + 2q_2 p_3.$$

In the same way, the symmetries of the form $g(B)$ produce two physically meaningful symmetries

$$T_4 = p_4, \qquad R_4 = q_4 p_1 + 2q_2 p_4,$$

while symmetries $h(C)$ produce a sole symmetry

$$T_1 = p_1.$$

It is easily checked that the symmetries T_1, T_2, M, and M_{34} preserve the condition of decay at infinity while the symmetry L does not.

On the other hand, it is easily checked that the following linear combinations of L and M are physically meaningful:

$$M_1 = 2(M + 2L) = 2q_1 p_1 + q_3 p_3 + q_4 p_4 - 2u,$$
$$M_2 = (M - L) = 2q_2 p_2 + q_3 p_3 + q_4 p_4 + 2u.$$

Thus, the subalgebra of physically meaningful symmetries is generated by the following generating functions:

$$
\begin{aligned}
T_i &= p_i, \ i = 1, 2, 3, 4 && \text{(translations)}, \\
M_{34} &= q_4 p_3 - q_3 p_4 && \text{(rotation)}, \\
M_1 &= 2q_1 p_1 + q_3 p_3 + q_4 p_4 - 2u && \\
M_2 &= 2q_2 p_2 + q_3 p_3 + q_4 p_4 + 2u && \text{(scale symmetries)}, \\
R_3 &= q_3 p_1 + 2q_2 p_3, && \\
R_4 &= q_4 p_1 + 2q_2 p_4. &&
\end{aligned}
$$

Translations $T_i = p_i$, $i = 1, 2, 3, 4$, are responsible for homogeneity of the four-dimensional space-time, the rotation M_{34} corresponds to isotropy of the space in the plane perpendicular to the beam propagation direction, M_1 and M_2 correspond to invariance of the Khokhlov–Zabolotskaya equation with respect to the scaling transformations

$$
A_\tau^{(1)} : u(q) \longmapsto e^{-2\tau} u(e^{2\tau} q_1, q_2, e^\tau q_3, e^\tau q_4),
$$
$$
A_\tau^{(2)} : u(q) \longmapsto e^{2\tau} u(q_1, e^{2\tau} q_2, e^\tau q_3, \ e^\tau q_4).
$$

Finally, R_3 and R_4 correspond to invariance of the equation with respect to the transformations

$$
A_\tau^{(3)} : u(q) \longmapsto u(q_1 + \tau q_3 + \tau^2 q_2, q_2, q_3 + 2\tau q_2, q_4),
$$
$$
A_\tau^{(4)} : u(q) \longmapsto u(q_1 + \tau q_4 + \tau^2 q_2, q_2, q_3, \ q_4 + 2\tau q_2).
$$

5.3.2. *Invariant solutions.* Let us find solutions invariant with respect to some subalgebras of the Lie algebra of all classical symmetries of the Khokhlov–Zabolotskaya equation (5.15).

Consider the subalgebra of symmetries $\mathcal{G}$ generated by the symmetries M_1, M_2, R_3, R_4. The Lie algebra structure in $\mathcal{G}$ with respect to the Jacobi bracket in the basis $e_1 = (M_1 + M_2)/2$, $e_2 = (-M_2 + M_1)/2$, $e_3 = R_3$, $e_4 = R_4$ is given by the relations

$$
[e_2, e_3] = e_3, \quad [e_2, e_4] = e_4, \quad [e_3, e_4] = 0, \quad [e_1, e_i] = 0, \qquad i = 2, 3, 4.
$$

Thus $\mathcal{G}$ is a solvable Lie algebra with the commutator subalgebra

$$
\mathcal{G}^{(1)} = [\mathcal{G}, \mathcal{G}] = \mathbb{R}e_3 \oplus \mathbb{R}e_4.
$$

Consider an arbitrary three-dimensional subalgebra $\mathcal{H} \subset \mathcal{G}$ containing $\mathcal{G}^{(1)}$. Any such a subalgebra is generated by the elements e_3, e_4, $\lambda_1 e_1 + \lambda_2 e_2$.

Let us describe $\mathcal{H}$-invariant solutions of equation (5.15). If $\lambda_1 + \lambda_2 \neq 0$, then $\mathcal{H}$-invariant solutions $u(q)$ are of the form

$$
u(q) = x^\lambda v(x^{-2-\lambda} y),
$$

where $x = q_2$, $y = 4q_1 q_2 - q_3^2 - q_4^2$, $\lambda = -2\lambda_2/(\lambda_1 + \lambda_2)$. Substituting this expression to the Khokhlov–Zabolotskaya equation, we obtain the equation for $v(t)$:

$$
vv'' + (v')^2 + \alpha t v'' = 0, \tag{5.18}
$$

where $t = x^{-2-\lambda} y$, $\alpha = -(\lambda_2 - \lambda_1)/4(\lambda_1 + \lambda_2)$.

Since the left-hand side of (5.18) is of the form $(vv' + \alpha t v' - \alpha v)'$, we obtain the equation

$$
vv' + \alpha t v' - \alpha v = \text{const}.
$$

In the case when the constant in the left-hand side vanishes, this equation possesses the solution

$$v \ln v + av = \alpha t,$$

where a is a constant.

5.3.3. *The Kadomtsev–Pogutse equations.* This system is a simplification of the general magnetohydrodynamic (MHD) system, where some inessential (from the point of view of maintaining high-temperature plasma in TOKAMAK-like systems) details are omitted. The authors of the paper [**48**] started from the ideal MHD equations, because in this context characteristic times of the most important physical processes are essentially less than the distinctive dissipation time. Dissipation here is caused by viscosity and electric resistance of plasma. Besides, it was taken into account that plasma stability needs: (a) plasma pressure and the cross-component of magnetic field pressure were much less than the pressure created by the longitudinal component of the magnetic field and (b) the smaller TOKAMAK radius was much less then the external radius.

As a result, a system of two scalar equations was obtained which in an adequate norming is of the form

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + [\nabla_\perp \varphi, \nabla_\perp \psi]_z = \dfrac{\partial \varphi}{\partial z}, \\[2mm] \dfrac{\partial}{\partial t} \Delta_\perp \varphi + [\nabla_\perp \varphi, \nabla_\perp \Delta_\perp \varphi]_z = \dfrac{\partial}{\partial t} \Delta_\perp \psi + [\nabla_\perp \psi, \nabla_\perp \Delta_\perp \psi]_z, \end{cases} \tag{5.19}$$

where $\nabla_\perp = \{\partial/\partial x, \partial/\partial y\}$, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ while $[u,v]_z = u_x v_y - u_y v_x$ is the z-component of the vector product $[u,v]$ of the vectors u and v.

Here the functions φ and ψ are the potentials of the velocity and of the cross-component of the magnetic field respectively (they also may be understood as the potential of the electric field and z-component of the vector potential of the magnetic field). The coordinate system is chosen in such a way that the z-axis is directed along the TOKAMAK axis.

We shall call equations (5.19) the Kadomtsev–Pogutse equations. They are also called reduced MHD equations.

Relative simplicity of the Kadomtsev–Pogutse equations made it possible to construct an experimentally confirmed theory of kink instability; they were also used in quantitative analysis of instability [**147**].

5.3.4. *Computation of symmetries.* We shall change here general notation for coordinates in jet spaces for specific ones to make formulas more understandable for reading. The coordinates in the base will be denoted by x, y, z, t, while for coordinates in the fiber of $J^0(4,2)$ the notations φ and ψ will be used. We shall write $\varphi_{x^i y^j z^k t^l}$ and $\psi_{x^i y^j z^k t^l}$ instead of p_σ^1, p_σ^2 respectively, where $\sigma = (i,j,k,l)$, $|\sigma| = i + j + k + l$. In this notation, (5.19) acquires the form

$$F_1 = \psi_t + \varphi_x \psi_y - \varphi_y \psi_x - \varphi_z = 0, \tag{5.20}$$

$$\begin{aligned} F_2 &= \varphi_{x^2 t} + \varphi_{y^2 t} + \varphi_x \varphi_{x^2 y} + \varphi_x \varphi_{y^3} - \varphi_y \varphi_{x^3} - \varphi_y \varphi_{xy^2} \\ &\quad - \psi x^2 z - \psi_{y^2 z} - \psi_x \psi_{xy^2} - \psi_x \psi_{y^3} + \psi_y \psi_{x^3} + \psi_y \psi_{xy^2} = 0. \end{aligned} \tag{5.21}$$

The system of Kadomtsev–Pogutse equations contains two dependent variables φ and ψ and consequently the generating function of a symmetry is two-component

in this case:

$$\mathcal{S} = S + A\varphi_x + B\varphi_y + C\varphi_z + E\varphi_t,$$
$$\mathcal{T} = T + A\psi_x + B\psi_y + C\psi_z + E\psi_t. \tag{5.22}$$

Here A, B, C, E, S, T are functions on $J^0(4,2)$, i.e., functions in x, y, z, t.

THEOREM 5.2 ([**98**]). *The algebra of classical symmetries of the Kadomtsev–Pogutse equations as a linear space over $\mathbb{R}$ is generated by symmetries with the generating functions of the form*

$$\mathcal{A}_\alpha = \begin{pmatrix} \alpha'(\varphi + \psi) + \alpha(\varphi_z + \varphi_t) \\ \alpha'(\varphi + \psi) + \alpha(\psi_z + \psi_t) \end{pmatrix}, \quad \mathcal{B}_\beta = \begin{pmatrix} \beta'(\varphi - \psi) + \beta(\varphi_z - \varphi_t) \\ -\beta'(\varphi - \psi) + \beta(\psi_z - \psi_t) \end{pmatrix},$$

$$\mathcal{C}_\gamma = \begin{pmatrix} \gamma'(x^2 + y^2) + 2\gamma(y\varphi_x - x\varphi_y) \\ \gamma'(x^2 + y^2) + 2\gamma(y\psi_x - x\psi_y) \end{pmatrix}, \quad \mathcal{D}_\delta = \begin{pmatrix} -\delta'(x^2 + y^2) + 2\delta(y\varphi_x - x\varphi_y) \\ \delta'(x^2 + y^2) + 2\delta(y\psi_x - x\psi_y) \end{pmatrix},$$

$$\mathcal{G}_G = \begin{pmatrix} yG_t + G\varphi_x \\ yG_z + G\psi_x \end{pmatrix}, \quad \mathcal{H}_H = \begin{pmatrix} -xH_t + H\varphi_y \\ -xH_z + H\psi_y \end{pmatrix}, \quad \mathcal{K}_K = \begin{pmatrix} K_t \\ K_z \end{pmatrix},$$

$$\mathcal{L} = \begin{pmatrix} \varphi + z\varphi_z + t\varphi_t \\ \psi + z\psi_z + t\psi_t \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} x\varphi_x + y\varphi_y + 2z\varphi_z + 2t\varphi_t \\ x\psi_x + y\psi_y + 2z\psi_z + 2t\psi_t \end{pmatrix}.$$

Here $\alpha = \alpha(z+t)$, $\beta = \beta(z-t)$, $\gamma = \gamma(z+t)$, $\delta = \delta(z-t)$, $G = G(z,t)$, $H = H(z,t)$, $K = K(z,t)$ are arbitrary functions and

$$\alpha'(\zeta) = \frac{d\alpha(\zeta)}{d\zeta}, \quad \beta'(\eta) = \frac{d\beta(\eta)}{d\eta}, \quad \gamma'(\xi) = \frac{d\gamma(\xi)}{d\xi}, \quad \delta'(\theta) = \frac{d\delta(\theta)}{d\theta}.$$

The vector fields on $J^0(4,2)$ corresponding to the above listed generating functions are of the form

$$X_{\mathcal{A}_\alpha} = \alpha'(\varphi + \psi)\left(\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\psi}\right) - \alpha\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right),$$

$$X_{\mathcal{B}_\beta} = \beta'(\varphi - \psi)\left(\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\psi}\right) - \beta\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right),$$

$$X_{\mathcal{C}_\gamma} = \gamma'(x^2 + y^2)\left(\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\psi}\right) - 2\gamma\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right),$$

$$X_{\mathcal{D}_\delta} = \delta'(x^2 + y^2)\left(\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\psi}\right) - 2\delta\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right),$$

$$X_{\mathcal{G}_G} = yG_t\frac{\partial}{\partial\varphi} + yG_z\frac{\partial}{\partial\psi} - G\frac{\partial}{\partial x},$$

$$X_{\mathcal{H}_H} = -xH_t\frac{\partial}{\partial\varphi} - xH_z\frac{\partial}{\partial\psi} - H\frac{\partial}{\partial y},$$

$$X_{\mathcal{K}_K} = K_t\frac{\partial}{\partial\varphi} + K_z\frac{\partial}{\partial\psi}$$

$$X_{\mathcal{L}} = \varphi\frac{\partial}{\partial\varphi} + \psi\frac{\partial}{\partial\psi} - z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t},$$

$$X_{\mathcal{M}} = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z} - 2t\frac{\partial}{\partial t}$$

The above listed symmetries is of the following physical meaning: $\mathcal{L}$ and $\mathcal{M}$ are scale symmetries, $\mathcal{C}_\gamma$ and $\mathcal{D}_\delta$ are generalized rotations in the (x, y)-plane (they become real rotations, when $\gamma = \delta = \mathrm{const}$), $\mathcal{A}_\alpha$ and $\mathcal{B}_\beta$ are generalized translations along z and t respectively (in the case $\alpha = \beta = 1$ real translations are obtained as

their linear combinations), the symmetries $\mathcal{G}_G$ and $\mathcal{H}_H$ are generalized translations along x and y respectively (real translations correspond to the case $G = H = 1$), finally $\mathcal{K}_K$ is the gauge by the trivial solution.

The Lie algebra structure of symmetries is described in Table 1.

TABLE 1. Commutation relations in the algebra of classical symmetries for the Kadomtsev–Pogutse equations

$[\,,\,]$	$\mathcal{A}_{\widehat{\alpha}}$	$\mathcal{B}_{\widehat{\beta}}$	$C_{\widehat{\gamma}}$	$\mathcal{D}_{\widehat{\delta}}$	$\mathcal{G}_{\widehat{G}}$
$\mathcal{A}_{\alpha}$	$2\mathcal{A}_{\widehat{\alpha}\alpha'-\widehat{\alpha}'\alpha}$	0	$-2\mathcal{G}_{\widehat{\gamma}'\alpha}$	0	$-\mathcal{G}_{\alpha(\widehat{G}_t+\widehat{G}_z)}$
$\mathcal{B}_{\beta}$	0	$2\mathcal{B}_{\beta'\widehat{\beta}-\widehat{\beta}'\beta}$	0	$-2\mathcal{D}_{\beta\widehat{\delta}'}$	$\mathcal{G}_{\beta(\widehat{G}_t-\widehat{G}_z)}$
$\mathcal{C}_{\gamma}$	$2\mathcal{C}_{\gamma'\widehat{\alpha}}$	0	0	0	$-2\mathcal{H}_{\delta\widehat{G}}$
$\mathcal{D}_{\delta}$	0	$2\mathcal{D}_{\delta'\widehat{\beta}}$	0	0	$-2\mathcal{G}_{\widehat{\delta}H}$
$\mathcal{G}_{G}$	$\mathcal{G}_{\widehat{\alpha}(G_t+G_z)}$	$-\mathcal{G}_{\widehat{\beta}(G_t-G_z)}$	$2\mathcal{H}_{\widehat{\gamma}G}$	$2\mathcal{H}_{\widehat{\delta}G}$	0
$\mathcal{H}_{H}$	$\mathcal{H}_{\widehat{\alpha}(H_t+H_z)}$	$\mathcal{H}_{\widehat{\beta}(H_t-H_z)}$	$-2\mathcal{G}_{\widehat{\gamma}H}$	$-2\mathcal{G}_{\widehat{\delta}H}$	$-\mathcal{K}_{\widehat{G}H}$
$\mathcal{K}_{K}$	$-c\mathcal{K}_{\widehat{\alpha}(K_t+K_z)}$	$-c\mathcal{K}_{\widehat{\beta}(K_t-K_z)}$	0	0	0
$l\mathcal{L}$	$-l\mathcal{A}_{(t+z)\widehat{\alpha}'-\widehat{\alpha}}$	$-l\mathcal{B}_{(z-t)\widehat{\beta}'-\widehat{\beta}}$	$-l\mathcal{C}_{(z+t)\widehat{\gamma}'}$	$-l\mathcal{D}_{(z-t)\widehat{\delta}'}$	$-l\mathcal{G}_{t\widehat{G}_t+z\widehat{G}_z}$
$m\mathcal{M}$	0	0	0	0	$m\mathcal{H}_{\widehat{H}}$

$[\,,\,]$	$\mathcal{H}_{\widehat{H}}$	$-\mathcal{K}_{\widehat{K}}$	$\widehat{l}\mathcal{L}$	$\widehat{m}\mathcal{M}$
$\mathcal{A}_{\alpha}$	$-\mathcal{H}_{\alpha(\widehat{H}_t+\widehat{H}_z)}$	$\mathcal{H}_{\alpha(\widehat{K}_t+\widehat{K}_z)}$	$\widehat{l}\mathcal{A}_{(t+x)\alpha'-\alpha}$	0
$\mathcal{B}_{\beta}$	$\mathcal{H}_{\beta(\widehat{H}_t-\widehat{H}_z)}$	$\mathcal{K}_{\beta(\widehat{K}_t-\widehat{K}_z)}$	$\widehat{l}\mathcal{B}_{(z-t)\beta'-\beta}$	0
$\mathcal{C}_{\gamma}$	$2\mathcal{G}_{\gamma\widehat{H}}$	0	$\widehat{l}\mathcal{C}_{(t+z)\gamma'}$	0
$\mathcal{D}_{\delta}$	$2\mathcal{G}_{\gamma\widehat{H}}$	0	$\widehat{l}\mathcal{D}_{(z-t)\delta'}$	0
$\mathcal{G}_{G}$	$\mathcal{K}_{G\widehat{H}}$	0	$\widehat{l}\mathcal{G}_{zG_z+tG_t}$	$-\widehat{m}\mathcal{G}_G$
$\mathcal{H}_{H}$	0	0	$\widehat{l}\mathcal{H}_{tH_t+zH_z}$	$-\widehat{m}\mathcal{H}_H$
$\mathcal{K}_{K}$	0	0	$\widehat{l}\mathcal{K}_{tK_t+zK_z}$	$-2\widehat{m}\mathcal{K}_K$
$l\mathcal{L}$	$-l\mathcal{H}_{t\widehat{H}_t+z\widehat{H}_z}$	$-l\mathcal{K}_{t\widehat{K}_t+z\widehat{K}_z}$	0	0
$m\mathcal{M}$	$m\mathcal{H}_{\widehat{H}}$	$2m\mathcal{K}_{\widehat{K}}$	0	0

5.3.5. *Invariant solutions.*

$\mathcal{A}_{\alpha}$-**invariant solutions.** The invariance conditions are of the form

$$\alpha'(\varphi + \psi) + \alpha(\varphi_z + \varphi_t) = 0, \quad \alpha'(\varphi + \psi) + \alpha(\psi_z + \psi_t) = 0.$$

Solving this system, we obtain

$$\varphi = (\alpha(z + t))^{-1} A(x, y, z - t) + B(x, y, z - t),$$
$$\psi = (\alpha(z + t))^{-1} A(x, y, z - t) + B(x, y, z - t),$$

where the functions A and B satisfy the following conditions implied by (5.20) and (5.21):

$$A_\eta = B_x A_y - A_x B_y,$$
$$\Delta_\perp A_\eta = B_x \Delta_\perp A_y + A_x \Delta_\perp B_y - A_y \Delta_\perp B_x - B_y \Delta_\perp A_x, \tag{5.23}$$

where $\eta = z - t$. Note that equations (5.23) are the same for all α.

$\mathcal{G}_G$-**invariant solutions.** Invariance conditions in this case lead to the following equations:

$$\varphi = -\frac{G_t}{G} xy + A(y, z, t), \quad \psi = -\frac{G_z}{G} xy + B(y, z, t),$$

where A and B satisfy the *linear* system

$$-B_t + A_z = \frac{y}{G}(G_z A - G_t B)_y,$$
$$(A_t - B_z)_{y^2} = \frac{y}{G}(G_t A - G_z B)_{y^3}.$$

$\mathcal{C}_\gamma$-**invariant solutions.** From the invariance conditions we obtain

$$\varphi = r^2 \theta \gamma'/2\gamma + A(r, z, t), \quad \psi = r^2 \theta \gamma'/2\gamma + B(r, z, t),$$

where r are θ are the polar coordinates on the plane x, y. The functions A and B satisfy the linear system

$$-B_t + A_z = (r\gamma'/2\gamma)(A - B)_r,$$
$$\Delta_\perp (A_t - B_z) = [(r\gamma'/2\gamma)\Delta_\perp + (2\gamma'/r\gamma)](A - B)_r.$$

It is interesting to note, that $\mathcal{C}_\gamma$-invariant solutions are uniquely defined only in the sector $|\theta - \theta_0| < \pi$ for some θ_0. Discontinuous solutions can be constructed of such pieces.

Let us consider examples of solutions invariant with respect to some two-dimensional subalgebras.

$\mathcal{A}_\alpha, \mathcal{B}_\beta$-**invariant solutions.** These solutions are of the form

$$\varphi = (\alpha(z+t))^{-1} A(x, y) + (\beta(z-t))^{-1} B(x, y),$$
$$\psi = (\alpha(z+t))^{-1} A(x, y) - (\beta(z-t))^{-1} B(x, y). \tag{5.24}$$

Thus these solutions represent an arbitrary (the so-called Alfvén) wave propagating along the z-axis with the velocity $c_A = 1$ and with the amplitude depending on x and y. The functions A and B must satisfy the system of equations

$$\begin{cases} A_x B_y - A_y B_x = 0, \\ A_x \Delta B_y - A_y \Delta B_x + B_x \Delta A_y - B_y \Delta A_x = 0. \end{cases}$$

From the first equation of the system it follows that $B = f(A)$. From the second one we obtain

$$f'(A)(A_x \Delta A_y - A_y \Delta A_x) + f''(A)[A_{xy}(A_x^2 - A_y^2) - A xy(A_{x^2} - A_{y^2})] = 0.$$

In particular, for $B = A$ we have

$$\Delta A = F(A). \tag{5.25}$$

Other examples of invariant solutions for the Kadomtsev–Pogutse equations, including the case of three-dimensional subalgebras, can be found in [**37**].

5.4. Reproduction of solutions. Let (φ_0, ψ_0) be a solution of system (5.20), (5.21) and $Q = (Q^1, Q^2)$ be its symmetry. Recall that solving the system

$$\frac{\partial}{\partial \tau} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix}$$

with the initial values $(\varphi, \psi)|_{\tau=0} = (\varphi_0, \psi_0)$, gives a family of solutions for equations (5.20), (5.21) depending on τ. Thus, the solution (φ_0, ψ_0) is "reproduced" or "deformed" by the symmetry Q. Consider some examples.

$\mathcal{C}_\gamma$-**reproduction.** In this case we need to solve the equations

$$\begin{aligned}
\varphi_\tau &= \gamma' r^2/2 + \gamma \varphi_\theta, & \varphi|_{\tau=0} &= \varphi_0(r, \theta, z, t), \\
\psi_\tau &= \gamma' r^2/2 + \gamma \psi_\theta, & \psi|_{\tau=0} &= \varphi_0(r, \theta, z, t).
\end{aligned} \tag{5.26}$$

The solutions are

$$\varphi_\tau = \gamma' r^2 \tau/2 + \varphi_0(r, \theta + \tau\gamma, z, t), \qquad \psi_\tau = \gamma' r^2 \tau/2 + \psi_0(r, \theta + \tau\gamma, z, t).$$

It can be noticed that the obtained solutions are single-valued while invariant solutions for this symmetry are multivalued (see above).

From the formulas for the family φ and ψ one can see that the phase θ can be deviated from the axis z by means of the Alfvén wave propagation $\gamma(z + t)$, but this causes the correction $\gamma r^2 \tau$ to the solution. Note that the $\mathcal{D}_\delta$-reproduction leads to similar formulas (and to the deviation $\tau\delta(z - t)$ of the phase). If the solution is periodic in z (for example, the TOKAMAK z-axis is closed and is of length l), then the "quantization conditions" for the parameter τ arise. For example, taking the phase deviation $\tau z = \tau[(z + t) + (z - t)]/2$, we shall obtain that $\tau l = 2\pi n$, $\tau = 2\pi n/l$. But this effect may not appear, if deviation is periodic (for example, if one has the "phase trembling" of the form $\gamma = \sin[\pi N(z + t)/l]$).

$\mathcal{G}_G$-**reproduction.** In this case

$$\varphi = y\tau G_t + \varphi_0(x + \tau G, y, z, t), \quad \psi = y\tau G_z + \psi_0(x + \tau G, y, z, t).$$

Here the solution deviates from the x-axis and the correction $(y\tau G_t, y\tau G_z)$ to the initial solution arises. Quite similar, the $\mathcal{H}_H$-reproduction is a shift along the y-axis by τH. Combining different shifts, one can bend the solution in a helical way.

Nevertheless, it should be noted that the indicated freedom in deformation choice is in a sense illusive: from the physical point of view, strong deformation seems to lead to necessity of taking into account the plasma pressure and consequently of adding new terms to the initial equation.

$\mathcal{A}_\alpha$-**reproduction.** To solve the system

$$\begin{aligned}
\varphi_\tau &= \alpha(\varphi_z + \varphi_t) + \alpha'(\varphi + \psi), \\
\psi_\tau &= \alpha(\psi_z + \psi_t) + \alpha'(\varphi + \psi),
\end{aligned}$$

let us add and subtract its equations:

$$\begin{aligned}
(\varphi + \psi)_\tau &= \alpha[(\varphi + \psi)_z + (\varphi + \psi)_t] + 2\alpha'(\varphi + \psi), \\
(\varphi - \psi)_\tau &= \alpha[(\varphi - \psi)_z + (\varphi - \psi)_t].
\end{aligned}$$

The transformed system is easy to solve with respect to the new unknown functions $(\varphi + \psi)$, $(\varphi - \psi)$. The answer is

$$\varphi = \frac{\alpha(\Sigma) + \alpha(z+t)}{2\alpha(z+t)}\varphi_0(x,y,Z,T) + \frac{\alpha(\Sigma) - \alpha(z+t)}{2\alpha(z+t)}\psi_0(x,y,Z,T),$$

$$\psi = \frac{\alpha(\Sigma) - \alpha(z+t)}{2\alpha(z+t)}\varphi_0(x,y,Z,T) + \frac{\alpha(\Sigma) + \alpha(z+t)}{2\alpha(z+t)}\psi_0(x,y,Z,T).$$

Here

$$\Sigma = \Gamma^{-1}(\tau + \Gamma(z+t)), \quad \Gamma(z+t) = \frac{1}{2}\int^{z+t}\frac{d\xi}{\alpha(\xi)},$$

$$Z = \frac{1}{2}(z - t + \Sigma), \quad T = \frac{1}{2}(t - z + \Sigma),$$

and Γ^{-1} is the inverse function. One can obtain explicit formulas for particular values of α. For example, for $\psi_0 \equiv 0$ and $\alpha = z+t$ we have the "rumpled" solution

$$\varphi = (1 + e^\tau)\frac{\varphi_0(x,y,e^\tau(z+t),z-t)}{2}.$$

It is interesting to note that if one takes a discontinuous symmetry, then the initially smooth solution includes in a family of discontinuous ones. For example, if $\psi_0 \equiv 0$, $\alpha = \xi^{-1}$, $\xi = z+t$, then

$$\varphi = 5(1 + |\xi|(\tau + \xi^2)^{-1/2})\frac{\varphi_0(x,y,\text{sgn}(\xi)\sqrt{\tau+\xi^2},z-t)}{2}.$$

For $\xi = 0$, $\tau \neq 0$ a traveling break arises proportional to

$$\varphi_0(x,y,\sqrt{\tau},z-t) - \varphi_0(x,y,-\sqrt{\tau},z-t).$$

Though the break of the potential φ does not admit a straightforward physical interpretation, when the function φ_0 is even a gradient break arises and it makes sense physically.

Consider another example of using invariant solutions for reproduction [37]. Choose an $\mathcal{A}_\alpha, \mathcal{B}_\beta$-invariant solution (5.24) taking $\alpha = 2$, $\beta = -2$ and functions A and B such that $A = B$ and $\Delta A = \exp A$ (cf. (5.25)). For the solution, we shall take the function

$$A(z,\theta) = -\ln\left[\frac{r^4}{2}\sinh^2\left(\frac{\cos\theta}{r} + 1\right)\right].$$

Then the corresponding invariant solution is of the form

$$\begin{cases} \varphi = 0, \\ \psi = A. \end{cases}$$

Thus obtained solution is static (independent of time) and constant along the z-axis. Let us apply the $\mathcal{C}_\alpha$-deformation to it, with $\alpha(z+t) = \exp(-(z+t))/2$. For $\tau = 1$ from formulas (5.26) we obtain

$$\varphi = -\frac{r^2}{2}e^{-(z+t)},$$

$$\psi = \frac{r^2}{2}e^{-(z+t)} - \ln\left\{\frac{r^4}{2}\sinh^2\left[\frac{\cos\theta - e^{-(z+t)}}{r} + 1\right]\right\}.$$

On Figure 3.11 one can see the magnetic level surface $\psi(x,y,z,0) = 5$. The
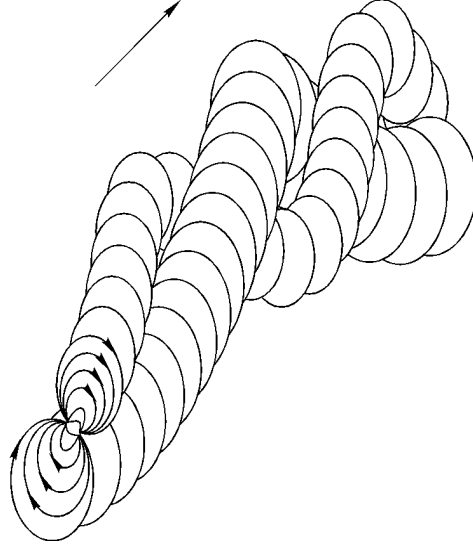
FIGURE 3.11
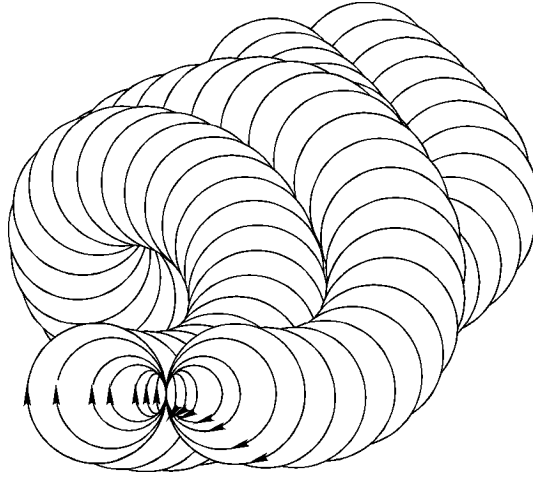
FIGURE 3.12

arrow shows the direction of Alfvén velocity along the z-axis. Thus, after defor-
mation the solution lost its static property and became inhomogeneous along the
z-axis.

On Figure 3.12 the same solution is exposed, first deformed by $\mathcal{L}_G$-deformation
and then by $\mathcal{H}_H$-deformation, where $G = (\sin 20z)/20$ and $H = (\cos 20z)/20$.

6. Factorization of equations by symmetries

The symmetry group of a differential equation consists of diffeomorphisms of
the jet space preserving the equation itself and taking solutions to solutions. What
will be the quotient object of this action, i.e., the orbit space? More general, what

will be the orbit space when we consider the action of a subgroup of the symmetry group?

Consider an equation $\mathcal{E} \subset J^k(\pi)$, whose symmetry group is a Lie group G, and assume that $H \subset G$ is a Lie subgroup. The Cartan distribution $\mathcal{C} = \mathcal{C}^k$ is invariant with respect to symmetries and the solutions are n-dimensional maximal integral manifolds of the distribution $\mathcal{C}^k(\mathcal{E})$. Consider the orbit space of the group H and denote by μ the factorization mapping[8]:

$$
\begin{array}{ccc}
\mathcal{E} & \longrightarrow & J^k(\mathbb{R}^n) \\
{\scriptstyle \mu|_\mathcal{E}} \downarrow & & \downarrow {\scriptstyle \mu} \\
\mathcal{E}' & \longrightarrow & J^k(\mathbb{R}^n)/H
\end{array}
$$

The quotient distribution $\mathcal{C}'$ is defined on $\mathcal{E}'$ and almost everywhere it is of the same dimension as $\mathcal{C}$. In fact, let $\mathcal{C}'_a = \mu_*(\mathcal{C}_{\mu^{-1}(a)})$ for $a \in J^k(\pi)/H$. It is a well-defined construction, since the distribution $\mathcal{C}$ is invariant with respect to H. Besides, vectors tangent to the orbits of H almost everywhere do not belong to the distribution $\mathcal{C}$. In fact, let $b \in J^k(\pi)$ and H_b be the orbit of the point b under the action of the group H. Then the mapping $H \to H_b$, $h \in H \mapsto h(b) \in H_b$, generates an epimorphism of the corresponding Lie algebra $\mathcal{H}$ to the tangent space $T_b(H_b)$. Therefore, any vector $X_b \in T_b(H_b)$ is of the form $X_\varphi|_b$, where φ is the generating section of some symmetry from $\mathcal{H}$. Let $X_b \in \mathcal{C}_b$. This means, in particular, that

$$
(X_\varphi \lrcorner (du^j - \sum_{i=1}^n p_i^j \, dx_i))_b = \varphi^j(b) = 0.
$$

Therefore, for almost all $b \in J^k(\pi)$ one has $\mathcal{C}_b \cap \ker \mu_*|_b = 0$ and consequently $\dim \mathcal{C} = \dim \mathcal{C}'$.

Solutions of the equation $\mathcal{E}$ are maximal integral n-dimensional manifolds of the distribution $\mathcal{C}$ and in general they project to n-dimensional manifolds of the distribution $\mathcal{C}'$ lying in $\mathcal{E}'$. It should be stressed that we are not interested in all integral manifolds of $\mathcal{C}'$, but in those ones which are images of solutions of the initial equation (since this equation is the ultimate object of our study). Therefore, the desired condition defining in $\mathcal{E}'$ the integral manifolds of $\mathcal{C}'$ will be the quotient equation[9].

Let $L' \subset \mathcal{E}'$ be the image of some solution L of the equation $\mathcal{E}$. Then

$$
\mu^{-1}(L') = H(L) = \bigcup_{h \in H} h(L). \tag{6.1}
$$

Denote by $i_{L'}$ the immersion $\mu^{-1}(L') \to J^k(\pi)$. Then the restriction of the Cartan distribution to $\mu^{-1}(L')$ is determined by the differential system $\mathcal{D}_{L'} = i_{L'}^* \mathcal{D}$, where $\mathcal{D} \subset \Lambda^1(J^k(\pi))$ is the differential system given by the Cartan forms and determining $\mathcal{C}$. It is easily seen that the distribution $\mathcal{D}_{L'}$ is completely integrable: any $h(L)$ is its integral manifold and the union of these manifolds is $\mu^{-1}(L')$. The Frobenius complete integrability condition $d\mathcal{D}_{L'} \subset \mathcal{D}_{L'}$ of the distribution $\mathcal{D}_{L'}$ is exactly the condition defining those L', which are the images of solutions of the initial

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[8] We assume that the orbit space is a smooth manifold and μ is a smooth bundle.

[9] Of course, if it may be realized, at least locally, as a submanifold in the jet space $J^{k'}(\pi')$ for some π' and k'.

equation under the factorization with respect to H. Thus, precisely these conditions constitute the quotient equation.

In the examples considered below such quotient equations are represented explicitly in coordinate form due to convenient relations between dimension of the objects under consideration; in more general situations, it is often difficult to obtain a coordinate description.

6.1. Second-order equations in two independent variables. Consider a second order equation in two independent and one dependent variable, where a two-dimensional Lie group H acts without fixed points. More exactly, let $\mathcal{E} \subset J^2(2,1)$ and q_1, q_2, u, p_1, p_2, $p_{(2,0)}$, $p_{(1,1)}$, $p_{(0,2)}$ be canonical coordinates in $J^2(2,1)$. Then $\dim \mathcal{E} = 7$, $\dim \mathcal{E}/H = 5$. In the same coordinates, a basis of the differential system $\mathcal{D}$ determining the Cartan distribution is

$$
\begin{aligned}
\omega_0 &= du - p_1\, dq_1 - p_2\, dq_2, \\
\omega_1 &= dp_1 - p_{(2,0)}\, dq_1 - p_{(1,1)}\, dq_2, \\
\omega_2 &= dp_2 - p_{(1,1)}\, dq_1 - p_{(0,2)}\, dq_2,
\end{aligned}
$$

and consequently dimension of Cartan planes at generic points is 4.

As it was already explained, the vector fields forming the Lie algebra $\mathcal{H}$ of the Lie group H does not intersect the Cartan distribution almost everywhere and therefore the quotient Cartan distribution on the five-dimensional manifold $\mathcal{E}/H$ is also four-dimensional. Hence, the differential system $\mathcal{D}'$ determining this distribution is one-dimensional (at least locally) and the distribution $\mathcal{C}'$ can be described by one differential form w. In the generic situation, a one-form on an even-dimensional manifold determines a contact structure and by the Darboux lemma it can be written as

$$
w = dv - y_1 dx_1 - y_2 dx_2
$$

in appropriate coordinates x_1, x_2, v, y_1, y_2 on $\mathcal{E}/H$. Thus, the quotient manifold $\mathcal{E}/H$ itself can be (locally) identified with $J^1(\mathbb{R}^2)$.

In these coordinates (also locally), "almost all" maximal integral manifolds of the form w can be represented as 1-jets of the graphs of functions in two variables:

$$
w = g(x_1, x_2), \quad y_1 = \frac{\partial g}{\partial x_1}, \quad y_2 = \frac{\partial g}{\partial x_2}.
$$

Let us denote such a manifold by V_g. The Frobenius conditions $d\mathcal{D}_{V_g} \subset \mathcal{D}_{V_g}$ of complete integrability are first-order partial differential equations for the functions determining the graph of 1-jet $j_1(g)$. Hence, they are second order equations for the function g. Actually, in the examples below a scalar second order equation arises and this is the desired quotient equation. Its solutions, as it follows by construction, correspond to two-parameter families of solutions of the initial equation.

EXAMPLE 6.1. Consider the Laplace equation $\mathcal{E} = \{p_{(2,0)} + p_{(0,2)} = 0\}$. Let the group H be generated by translations along q_1 and q_2; respectively, its Lie algebra $\mathcal{H}$ possesses the basis $\partial/\partial q_1$, $\partial/\partial q_2$.

The form ω determining the contact structure on $\mathcal{E}/H$ is uniquely, up to a factor, defined by the conditions

$$\frac{\partial}{\partial q_1} \lrcorner \mu^*\omega = 0, \quad \frac{\partial}{\partial q_2} \lrcorner \mu^*\omega = 0,$$
$$\frac{\partial}{\partial q_1}(\mu^*\omega) = 0, \quad \frac{\partial}{\partial q_2}(\mu^*\omega) = 0, \tag{6.2}$$

where $\mu^*\omega = \alpha_0\omega_0 + \alpha_1\omega_1 + \alpha_2\omega_2$.

By last condition, equations (6.2) imply

$$\left.(\alpha_0 p_1 + \alpha_1 p_{(2,0)} + \alpha_2 p_{(1,1)})\right|_{\mathcal{E}} = 0,$$
$$\left.(\alpha_0 p_2 + \alpha_1 p_{(1,1)} + \alpha_2 p_{(0,2)})\right|_{\mathcal{E}} = 0,$$

from where, up to a factor, one obtains

$$\alpha_0 = \left.(p_{(1,1)}^2 - p_{(2,0)}p_{(0,2)})\right|_{\mathcal{E}}, \qquad \alpha_1 = \left.(p_1 p_{(0,2)} - p_2 p_{(1,1)})\right|_{\mathcal{E}},$$
$$\alpha_2 = \left.(p_2 p_{(2,0)} - p_1 p_{(1,1)})\right|_{\mathcal{E}}.$$

The second pair of conditions (6.2) means that $\mu^*\omega$ is independent of q_1, q_2. Therefore, the factorization mapping μ can be identified with the projection to the plane $\{q_1 = q_2 = 0\} \subset J^2(\mathbb{R}^2)$. Thus,

$$\mu^*\omega = f \cdot [(p_{(1,1)}^2 - p_{(2,0)}p_{(0,2)})\,du + (p_1 p_{(0,2)} - p_2 p_{(1,1)})\,dp_1$$
$$\left. + (p_2 p_{(2,0)} - p_1 p_{(1,1)})\,dp_2]\right|_{\mathcal{E}}$$

where f is a function (coefficient of proportionality).

Set $f = (p_{(1,1)}^2 - p_{(2,0)}p_{(0,2)})^{-1}$. Since the equality $p_{(0,2)} = -p_{(2,0)}$ holds on the equation, we have

$$\omega = d\overline{u} - \frac{\overline{p}_2\overline{p}_{(1,1)} + \overline{p}_1\overline{p}_{(2,0)}}{\overline{p}_{(2,0)}^2 + \overline{p}_{(1,1)}^2}\,d\overline{p}_1 - \frac{\overline{p}_1\overline{p}_{(1,1)} - \overline{p}_2\overline{p}_{(2,0)}}{\overline{p}_{(2,0)}^2 + \overline{p}_{(1,1)}^2}\,d\overline{p}_2,$$

where $\overline{\xi}$ denotes the restriction of ξ to the equation. Taking into account that $\omega = dv - y_1\,dx_1 - y_2\,dx_2$, we obtain

$$v = \overline{u}, \quad x_i = \overline{p}_i, \quad y_1 = \frac{\overline{p}_2\overline{p}_{(1,1)} + \overline{p}_1\overline{p}_{(2,0)}}{\overline{p}_{(2,0)}^2 + \overline{p}_{(1,1)}^2}, \quad y_2 = \frac{\overline{p}_1\overline{p}_{(1,1)} - \overline{p}_2\overline{p}_{(2,0)}}{\overline{p}_{(2,0)}^2 + \overline{p}_{(1,1)}^2}.$$

From here, in turn, one can see that

$$\overline{u} = v, \quad \overline{p}_i = x_i, \quad \overline{p}_{(2,0)} = \frac{x_1 y_1 - x_2 y_2}{y_1^2 + y_2^2}, \quad \overline{p}_{(1,1)} = \frac{x_1 y_2 + x_2 y_1}{y_1^2 + y_2^2}.$$

In these new coordinates, restrictions of the Cartan forms to the equation are of the form

$$\overline{\omega}_0 = dv - x_1\,dq_1 - x_2\,dq_2,$$
$$\overline{\omega}_1 = dx_1 - \frac{x_1 y_1 - x_2 y_2}{y_1^2 + y_2^2}\,dq_1 - \frac{x_1 y_2 + x_2 y_1}{y_1^2 + y_2^2}\,dq_2, \tag{6.3}$$
$$\overline{\omega}_2 = dx_2 - \frac{x_1 y_2 + x_2 y_1}{y_1^2 + y_2^2}\,dq_1 + \frac{x_1 y_1 - x_2 y_2}{y_1^2 + y_2^2}\,dq_2.$$

Consider now in $\mathcal{E}/H$ a submanifold of the form V_g, which is the graph of the 1-jet of the function $g(x_1, x_2)$:

$$v = g(x_1, x_2), \quad y_i = \frac{\partial g}{\partial x_i}.$$

The same equations describe the manifold $\widetilde{V}_g = \mu^{-1}V_g \subset J^2(\mathbb{R}^2)$. Introduce the notation $\omega_i^g = \omega_i|_{\widetilde{V}_g}$. Since $\omega|_{V_g} = 0$, we have

$$\alpha_0\omega_0^g + \alpha_1\omega_1^g + \alpha_2\omega_2^g = 0,$$

i.e., the forms $\{\omega_i^g\}$ are linearly independent[10]. Therefore, one can assume, for example, that in the domain $\alpha_2 = -y_2 \neq 0$ the restriction of the Cartan distribution to $\widetilde{V}_g$ is determined by the forms ω_0^g, ω_1^g. Consequently, integrability conditions for this distribution are

$$d\omega_0^g \wedge \omega_0^g \wedge \omega_1^g = 0, \qquad d\omega_1^g \wedge \omega_0^g \wedge \omega_1^g = 0. \tag{6.4}$$

On the other hand, using (6.3), one can obtain the following expressions for the basis forms ω_0^g, ω_1^g:

$$\omega_0^g = g_{x_1}\, dx_1 + g_{x_2}\, dx_2 - x_1\, dq_1 - x_2\, dq_2,$$
$$\omega_1^g = dx_1 - \frac{x_1y_1 - x_2y_2}{y_1^2 + y_2^2}\, dq_1 - \frac{x_1y_2 + x_2y_1}{y_1^2 + y_2^2}\, dq_2.$$

Rather cumbersome computations show that when one substitutes these expressions to (6.4), the first condition in (6.4) fulfills identically while the second one is equivalent to the equation $y_2(g_{x_1x_1} + g_{x_2x_2}) = 0$. But in the domain we chose one has $y_2 \neq 0$ and therefore the equality

$$g_{x_1x_1} + g_{x_2x_2} = 0$$

coincides with the condition describing our functions. In other words, this is exactly the quotient equation for the Laplace equation[11].

Thus, the factorization procedure leads us to the Laplace equation again (note that the Laplace equation is linear and consequently its total symmetry algebra is infinite dimensional). From the above exposed theoretical reasons and from coincidence of the initial equation with the quotient one it follows that to any solution of the Laplace equation there corresponds a two-parameter family of solutions of the same equation.

In fact, if $g(x_1, x_2)$ is a solution of the Laplace equation, then the manifold $\widetilde{V}_g$ is an integral manifold of the Cartan distribution and the restrictions $D_{q_i}^g$ of the total derivatives D_{q_i} to $\widetilde{V}_g$ are of the form

$$
\begin{aligned}
D_{q_1}^g &= \left(\frac{\partial}{\partial q_1} + p_1\frac{\partial}{\partial u} + p_{(2,0)}\frac{\partial}{\partial p_1} + p_{(1,1)}\frac{\partial}{\partial p_2} \right)\Bigg|_{\widetilde{V}_g} \\
&= \frac{\partial}{\partial q_1} + \frac{x_1y_1 - x_2y_2}{y_1^2 + y_2^2}\frac{\partial}{\partial x_1} + \frac{x_1y_2 + x_2y_1}{y_1^2 + y_2^2}\frac{\partial}{\partial x_2}, \\
D_{q_2}^g &= \left(\frac{\partial}{\partial q_2} + p_1\frac{\partial}{\partial u} + p_{(1,1)}\frac{\partial}{\partial p_1} + p_{(0,2)}\frac{\partial}{\partial p_2} \right)\Bigg|_{\widetilde{V}_g} \\
&= \frac{\partial}{\partial q_2} + \frac{x_1y_2 + x_2y_1}{y_1^2 + y_2^2}\frac{\partial}{\partial x_1} - \frac{x_1y_1 - x_2y_2}{y_1^2 + y_2^2}\frac{\partial}{\partial x_2},
\end{aligned}
\tag{6.5}
$$

where $y_i = \partial g/\partial x_i$. We shall look for integral manifolds of the form

$$q_1 = \varphi(x_1, x_2), \qquad q_2 = \psi(x_1, x_2)$$

---

[10]It is easily seen that $\alpha_0 = 1$, $\alpha_1 = -y_1$, $\alpha_2 = -y_2$, since $\omega = dv - y_1\, dx_1 - y_2\, dx_2$.

[11]Note that the result is independent of the choice of the domain $y_2 \neq 0$, which was convenient for computations; for another domain the quotient equation will be the same.

in $\widetilde{V}_g$. The functions φ are ψ obtained from the conditions

$$D_{q_i}^g(q_1 - \varphi) = 0, \quad D_{q_i}^g(q_2 - \psi) = 0, \qquad i = 1, 2.$$

The last system can be rewritten explicitly using formulas (6.5). Resolving it with respect to $\partial\varphi/\partial x_i$, $\partial\psi/\partial x_i$, we obtain the following formulas

$$\frac{\partial\varphi}{\partial x_1} = \left(x_1\frac{\partial g}{\partial x_1} - x_2\frac{\partial g}{\partial x_2}\right)r, \qquad \frac{\partial\varphi}{\partial x_2} = \left(x_1\frac{\partial g}{\partial x_2} + x_2\frac{\partial g}{\partial x_1}\right)r,$$

$$\frac{\partial\psi}{\partial x_1} = \left(x_1\frac{\partial g}{\partial x_2} + x_2\frac{\partial g}{\partial x_1}\right)r, \qquad \frac{\partial\psi}{\partial x_2} = \left(x_2\frac{\partial g}{\partial x_2} - x_1\frac{\partial g}{\partial x_1}\right)r,$$

where $r = (x_1^2 + x_2^2)^{-1}$. Thus, when g is given, the functions φ, ψ are obtained by straightforward integration.

For example, if $g = x_2$, then $\varphi = \arctan(x_2/x_1) + C_1$, $\psi = \ln(x_1^2 + x_2^2)/2 + C_2$. Coming back to the standard coordinates on $J^2(\mathbb{R}^2)$, we shall have the following relations valid on $\widetilde{V}_g$ (where $g = x_2$):

$$u = p_2, \quad p_{(2,0)} = -p_{(0,2)}, \quad p_{(2,0)} = -p_2, \quad p_{(1,1)} = p_1,$$

$$C_1 + q_1 = \arctan\frac{p_2}{p_1}, \quad C_2 + q_2 = \frac{\ln(p_1^2 + p_2^2)}{2}, \qquad C_1, C_2 \in \mathbb{R}.$$

Expressing $p_2 = u$ via q_i, we shall obtain the desired two-parameter family of the Laplace equation solutions:

$$u = e^{q_2 + C_2}\sin(q_1 + C_1).$$

EXAMPLE 6.2. Consider the Laplace equation $\mathcal{E} = \{p_{(2,0)} + p_{(0,2)} = 0\}$ again. Let the group H correspond to the Lie algebra $\mathcal{H}$ generated by the fields $X_{q_1} = \partial/\partial p_1 + q_1\partial/\partial u$ and $X_{q_2} = \partial/\partial p_2 + q_2\partial/\partial u$.

Acting in the same way as we did in Example 6.1, we shall see that the quotient equation again coincides with the Laplace equation. To any solution g of the quotient equation there corresponds a two-parameter family of the initial one. These families can be found by solving the following system of equations:

$$\begin{cases} u = g(q_1, q_2) + q_1 p_1 + q_2 p_2, \\ p_{(0,2)} = -p_{(2,0)}, \\ p_{(2,0)} = -\dfrac{q_1 g_{q_1} - q_2 g_{q_2}}{q_1^2 + q_2^2}, \\ p_{(1,1)} = -\dfrac{q_2 g_{q_1} + q_1 g_{q_2}}{q_1^2 + q_2^2}, \\ p_1 = \varphi(q_1, q_2) + C_1, \\ p_2 = \psi(q_1, q_2) + C_2, \end{cases}$$

where C_1 and C_2 are arbitrary constants. In turn, the functions φ and ψ are determined by the following relations:

$$g_{q_1}\,dq_1 + g_{q_2}\,dq_2 + q_1\,dp_1 + q_2\,dp_2 = q_1\,d(p_1 - \varphi) + q_2\,d(p_2 - \psi),$$

$$(q_1^2 + q_2^2)\,dp_1 + (g_{q_1}q_1 - g_{q_2}q_2)\,dq_1 + (g_{q_2}q_1 + g_{q_1}q_2)\,dq_2 = (q_1^2 + q_2^2)\,d(p_1 - \varphi).$$

In particular, for $g = q_1$ we obtain the following two-parameter family of the Laplace equation solutions:

$$u = q_1\left(C_1 - \frac{\ln(q_1^2 + q_2^2)}{2}\right) + q_2\left(C_2 + \arctan\frac{q_2}{q_1}\right),$$

EXAMPLE 6.3. Consider the wave equation $p_{(2,0)} - p_{(0,2)} = 0$ and the group H of translations along the independent variables. The corresponding Lie algebra $\mathcal{H}$ is generated by the fields $\partial/\partial q_1$ and $\partial/\partial q_2$.

In this case, the quotient equation also coincides with the initial wave equation. Any solution g of the quotient equation corresponds to a two-parameter family of solutions of the initial equation. This family can be found from the relations

$$
\begin{cases}
u = g(p_1, p_2), \\
p_{(2,0)} = p_{(0,2)}, \\
p_{(2,0)} = \dfrac{p_1 g_{p_1} - p_2 g_{p_2}}{g_{p_1}^2 - g_{p_2}^2}, \\
p_{(1,1)} = \dfrac{p_2 g_{p_1} - p_1 g_{p_2}}{g_{p_1}^2 - g_{p_2}^2}, \\
q_1 = C_1 + \varphi(p_1, p_2), \\
q_2 = C_2 + \psi(p_1, p_2),
\end{cases}
$$

where C_1 and C_2 are arbitrary real constants. In turn, the functions φ and ψ are determined from the relations

$$
d\varphi = \frac{p_2 g_{p_2} - p_1 g_{p_1}}{p_2^2 - p_1^2}\, dp_1 + \frac{p_2 g_{p_1} - p_1 g_{p_2}}{p_2^2 - p_1^2}\, dp_2,
$$

$$
d\psi = \frac{p_2 g_{p_1} - p_1 g_{p_2}}{p_2^2 - p_1^2}\, dp_1 + \frac{p_2 g_{p_2} - p_1 g_{p_1}}{p_2^2 - p_1^2}\, dp_2.
$$

In particular, for $g = q_1$ we obtain the following two-parameter family of the wave equation solutions:

$$
u = e^{q_1 + C_1} \cosh(q_2 + C_2).
$$

EXAMPLE 6.4. Let us consider the heat equation $p_1 = p_{(0,2)}$. Choose the group H of translations along the independent variables with the Lie algebra $\mathcal{H}$ generated by the fields $\partial/\partial q_1$ and $\partial/\partial q_2$.

In suitable coordinates x, y, v, the quotient equation is a quasilinear parabolic equation of the form

$$
(xv_x)^2 v_{xx} - 2(xv_x)(y - xv_y) v_{xy} + (y - xv_y)^2 v_{yy} = 0.
$$

EXERCISE 6.1. Show that the quotient equation of the heat equation over the group of scale symmetries $u \mapsto \tau u$ coincides with the Burgers equation.

EXAMPLE 6.5. Consider the Burgers equation $p_2 + u p_1 + p_{(2,0)} = 0$. Let H be the group whose Lie algebra is generated by the fields $\partial/\partial q_1$ and $-q_2 \partial/\partial q_1 + \partial/\partial u$, i.e., by the translation along q_1 and by the Galilean symmetry.

In suitable coordinates x, y, v the quotient equation is of the form

$$
v_{xx} - \frac{v_x^2}{2v} + v_x \frac{x^2}{2v} + \frac{v_y}{2v} - 3x = 0.
$$

EXAMPLE 6.6 (evolution equations in one spatial variable). We expose here the results of the paper [116], where evolution equations with one space variable were studied. Most part of examples considered in this paper are particular cases of the above described general construction. But specifics of the equation $u_t = F(t, x, u, u_x, u_{xx}, \dots)$ allows one to obtain complete answers related to the factorization procedure.

The functions constant on the orbits of the group H or of its action on higher order jets are called *scalar differential invariants* [121, 138, 3]. In [116], it was

proved that if $\dim \mathcal{H} = m$, then in the generic situation the number of independent differential invariants on $J^m(\mathbb{R}^2)$ equals 3.

These three invariants are chosen as one dependent and two independent variables (i.e., as coordinates in $J^0(\mathbb{R}^2)$) for the quotient equation. For total derivatives along new coordinates "invariant derivations" are chosen, i.e., linear combinations $pD_t + qD_x$, such that $[pD_t + qD_x, h] = 0$ for an arbitrary generating function $h \in \mathcal{H}$. Naturally, the choice both of invariants and of invariant derivations is not unique, but reasonable rules are given to obtain quotient equations of the simplest form. In particular, using this approach, the following results were obtained.

1. Let a Lie algebra $\mathcal{G}_0$ consisting of symmetries of the equation $\mathcal{E}$ contain an ideal $\mathcal{G}_1$. If μ_i denotes the factorization determined by the algebra $\mathcal{G}_i$ and $\mathcal{E}_i$ is the corresponding quotient equation, then the algebra of all symmetries of the equation $\mathcal{E}_0$ contains a subalgebra $\mathcal{H} \simeq \mathcal{G}_0/\mathcal{G}_1$. For the factorization μ over the subalgebra $\mathcal{H}$ one has $\mu_0 = \mu \circ \mu_1$.

2. The result of a second order evolution equation factorization is either an evolution equation, or an equation, which can be transformed to the equation

$$v_{tt} + 2vv_{xt} + v^2 v_{xx} + \Phi(v_t, v_x, v, x, t) = 0.$$

by point transformations. In particular, it is always parabolic.

3. Equations $u_t = u_{xx} + h(t,x)u$ always possess symmetries of the form $\alpha(t,x)\partial/\partial u$, where $\alpha(t,x)$ is a solution (gauge by a solution). Factorization over any finite-dimensional algebra consisting of such symmetries leads to a linear equation of the same type, but, in general, with a different potential h. In particular, when the algebra is one-dimensional, the quotient equation is of the form

$$v_t = v_{xx} + (h + 2(\ln \alpha)_{xx})v.$$

It turns out that factorization in this case coincides with the "dressing procedure" used in the inverse scattering problem. The pass from h to $h + 2(\ln \alpha)_{xx}$ is exactly the Darboux transformation.

7. Exterior and interior symmetries

Up to now, we understood symmetries as Lie transformations in $J^k(\pi)$. Such an approach correspond to studying equations as if from "outside". Conceptually, the other approach is more reasonable. It is based on study of the restriction $\mathcal{C}(\mathcal{E})$ of the Cartan distribution to the equation $\mathcal{E} \subset J^k(\pi)$ and of diffeomorphisms of $\mathcal{E}$ preserving $\mathcal{C}(\mathcal{E})$. This point of view corresponds to interior equation geometry and seems to be more adequate to study individual equations, though far less convenient with regards to computations. Luckily, for a very broad class of equations these two approaches happen to be equivalent. In this section, we describe sufficient conditions for this equivalence. We say that equations, for which the equivalence holds, are *rigid*. The exposition below is based on the results from [**60, 130**].

Let us introduce some notions needed below. Recall that the distribution $\mathcal{C}(\mathcal{E})$ is defined at each point as follows: $\mathcal{C}(\mathcal{E})_\theta = \mathcal{C}^k \cap T_\theta(\mathcal{E})$.

DEFINITION 7.1. A diffeomorphism of the manifold $\mathcal{E}$ is called an *interior symmetry*, if it preserves the distribution $\mathcal{C}(\mathcal{E})$. A vector field on $\mathcal{E}$ is called an *infinitesimal interior symmetry*, if the corresponding one-parameter group preserves $\mathcal{C}(\mathcal{E})$.

Denote by $\mathrm{Sym}_e(\mathcal{E})$ the group of *exterior* symmetries defined in §4 and by $\mathrm{Sym}_i(\mathcal{E})$ the group of interior symmetries of the equation $\mathcal{E}$. The restriction operation leads to the natural mapping

$$\mathrm{Sym}_e\,\mathcal{E} \longrightarrow \mathrm{Sym}_i\,\mathcal{E}.$$

Equations for which this mapping is an epimorphism, are called *common*. An equation is rigid, if the mapping is bijective. Of course, rigid equations are common, but the converse is not true (for example, first order equations in one dependent variable, see below).

DEFINITION 7.2. An equation $\mathcal{E} \subset J^k(\pi)$ is called $\mathcal{C}$-*general*, if:

1. The set $\pi_{k,k-1}(\mathcal{E})$ is everywhere dense in $J^{k-1}(\pi)$.
2. The fibers of the bundle $\pi_{\mathcal{E}} = \pi_{k,k-1}|_{\mathcal{E}} : \mathcal{E} \to \pi_{k,k-1}(\mathcal{E})$ are connected and they and only they are integral manifolds of maximal dimension for the distribution $\mathcal{C}(\mathcal{E})$.

In fact, any interior symmetry of a $\mathcal{C}$-general equation $\mathcal{E} \subset J^k(\pi)$ determines a transformation in the space $J^{k-1}(\pi)$. Namely, the following fact is valid:

PROPOSITION 7.1. *If $\mathcal{E} \subset J^k(\pi)$ is a $\mathcal{C}$-general equation and A is its interior symmetry, then there exists a diffeomorphism A' of the space $J^{k-1}(\pi)$ such that the diagram*

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\ \ A\ \ } & \mathcal{E} \\
{\scriptstyle \pi_{\mathcal{E}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\
J^{k-1}(\pi) & \xrightarrow{\ \ A'\ \ } & J^{k-1}(\pi)
\end{array}
\tag{7.1}
$$

is commutative.

PROOF. Let $\theta_{k-1} \in \pi_{k,k-1}(\mathcal{E})$. Then by definition, the set $\pi_{k,k-1}^{-1}(\theta_{k-1})$ is an integral manifold of maximal dimension for the distribution $\mathcal{C}(\mathcal{E})$. Since A is an automorphism of the distribution $\mathcal{C}(\mathcal{E})$, the set $A(\pi_{k,k-1}^{-1}(\theta_{k-1}))$ is also an integral manifold of maximal dimension and consequently is of the form $\pi_{k,k-1}^{-1}(\theta'_{k-1})$ for some point $\theta'_{k-1} \in \pi_{k,k-1}(\mathcal{E})$. Let us set $A'(\theta_{k-1}) = \theta'_{k-1}$. Commutativity of the diagram is obvious. $\qquad\square$

Let us understand now, when the mapping A' is a Lie transformation of the space $J^{k-1}(\pi)$. To this end, we shall need another two notions.

DEFINITION 7.3. An equation $\mathcal{E} \subset J^k(\pi)$ is called $\mathcal{C}$-*complete*, if for any point $\theta_{k-1} \in \pi_{k,k-1}(\mathcal{E})$ the span of the set[12]

$$\bigcup_{\theta_k \in \pi_{\mathcal{E}}^{-1}(\theta_{k-1})} L_{\theta_k} \subset T_{\theta_{k-1}} J^{k-1}(\pi)$$

coincides with the Cartan plane $\mathcal{C}_{\theta_{k-1}}$.

Thus, an equation is $\mathcal{C}$-complete, if it completely determines Cartan planes at the points of $\pi_{k,k-1}(\mathcal{E})$.

Denote by $\mathcal{E}^{(l)} \subset J^{k+l}(\pi)$ the *l-prolongation*[13] of the equation $\mathcal{E}$, i.e., the equation consisting of all *differential consequences* of order $\leq l$ of the equation $\mathcal{E}$.

---

[12]Recall that by L_{θ_k} we denote the R-plane in $J^{k-1}(\pi)$ determined by the point $\theta_k \in J^k(\pi)$, see §2.

[13]Geometric definition see in §3 of Ch. 4.

DEFINITION 7.4. An equation $\mathcal{E} \subset J^k(\pi)$ is called *l-solvable* (at a point $\theta_k \in \mathcal{E}$), if the mapping $\pi_{l+k,k} \colon \mathcal{E}^{(l)} \to \mathcal{E}$ is a surjection (resp., if $\mathcal{E}^{(l)} \cap \pi_{l+k,k}^{-1}(\theta_k) \neq \varnothing$).

PROPOSITION 7.2. *If a $\mathcal{C}$-general equation is* 1*-solvable and $\mathcal{C}$-complete, then the above constructed mapping A' is a Lie transformation. Moreover, the* 1*-lifting of the mapping A' restricted to $\mathcal{E}$ coincides with A.*

PROOF. Since the equation $\mathcal{E}$ is $\mathcal{C}$-complete, to prove the first statement it suffices to show that the conditions $v \in L_{\theta_k}$, $\theta_k \in \pi_{\mathcal{E}}^{-1}(\theta_{k-1})$, $\theta_{k-1} \in \pi_{k,k-1}(\mathcal{E})$ imply that $A'_*(v) \in \mathcal{C}_{A'(\theta_{k-1})}$.

Let a point $\theta_{k+1} \in \mathcal{E}^{(1)}$ be such that $\pi_{k+1,k}(\theta_{k+1}) = \theta_k$. Then $L_{\theta_{k+1}} \subset T_{\theta_k}(\mathcal{E})$ and $(\pi_{k,k-1})_*(L_{\theta_{k+1}}) = L_{\theta_k}$. Therefore, there exists a vector $v_1 \in L_{\theta_{k+1}}$ such that $(\pi_{\mathcal{E}})_*(v_1) = v$. Then $A'_*(v) = A'_*((\pi_{\mathcal{E}})_*(v_1)) = (\pi_{\mathcal{E}})_*(A_*(v_1))$. But A is an automorphism of the distribution $\mathcal{C}(\mathcal{E})$ and consequently $A_*(v_1) \in \mathcal{C}_{A(\theta_k)}(\mathcal{E}) \subset \mathcal{C}_{A(\theta_k)}$.

On the other hand, $(\pi_{k,k-1})_*(\mathcal{C}_{A(\theta_k)}) = L_{A(\theta_k)}$, from where it follows that $A'_*(v) \in L_{A(\theta_k)} \subset \mathcal{C}_{A'(\theta_{k-1})}$.

The diffeomorphism A coincides with the restriction of the lifting of the Lie transformation A' to $\mathcal{E}$. This means that $A'_*(L_{\theta_k}) = L_{A(\theta_k)}$, where $\theta_k \in \mathcal{E}$. But the above arguments show that $A'_*(L_{\theta_k}) \subset L_{A(\theta_k)}$. Since A' is a diffeomorphism and $\dim L_{\theta_k} = \dim L_{A(\theta_k)} = n$, the second statement is also proved. $\square$

Equations satisfying the assumptions of Proposition 7.2 will be called *normal*.

Let us explain why normal equations are rigid. If an equation is 1-solvable, the condition of $\mathcal{C}$-completeness can be changed by the following one: the span of the spaces $(\pi_{k,k-1})_*(\mathcal{C}_{\theta_k}(\mathcal{E})) \subset \mathcal{C}_{\theta_{k-1}}$, where $\theta_k \in \pi_{\mathcal{E}}^{-1}(\theta_{k-1})$, coincides with $\mathcal{C}_{\theta_{k-1}}$. In fact, in this case we have $L_{\theta_{k+1}} \subset \mathcal{C}_{\theta_k}(\mathcal{E})$ for $\theta_{k+1} \in \mathcal{E}^{(1)}$ and $(\pi_{k,k-1})_*(L_{\theta_{k+1}}) = L_{\theta_k} = (\pi_{k,k-1})_*(\mathcal{C}_{\theta_k})$. Therefore, $(\pi_{k,k-1})_*(\mathcal{C}_{\theta_k}(\mathcal{E})) = L_{\theta_k}$.

The last remark shows that normality is an "interior" property, i.e., it can be formulated only in terms of the manifold $\mathcal{E}$ and of the distribution $\mathcal{C}(\mathcal{E})$ on this manifold. In fact, the integral manifolds of maximal dimension determine in this case a bundle ν, whose base B can be endowed with the distribution

$$B \ni b \longmapsto \mathcal{C}_b = \text{span of the planes } \{\, \nu_*(\mathcal{C}_y(\mathcal{E})) \mid y \in \nu^{-1}(b) \,\}.$$

In the situation under consideration, the base $B = J^{k-1}(\pi)$ and the distribution described is $J^{k-1}(\pi)$ and the Cartan distribution respectively. If now one considers the manifold B^1 of all n-dimensional maximal integral planes of the distribution constructed on B, then $\mathcal{E} \subset B^1$ and $B^1 = J^k(\pi)$. This procedure shows, that normality allows one to reconstruct the "environment" of the equation, i.e., to reconstruct the embedding $\mathcal{E} \subset J^k(\pi)$ using the distribution $\mathcal{C}(\mathcal{E})$ on $\mathcal{E}$ only. Thus we proved the following theorem.

THEOREM 7.3. *Normal equations are rigid.*

Let us now write down explicitly the conditions of 1-solvability and $\mathcal{C}$-generality constituting the normality property.

Let the equation $\mathcal{E}$ be given as the zero set of the vector function $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{\theta}, \boldsymbol{u}, \boldsymbol{p}_\sigma)$, or $\mathcal{E} = \{\, F_i(\boldsymbol{\theta}, \boldsymbol{u}, \boldsymbol{p}_\sigma) = 0 \mid 1 \leq i \leq r \,\}$. Then locally $\boldsymbol{F}$ can be considered as the mapping $\boldsymbol{F} \colon J^k(\pi) \to J^0(\pi')$, where π' is an r-dimensional bundle over the same base. Let a point $\theta_{k+1} \in J^{k+1}(\pi)$ be represented as a pair (θ, L), where L is an R-plane at the point $\theta_k = \pi_{k+1,k}(\theta_{k+1}) \in J^k(\pi)$. Then for almost all θ_{k+1}

the plane $\boldsymbol{F}_*(L)$ will be an R-plane at the point $G(\theta_k)$. Consequently, setting $\boldsymbol{F}^{(1)}(\theta_{k+1}) = (F(\theta_k), F_*(L))$, we obtain a mapping $J^{k+1}(\pi) \to J^1(\pi')$. By definitions, the following diagram is commutative

$$
\begin{array}{ccccc}
J^{k+1}(\pi) & \xrightarrow{\pi_{k+1,k}} & J^k(\pi) & \xrightarrow{\pi_k} & M \\
\boldsymbol{F}^{(1)} \downarrow & & \boldsymbol{F} \downarrow & & \downarrow \text{id} \\
J^1(\pi') & \xrightarrow{\pi'_{1,0}} & J^0(\pi') & \xrightarrow{\pi'_0} & M
\end{array}
\qquad (7.2)
$$

From commutativity of this diagram it follows that the intersection $\mathcal{E}^{(1)} \cap \pi_{k+1,k}^{-1}(\theta_k)$ is not empty, if the restriction of $F^{(1)}$ to the fiber,

$$
\boldsymbol{F}^{(1)} \colon \pi_{k+1,k}^{-1}(\theta_k) \longrightarrow (\pi'_{1,0})^{-1}(\boldsymbol{F}(\theta_k)),
$$

is surjective. But this restriction is exactly the leading symbol of the mapping $\boldsymbol{F}$. In coordinates, it is represented by the matrix $\partial G_i / \partial p_\sigma^j$, $|\sigma| = k$. Hence, the equation is 1-solvable, if the senior symbol of the determining vector function is surjective.

If the equation is determined, i.e., the number of equations coincides with the number of dependent variables, then the dimension of fibers of the projection $J^{k+1}(\pi) \to J^k(\pi)$ is greater than the dimension of fibers of the projection $J^1(\pi') \to J^0(\pi')$. Therefore, in a generic situation the needed surjectivity always takes place. If, for example, $\dim \pi = \dim \pi' = 1$ and $\mathcal{E} = \{g = 0\}$, then the symbol is surjective provided the equality $\partial g / \partial p_\sigma \neq 0$ holds at least for one multi-index σ, $|\sigma| = k$. Thus, 1-solvability is a rather weak condition.

The $\mathcal{C}$-generality condition relates to simple dimension reasons.

PROPOSITION 7.4. *Let $\mathcal{E} \subset J^k(\pi)$, $\dim \pi = m$, $\dim M = n$, and fibers of the projection $\pi_{\mathcal{E}}$ be connected. Then $\mathcal{E}$ is a $\mathcal{C}$-general equation, if the inequality*

$$
\operatorname{codim} \mathcal{C} \leq m \frac{(n+k-2)!}{(k-1)!(n-1)!} - 2
$$

holds.

PROOF. Indeed, the number

$$
\lambda = m \frac{(n+k-2)!}{(k-1)!\,(n-1)!} - 1
$$

equals the difference between dimensions of integral manifolds projecting to $J^{k-1}(\pi)$ with dimensions $r = 0$ and 1 (see Proposition 2.5). Therefore, when restricting the Cartan distribution to $\mathcal{E}$, dimension of maximal integral manifolds with $r = 0$ will be diminished at least by λ. □

If $\mathcal{E}$ is a determined equation, then the assumptions of the previous proposition are satisfied in all cases except for: (a) $k = 1$, (b) $n = 1$, (c) $m = 1$, $k = n = 2$.

Let us consider examples of equations, where exterior and interior symmetries do not coincide.

EXAMPLE 7.1. Let $\theta_k \in J^k(\pi)$, where $k \geq 0$ for $\dim \pi > 1$ or $k \geq 1$ for $\dim \pi = 1$. Let further $\mathcal{E} = \pi_{k+1,k}^{-1}(\theta_k)$. Then $\mathcal{E}$ is not a rigid and even not a common equation.

In fact, for any point $\theta_{k+1} \in \mathcal{E}$ the tangent plane $T_{\theta_{k+1}}(\mathcal{E})$ lies in the Cartan distribution and consequently $\mathcal{C}_{\theta_{k+1}}(\mathcal{E}) = T_{\theta_{k+1}}(\mathcal{E})$. Therefore, the set of interior

symmetries of this equation coincides with the group of its diffeomorphisms. But any exterior symmetry of the equation $\mathcal{E}$ is determined by some Lie transformation of the space $J^k(\pi)$ leaving the point θ_k fixed.

EXAMPLE 7.2. First order equations in one dependent variable are common, but not rigid.

Let $k = \dim \pi = 1$. We confine ourselves with the case of determined equations, i.e., when $\mathrm{codim}\,\mathcal{E} = 1$. Let us first recall some facts from Ch. 2.

The contact structure on $J^1(\pi)$ is determined by any 1-form $\omega_\lambda = \lambda U_1(\pi)$, where λ is a function from $C^\infty(J^1(\pi))$ vanishing nowhere. To escape topological problems, we shall assume that there exists a function λ such that the form $d\,\omega_\lambda|_{\mathcal{E}}$ is nondegenerate. This is always true locally. In what follows, we fix λ and set $\omega = \omega_\lambda$, $\widetilde{\omega} = \omega_\lambda|_{\mathcal{E}}$.

It is not difficult now to describe the structure of interior symmetries of the equation $\mathcal{E}$. Obviously, $X \in \mathrm{Sym}_i\,\mathcal{E}$ if and only if $X(\widetilde{\omega}) = \mu\widetilde{\omega}$, where $\mu \in C^\infty(\mathcal{E})$. Since $X(\widetilde{\omega}) = X \lrcorner\, d\widetilde{\omega} + d(X \lrcorner\, \widetilde{\omega})$, it can be easily seen that $X \in \mathrm{Sym}_i\,\mathcal{E}$ if and only if

$$\mu\widetilde{\omega} = X \lrcorner\, d\widetilde{\omega} + d\widetilde{f}, \tag{7.3}$$

where $\widetilde{f}$ denotes $X \lrcorner\, \widetilde{\omega}$.

The form $d\widetilde{\omega}$ is nondegenerate and therefore the mapping

$$G \colon\ \mathrm{D}(\mathcal{E}) \longrightarrow \Lambda^1(\mathcal{E}), \qquad G(Y) = -Y \lrcorner\, d\widetilde{\omega},$$

is an isomorphism of $C^\infty(\mathcal{E})$-modules of vector fields and 1-forms on $\mathcal{E}$. Applying the mapping G^{-1} to equation (7.3), we find that $X \in \mathrm{Sym}_i\,\mathcal{E}$ if and only if

$$\mu Y_{\widetilde{\omega}} = Y_{d\widetilde{f}} - X,$$

where $Y_\theta = G^{-1}(\theta)$, $\theta \in \Lambda^1(\mathcal{E})$.

Note now that $Y_\theta \lrcorner\, \theta = -Y_\theta \lrcorner\, (Y_\theta \lrcorner\, d\widetilde{\omega}) = 0$. Therefore,

$$(\mu Y_{\widetilde{\omega}})(\widetilde{\omega}) = (\mu Y_{\widetilde{\omega}}) \lrcorner\, d\widetilde{\omega} + d(\mu Y_{\widetilde{\omega}} \lrcorner\, \widetilde{\omega}) = -\mu\widetilde{\omega},$$

i.e., $\mu Y_{\widetilde{\omega}} \in \mathrm{Sym}_i\,\mathcal{E}$ (as it is easily seen, the fields of the form $\mu Y_{\widetilde{\omega}}$ are directed along the characteristics of the equation $\mathcal{E}$). Consequently, $X \in \mathrm{Sym}_i\,\mathcal{E}$ if and only if $Y_{d\widetilde{f}} \in \mathrm{Sym}_i\,\mathcal{E}$. But

$$Y_{d\widetilde{f}}(\widetilde{\omega}) = Y_{d\widetilde{f}} \lrcorner\, d\widetilde{\omega} + d(Y_{d\widetilde{f}} \lrcorner\, \widetilde{\omega}) = d((X + \mu Y_{\widetilde{\omega}}) \lrcorner\, \widetilde{\omega}) - d\widetilde{f} = d(X \lrcorner\, \widetilde{\omega}) - d\widetilde{f} = 0.$$

Hence, $Y_{d\widetilde{f}} \in \mathrm{Sym}_i\,\mathcal{E}$ if and only if $Y_{d\widetilde{f}} \lrcorner\, \widetilde{\omega} = \widetilde{f}$.

Let us rewrite the last condition in a more clear manner:

$$Y_{d\widetilde{f}} \lrcorner\, \widetilde{\omega} = -Y_{d\widetilde{f}} \lrcorner\, Y_{\widetilde{\omega}} \lrcorner\, d\widetilde{\omega} = Y_{\widetilde{\omega}} \lrcorner\, Y_{d\widetilde{f}} \lrcorner\, d\widetilde{\omega} = Y_{\widetilde{\omega}} \lrcorner\, (-d\widetilde{f}) = -Y_{\widetilde{\omega}}(\widetilde{f}).$$

Thus, we have proved the following result:

PROPOSITION 7.5. *A field $X \in \mathrm{D}(\mathcal{E})$ is an interior symmetry of the equation $\mathcal{E}$ if and only if it is of the form*

$$X = \mu Y_{\widetilde{\omega}} + Y_{d\widetilde{f}},$$

where μ is a function and $Y_{\widetilde{\omega}}(\widetilde{f}) = -\widetilde{f}$. We also have $X(\widetilde{\omega}) = \mu\widetilde{\omega}$.

Recall that the 2-form $d\widetilde{\omega}$ determines a symplectic structure on $\mathcal{E}$ and the field Y_{dh}, $h \in C^\infty(\mathcal{E})$, is a Hamiltonian vector field with the Hamiltonian h. Therefore, a Hamiltonian field on $\mathcal{E}$ is an interior, if the Hamiltonian h satisfies the condition $Y_{\widetilde{\omega}}(h) = -h$.

Let us now describe the relations between the algebras $\mathrm{Sym}_i \mathcal{E}$ and $\mathrm{Sym}_e \mathcal{E}$. Recall that a contact field X on $J^1(\pi)$ is uniquely determined by its generating function $f = X \lrcorner \omega$ and is a unique solution of the system consisting of the following two equations:

$$X \lrcorner d\omega + df = X_1(f)\omega, \qquad f = X \lrcorner \omega, \tag{7.4}$$

where the field $X_1 = \partial/\partial x_1$ is uniquely determined by the following conditions:

$$X_1 \lrcorner \omega = 1, \qquad X_1 \lrcorner d\omega = 0.$$

THEOREM 7.6. *A contact field X_f is an exterior symmetry of a first order equation $\mathcal{E}$ in one dependent variable if and only if $Y_{\widetilde{\omega}}(\widetilde{f}) = -\widetilde{f}$.*

PROOF. From (7.4) one obtains that

$$X_g(f) + X_g \lrcorner X_f \lrcorner d\omega = X_g(df + X_f \lrcorner d\omega) = X_g \lrcorner X_1(f)\omega = gX_1(f).$$

In a similar way, the equality $X_f(g) + X_f \lrcorner X_g \lrcorner d\omega = fX_1(g)$ holds. Since $X_f \lrcorner X_g \lrcorner d\omega + X_g \lrcorner X_f \lrcorner d\omega = 0$, we have

$$X_f(g) + X_g(f) = fX_1(g) + gX_1(f) = X_1(fg).$$

Let locally $\mathcal{E} = \{g = 0\}$, $dg_\theta \neq 0$, $\theta \in \mathcal{E}$. Then, if X is tangent to $\mathcal{E}$, from the last equality it follows that

$$X_f(g)|_{\mathcal{E}} = -\widetilde{X}_g(\widetilde{f}) + \widetilde{f}\, X_1(g)|_{\mathcal{E}}\,, \tag{7.5}$$

where $\widetilde{X}$ denotes the restriction of the field X to $\mathcal{E}$ (recall that the field X_g is always tangent to the hypersurface $\{g = 0\}$). Restricting the equality $-X_g \lrcorner d\omega = dg - X_1(g)\omega$ to $\mathcal{E}$, we obtain

$$-\widetilde{X}_g \lrcorner d\widetilde{\omega} = \alpha\widetilde{\omega}, \qquad \alpha = -X_1(g)|_{\mathcal{E}}\,.$$

Therefore, using the above introduced notation, one has

$$\widetilde{X}_g = \alpha Y_{\widetilde{\omega}}$$

and consequently (7.5) can be rewritten in the form

$$X_f(g)|_{\mathcal{E}} = -\alpha Y_{\widetilde{\omega}}(\widetilde{f}) - \alpha\widetilde{f}.$$

Thus, the equality $X_f(g)|_{\mathcal{E}} = 0$ (i.e., the condition that X_f is tangent to the manifold $\mathcal{E}$) will hold if and only if $Y_{\widetilde{\omega}}(\widetilde{f}) = -\widetilde{f}$, since $\alpha = -X_1(g)|_{\mathcal{E}} \neq 0$. The last fact is implied by $X_1|_\theta \notin T_\theta(\mathcal{E})$, since $X_1 \in \ker d\omega|_\theta$, while the form $d\omega|_{\mathcal{E}}$ is nondegenerate. $\square$

COROLLARY 7.7. *For first order equations in one dependent variable, the mapping $\mathrm{Sym}_e \mathcal{E} \to \mathrm{Sym}_i \mathcal{E}$ is always epimorphic.*

PROOF. Consider the function $h = f + \lambda g$, $\lambda \in C^\infty(J^1(\pi))$, $\lambda \neq 0$, and let us try to find the function λ to satisfy the equality $\widetilde{X}_h = \mu Y_{\widetilde{w}} + Y_{d\widetilde{f}}$. By the theorem proved, the field $\widetilde{X}_f$ is an interior symmetry of the equation $\mathcal{E}$ and consequently, by Proposition 7.5, the equality

$$\widetilde{X}_f = \nu Y_{\widetilde{\omega}} + Y_{d\widetilde{f}}, \qquad \nu \in C^\infty(\mathcal{E})$$

holds. Further, as it was shown in Ch. 2, one has $X_{\varphi\psi} = \varphi X_\psi + \psi X_\varphi - \varphi\psi X_1$. From this equality and from (7.5) we obtain

$$\widetilde{X}_{\lambda g} = \widetilde{\lambda}\widetilde{X}_g = \alpha\widetilde{\lambda}Y_{\widetilde{\omega}}, \qquad \widetilde{\lambda} = \lambda|_{\mathcal{E}}.$$

Therefore, choosing λ to satisfy

$$\widetilde{\lambda} = \frac{\mu - \nu}{\alpha}, \qquad \alpha = -\left.X_1(g)\right|_{\mathcal{E}},$$

we can assume that $\widetilde{X}_h = \mu Y_{\widetilde{\omega}} + Y_{d\widetilde{f}}$. $\qquad\qquad\qquad\qquad\qquad\square$

Let $\widetilde{X}_f = \mu Y_{\widetilde{\omega}} + Y_{d\widetilde{f}}$. Then, restricting the equality

$$X_f(\omega) = X_f \lrcorner\, d\omega + df = X_1(f)\omega$$

to the equation $\mathcal{E}$ and taking into account that X_f is tangent to $\mathcal{E}$, we obtain

$$\widetilde{X}_f(\widetilde{\omega}) = \widetilde{X}_f \lrcorner\, d\widetilde{\omega} + d\widetilde{f} = \beta\widetilde{\omega}, \qquad \beta = \left.X_1(f)\right|_{\mathcal{E}}.$$

On the other hand, as Proposition 7.5 shows, if $Y = \mu Y_{\widetilde{\omega}} + Y_{d\widetilde{f}}$, then $Y(\widetilde{\omega}) = \mu\widetilde{\omega}$. Therefore, $\beta = \mu$. Thus, the following result is valid:

COROLLARY 7.8. *Let* $Y = \mu Y_{\widetilde{\omega}} + Y_{d\widetilde{f}} \in \mathrm{Sym}_i\,\mathcal{E}$. *Then* $\widetilde{X}_f = Y$ *if and only if* $f|_{\mathcal{E}} = \widetilde{f}$ *and* $\left.X_1(f)\right|_{\mathcal{E}} = -\mu$.

The last two corollaries imply the following result:

COROLLARY 7.9. *First order equations in one dependent variable are common but not rigid.*

CHAPTER 4

Higher Symmetries

In Ch. 3 we considered nonlinear partial differential equations as submanifolds of spaces $J^k(\pi)$ endowed with the Cartan distribution. Basing on this approach, we constructed the symmetry theory. In fact, if one considers a differential equation not "as is" but together with all its differential consequences[1], the symmetry theory can be naturally generalized. These differential consequences form the so-called *infinite prolongation* of an equation, while the spaces of *infinite jets* are a natural environment for these prolongations. The Cartan distribution exists on infinite jets as well and, contrary to the finite case, is completely integrable. Studying automorphisms of this distribution, one comes to the concept of a *higher symmetry* of a differential equation. A widely known example of such symmetries is the series of the higher Korteweg–de Vries equations and similar series for other integrable nonlinear systems (see for example [**22**]).

Spaces of infinite jets and infinite prolongations of most equations are infinite-dimensional manifolds. By this reason, we start our exposition with a description of basic differential geometric constructions (such as smooth functions, vector fields, differential forms, etc.) on these manifolds (§1). In §2, the Cartan distribution on the space of infinite jets is introduced and infinitesimal automorphisms of this distribution are studied. A complete description of these automorphisms is given in terms of *evolutionary derivations* (a generalization of Lie fields) and the corresponding generating functions. The next section deals with the geometric definition of infinite prolongations. Determining equations for higher symmetries are also obtained here. Finally, in §4, some computational examples are considered.

1. Spaces of infinite jets and basic differential geometric structures on them

The space of infinite jets of sections of a fiber bundle $\pi\colon P \to M$ is the inverse limit of the tower of finite jets with respect to the projections $\pi_{k+1,k}\colon J^{k+1}(\pi) \to J^k(\pi)$. The aim of this section is to introduce the basic elements of calculus and differential geometry on $J^\infty(\pi)$ (smooth functions, vector fields, differential forms, etc.) needed to construct the theory of higher symmetries. Solving this problem meets some methodological difficulties caused by infinite dimension of $J^\infty(\pi)$. To overcome these difficulties, we pass to a "dual", algebraic language of smooth function rings on these manifolds. The algebraic-geometrical dualism considerably simplifies the definition of necessary concepts and makes all considerations sufficiently transparent. A good way to check efficiency of the algebraic language is to try to rewrite the material below using a purely geometrical approach.

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[1]Note that in §7 of Ch. 3, when discussing exterior and interior symmetries, we had already met with the necessity to consider differential consequences of an equation.

1.1. The manifolds $J^\infty(\pi)$. Let M be a smooth manifold of dimension n and $\pi\colon P \to M$ be a smooth locally trivial vector bundle over M whose fiber is of dimension m.

Let us consider the chain of projections (see Ch. 3)

$$M \xleftarrow{\ \pi\ } P \xleftarrow{\ \pi_{1,0}\ } J^1(\pi) \longleftarrow \cdots \longleftarrow J^k(\pi) \xleftarrow{\ \pi_{k+1,k}\ } J^{k+1}(\pi) \longleftarrow \cdots \qquad (1.1)$$

and for any point $x \in M$ let us choose a sequence of points $\theta_l \in J^l(\pi)$, $l = 0, 1, \ldots, k, \ldots$, such that the equalities $\pi_{l+1,l}(\theta_{l+1}) = \theta_l$, $\pi(\theta_0) = x$ take place. Due to these equalities and by the Borel lemma [**80**], using the definition of the spaces $J^l(\pi)$ one can choose a local section s of the bundle π such that $\theta_k = [s]_x^l$ for any l. Thus any point θ_l is determined by the partial derivatives up to order l of the section s at the point x, while the whole sequence of points $\{\theta_l\}$ contains information on all partial derivatives of the section s at x. Denote by $J^\infty(\pi)$ the set of all such sequences. Points of the space $J^\infty(\pi)$ may be obviously understood as classes of sections of the bundle π tangent to each other with infinite order or, which is the same, as infinite Taylor series of these sections.

For any point $\theta_\infty = \{x, \theta_k\}_{k\in\mathbb{N}} \in J^\infty(\pi)$, let us set $\pi_{\infty,k}(\theta_\infty) = \theta_k$ and $\pi_\infty(\theta_\infty) = x$. Then for all $k \geq l \geq 0$ one has the following commutative diagrams

$$
\begin{array}{ccc}
J^\infty(\pi) \xrightarrow{\ \pi_{\infty,k}\ } J^k(\pi) & \qquad & J^\infty(\pi) \xrightarrow{\ \pi_{\infty,k}\ } J^k(\pi) \\
{\scriptstyle \pi_\infty}\searrow \quad \swarrow{\scriptstyle \pi_k} & & {\scriptstyle \pi_{\infty,l}}\searrow \quad \swarrow{\scriptstyle \pi_{k,l}} \\
M & & J^l(\pi)
\end{array}
\qquad (1.2)
$$

i.e., the equalities $\pi_k \circ \pi_{\infty,k} = \pi_\infty$ and $\pi_{k,l} \circ \pi_{\infty,k} = \pi_{\infty,l}$ are valid. In addition, if s is a section of the bundle π, then the mapping $j_\infty(s)\colon M \to J^\infty(\pi)$ is defined by the equality $j_\infty(s)(x) = \{x, [s]_x^k\}_{k\in\mathbb{N}}$. One has the following identities: $\pi_{\infty,k} \circ j_\infty(s) = j_k(s)$ and $\pi_\infty \circ j_\infty(s) = \mathrm{id}_M$, where id_M is the identical diffeomorphism of the manifold M.

DEFINITION 1.1. The section $j_\infty(s)$ of the bundle $\pi_\infty\colon J^\infty(\pi) \to M$ is called the *infinite jet of the section* $s \in \Gamma(\pi)$.

Similar to the spaces $J^k(\pi)$, $k \leq \infty$, the set $J^\infty(\pi)$ is endowed with a natural structure of a smooth manifold, but, in contrast to the former, it is infinite-dimensional. Local coordinates arising in $J^\infty(\pi)$ over a neighborhood $\mathcal{U} \subset M$ are $x_1, \ldots, x_n$ together with all functions p_σ^j, where $|\sigma|$ is of an arbitrary (but finite) value.

DEFINITION 1.2. The bundle $\pi_\infty\colon J^\infty(\pi) \to M$ is called the *bundle of infinite jets*, while the space $J^\infty(\pi)$ is called the *manifold of infinite jets* of the bundle π.

Our nearest aim is to construct on $J^\infty(\pi)$ the analogs of the basic differential-geometric concepts one meets in calculus over finite-dimensional manifolds. When doing this, we shall use the following, yet informal, but highly important principle: any natural construction on $J^\infty(\pi)$ is to "remember" the fact that the manifold of infinite jets is the inverse limit of the tower of projections of finite-dimensional manifolds (1.1). Let us begin with the notion of a smooth function on $J^\infty(\pi)$.

1.2. Smooth functions on $J^\infty(\pi)$. Let M be a smooth manifold and $C^\infty(M)$ be the set of smooth functions on it. If M' is another smooth manifold and $G\colon M' \to M$ is a smooth mapping, then the latter generates the mapping $G^*\colon C^\infty(M) \to$

$C^\infty(M')$ defined by $G^*(f)(x') = f(G(x'))$, $f \in C^\infty(M)$, $x' \in M'$, and satisfying the identities $G^*(f_1 + f_2) = G^*(f_1) + G^*(f_2)$, $G^*(f_1 f_2) = G^*(f_1)G^*(f_2)$, and $G^*(\alpha f) = \alpha G^*(f)$ for any $f_1, f_2, f \in C^\infty(M)$ and $\alpha \in \mathbb{R}$. In other words, G^* is an homomorphism of $\mathbb{R}$-algebras. If G is a submersion, then it is easily seen that G^* is a monomorphism, i.e., $\ker G^* = 0$. Let us consider now the sequence of submersions (1.1) and denote by $\mathcal{F}_k(\pi)$ the ring of smooth functions on the manifold $J^k(\pi)$, $k \geq 0$, and by $\mathcal{F}_{-\infty}(\pi)$ the ring $C^\infty(M)$. Then (1.1) determines the sequence of $\mathbb{R}$-algebra embeddings

$$\mathcal{F}_{-\infty}(\pi) \xrightarrow{\nu} \mathcal{F}_0(\pi) \xrightarrow{\nu_{1,0}} \cdots \longrightarrow \mathcal{F}_k(\pi) \xrightarrow{\nu_{k+1,k}} \mathcal{F}_{k+1}(\pi) \longrightarrow \cdots,$$

where $\nu = \pi^*$ and $\nu_{k+1,k} = \pi_{k+1,k}^*$.

Let us define now the algebra $\mathcal{F} = \mathcal{F}(\pi)$ of smooth functions on $J^\infty(\pi)$. From the existence of the projections $\pi_{\infty,k} \colon J^\infty(\pi) \to J^k(\pi)$ it follows that any algebra $\mathcal{F}_k = \mathcal{F}_k(\pi)$ is embedded to the algebra $\mathcal{F}$ by some homomorphism ν_k. From the equalities $\pi_l \circ \pi_{k,l} = \pi_k$, $k \geq l$ (see Ch. 3), and from the commutative diagram (1.2) it follows that the diagram

$$\mathcal{F}_{-\infty} \xrightarrow{\nu} \mathcal{F}_0 \longrightarrow \cdots \longrightarrow \mathcal{F}_k \xrightarrow{\nu_{k+1,k}} \mathcal{F}_{k+1} \longrightarrow \cdots$$

with maps $\nu_{-\infty}$, ν_0, ν_k, ν_{k+1} to $\mathcal{F}$

is also commutative. Consequently, the algebra $\mathcal{F}$ must contain the union of all algebras $\mathcal{F}_k$, $k = -\infty, 0, 1 \dots$. On the other hand, since $J^\infty(\pi)$ is completely determined by its projections to the manifolds $J^k(\pi)$, it is natural to assume that $\mathcal{F}$ is exhausted by this union. Thus, let us set $\mathcal{F}(\pi) = \bigcup_k \mathcal{F}_k(\pi)$. From this definition it follows that any function φ on $J^\infty(\pi)$ is uniquely determined by its restriction to some manifold $J^k(\pi)$ with a sufficiently big k. This number k is called the *filtration degree* of the function φ and is denoted by $\deg(\varphi)$. Obviously, the set $\mathcal{F}$ is a commutative $\mathbb{R}$-algebra, the operations in which are related to filtrations in the following way:

$$\deg(\varphi_1 + \varphi_2) \leq \max(\deg(\varphi_1), \deg(\varphi_2)),$$
$$\deg(\varphi_1 \varphi_2) = \max(\deg(\varphi_1), \deg(\varphi_2)), \qquad (1.3)$$
$$\deg(\alpha \varphi) = \deg(\varphi),$$

where φ_1, φ_2, $\varphi \in \mathcal{F}$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

The algebras $\mathcal{F}_k$ are subalgebras of $\mathcal{F}$ determined by the conditions $\mathcal{F}_k = \{\varphi \in \mathcal{F} \mid \deg(\varphi) \leq k\}$. The algebra $\mathcal{F}$, together with the function $\deg \colon \mathcal{F} \to \mathbb{Z}$ taking integer values (and, if necessary, the values $\pm\infty$) and possessing the properties (1.3), is called *filtered*, or an *algebra with filtration*. Filtration of the algebra $\mathcal{F}$ is the algebraic counterpart of the tower of projections (1.1). We can now revise the above formulated informal principle:

> *All natural differential-geometric constructions on $J^\infty(\pi)$ are to be consistent with the structure of a filtered algebra in $\mathcal{F}(\pi)$.*

EXAMPLE 1.1. Let $G \colon J^k(\pi) \to J^k(\pi)$ be a Lie transformation of the space of k-jets. Then, as it was shown in the previous chapter, G is the k-th lifting of some diffeomorphism G_0 of the space $J^0(\pi)$, if $\dim(\pi) > 1$, and is the $(k-1)$-st lifting of some contact transformation G_1 of the manifold $J^1(\pi)$, if $\dim(\pi) = 1$. The family

$\left\{ G_\varepsilon^{(l)} \right\}$ of all liftings of the mapping G_ε (ε equals 0 or 1 depending on dimension of π) is in compatible with the projections $\pi_{l,l-1}$ and determines an automorphism G_ε^* of the algebra $\mathcal{F}$. Moreover, if $\varphi \in \mathcal{F}$ and $\deg(\varphi) \geq \varepsilon$, then

$$\deg G(\varphi) = \deg(\varphi).$$

Therefore one can assume that any Lie transformation (together with all its liftings) determines a diffeomorphism of the space $J^\infty(\pi)$.

DEFINITION 1.3. A mapping $G\colon J^\infty(\pi) \to J^\infty(\xi)$, where $\pi\colon E \to V$ and $\xi\colon Q \to M$ are vector bundles, is called *smooth*, if for any smooth function $\varphi \in \mathcal{F}(\xi)$ the element $G^*(\varphi) = \varphi \circ G \in \mathcal{F}(\pi)$ is a smooth function on the space $J^\infty(\pi)$ and there exist integers l and l_0 such that

$$\deg G^*(\varphi) = \deg(\varphi) + l$$

for all $\varphi \in \mathcal{F}(\xi)$, $\deg(\varphi) \geq l_0$.

Below we shall describe a broad class of smooth mappings which, unlike the mappings described in Example 1.1, raise filtration degree. To do this, we shall need the following interpretation of elements $\varphi \in \mathcal{F}(\pi)$.

Let $\varphi \in \mathcal{F} = \mathcal{F}(\pi)$ and $\deg(\varphi) = k$, i.e., $\varphi \in \mathcal{F}_k$. Consider an arbitrary section s of the bundle π. Then, as we already know, the section $j_k(s)$ of the bundle π_k corresponds to s, while the composition $\varphi \circ j_k(s) = \varphi(s)$ is a smooth function on the manifold M. In other words, the function φ puts into correspondence to smooth sections of the bundle π elements of $C^\infty(M)$ and, since the points lying on the graph of the section $j_k(s)$ are the Taylor expansions of order k for this section, the values of the function $\varphi(s)$ are determined by the partial derivatives of the section s up to order k. Consequently, the function φ determines in a canonical way a scalar nonlinear differential operator on the set of sections of the bundle π and the order of this operator coincides with the filtration degree of the element φ. This correspondence becomes even more clear when one passes to coordinate representation: if $\mathcal{U}$ is a neighborhood in M and $x_1, \ldots, x_n, \ldots, p_\sigma^j, \ldots$ are the corresponding local coordinates in $J^\infty(\pi)$, then φ is a function in the variables x_i, p_σ^j, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $|\sigma| \leq k$, and if $s = (s^1(x), \ldots, s^m(x))$, then $\varphi(s) = \varphi(x_1, \ldots, x_n, \ldots, \partial^{|\sigma|} s^j / \partial x_1^{i_1} \cdots \partial x_n^{i_n}, \ldots)$. Sometimes it is convenient to distinguish between functions $\varphi \in \mathcal{F}$ and the corresponding differential operators. In this case, the operator determined by the function φ will be denoted by Δ_φ, while the function corresponding to the operator Δ will be denoted by φ_Δ.

Let us construct now a similar correspondence for matrix differential operators. Locally, over a neighborhood $\mathcal{U} \subset M$, any matrix operator defined on sections of the bundle $\pi|_\mathcal{U}$ is to take its values in vector functions, i.e., in sections of the direct product $\pi'|_\mathcal{U} : \mathcal{U} \times \mathbb{R}^{m'} \to \mathcal{U}$ and these values should be compatible on the intersection $\mathcal{U} \cap \mathcal{U}'$ of two such neighborhoods. Passing to the global point of view, i.e., taking into consideration operators defined on the entire manifold M, one needs to consider a bundle π' whose restriction to $\mathcal{U}$ coincides with the bundles $\pi'|_\mathcal{U} : \mathcal{U} \times \mathbb{R}^{m'} \to \mathcal{U}$. In other words, global counterparts of matrix differential operators are operators acting from sections of a locally trivial vector bundle π to sections of another locally trivial vector bundle π' over the same manifold M.

Let Δ be a matrix differential operator of order k, s be a section of the bundle π and $x \in M$. Then the value of the section $\Delta(s)$ at x is determined by the values of partial derivatives of the section s, or by the k-th jet of s at x. In other words, the
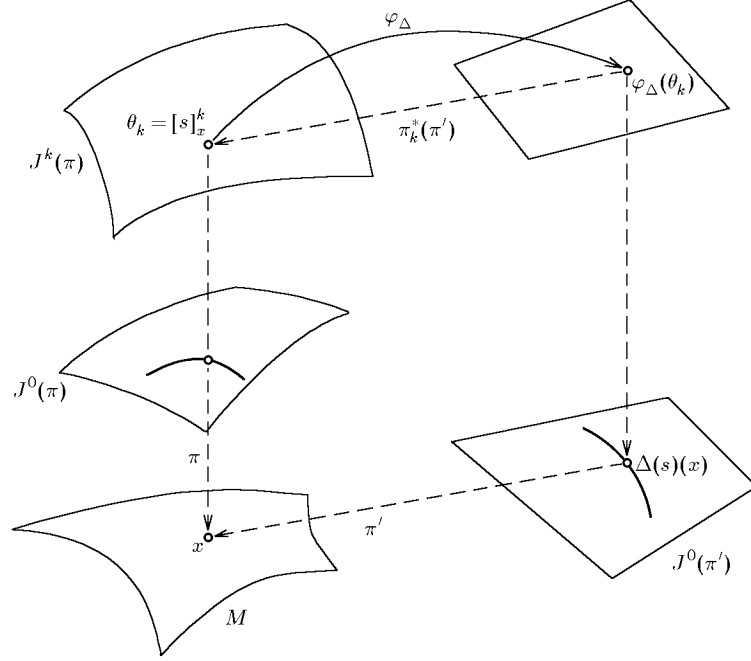
FIGURE 4.1. Construction of the section φ_Δ

operator Δ makes it possible to put into correspondence to every point $\theta_k \in J^k(\pi)$ a point of the fiber of the bundle π' growing over $x = \pi_k(\theta_k)$. This can be understood in two ways.

First, this means that the operator Δ determines a section φ_Δ of the bundle $\pi_k^*(\pi')$ over $J^k(\pi)$. This bundle is the pullback of the bundle π' with respect to the projection π_k. In fact, the total space of the pullpack $\pi_k^*(\pi')$ consists of points $(\theta_k, \theta_0') \in J^k(\pi) \times J^0(\pi')$ such that $\pi_k(\theta_k) = \pi'(\theta_0')$ and we can set $\varphi_\Delta(\theta_k) = (\theta_k, (\Delta(s))(x))$, where $\theta_k = [s]_x^k$, $s \in \Gamma(\pi)$, see Figure 4.1.

Second, from the above said it follows that a mapping $\Phi_\Delta \colon J^k(\pi) \to J^0(\pi')$ corresponds to the operator Δ and this mapping is compatible with the projections to the manifold M, i.e., $\pi' \circ \Phi_\Delta = \pi_k$. If $\theta_k = [s]_x^k \in J^k(\pi)$, $s \in \Gamma(\pi)$, then we set $\Phi(\theta_k) = [\Delta(s)]_x^0 = (\Delta(s))(x)$. Thus the following commutative diagram takes place

$$
\begin{array}{ccc}
J^k(\pi) & \xrightarrow{\ \Phi_\Delta\ } & J^0(\pi') \\
& \searrow{\scriptstyle \pi_k} \quad \swarrow{\scriptstyle \pi'} & \\
& M &
\end{array}
\tag{1.4}
$$

i.e., Φ_Δ is a morphism of the bundle π_k to the bundle π'. Note that for any operator $\Delta \colon \Gamma(\pi) \to \Gamma(\pi')$ of order k the identity

$$\Delta = \Phi_\Delta^* \circ j_k,$$

holds, and it can be taken for the definition. Here $\Phi_\Delta^* \colon \Gamma(\pi_k) \to \Gamma(\pi')$ is the mapping induced by the morphism Φ_Δ.

EXAMPLE 1.2. The operator j_k constructed above is an operator of order k acting from the bundle π to the bundle $\pi_k \colon J^k(\pi) \to M$. The corresponding section

φ_{j_k} puts into correspondence to any point $\theta_k \in J^k(\pi)$ the same point considered as an element of the fiber of the pullback $\pi_k^*(\pi_k)$ growing over θ_k. In other words, φ_{j_k} is the diagonal. It is also easily seen that the mapping $\Phi_{j_k} : J^k(\pi) \to J^k(\pi) = J^0(\pi_k)$ is the identity.

1.3. Prolongations of differential operators. Let $\Delta : \Gamma(\pi) \to \Gamma(\pi')$ and $\Delta' : \Gamma(\pi') \to \Gamma(\pi'')$ be two differential operators of orders k and k' respectively. Then their composition $\Delta' \circ \Delta$ is a differential operator of order $\leq k + k'$. To establish this fact (which is obvious locally), let us note the following.

First, for any morphism Φ of the bundles $\pi : P \to M$ and $\xi : Q \to M$ its liftings

$$\Phi^{(k)} : J^k(\pi) \to J^k(\xi), \quad k \geq 0, \qquad \Phi^{(k)}([s]_x^k) = [\Phi \circ s]_x^k$$

are defined and the following equalities are valid:

$$\Phi^{(0)} = \Phi, \qquad \Phi^{(l)} \circ \pi_{k,l} = \xi_{k,l} \circ \Phi^{(k)}, \qquad \xi_k \circ \Phi^{(k)} = \pi_k, \quad k \geq l.$$

Second, for any $k, k' \geq 0$ one can construct the map

$$\Phi_{k,k'} : J^{k+k'}(\pi) \to J^{k'}(\pi_k), \qquad \Phi_{k,k'}([s]_x^{k+k'}) = [j_k(s)]_{\theta_k}^{k'},$$

where $\theta_k = [s]_x^k$. Then one has $\pi_k = \Phi_{k,k'} \circ (\pi_k)_{k'}$ and, as it is easily seen, $\Phi_{k,k'}(j_{k+k'}(s)) = j_{k'}(j_k(s))$. Thus, $\Phi_{k,k'} \circ j_{k+k'} = j_{k'} \circ j_k$. In other words, $\Phi_{k,k'} = \Phi_{j_{k'} \circ j_k}$, and it proves that the composition $j_{k'} \circ j_k$ is a differential operator of order $k + k'$.

Let us return back to the case of general operators Δ, Δ' and consider the diagram

$$
\begin{array}{ccccc}
& J^{k'}(\pi_k) & \xrightarrow{\Phi_\Delta^{(k')}} & J^{k'}(\pi') & \\
\Phi_{k,k'} \nearrow & \downarrow (\pi_k)_{k',0} & & \downarrow \pi'_{k',0} & \searrow \Phi_{\Delta'} \\
J^{k+k'}(\pi) \quad J^k(\pi) & \xrightarrow{\Phi_\Delta} & J^0(\pi') & & J^0(\pi'') \\
\searrow \pi_k & \downarrow \pi' & \swarrow \pi'' & & \\
& M & & &
\end{array}
$$

From commutativity of this diagram it follows that for any section $s \in \Gamma(\pi)$ the equality

$$\Delta'(\Delta(s)) = \Phi_{\Delta'}(\Phi_\Delta^{(k')}(\Phi_{k,k'}(j_{k+k'}(s))))$$

holds, i.e.,

$$\Phi_{\Delta' \circ \Delta} = \Phi_{\Delta'} \circ \Phi_\Delta^{(k')} \circ \Phi_{k,k'},$$

which proves the desired result.

In particular, for any differential operator Δ its composition Δ_l with the operator $j_l : \Gamma(\pi') \to \Gamma(\pi'_l)$, $l \geq 0$ is defined. Let $\Phi_\Delta^l = \Phi_{\Delta_l} : J^{k+l}(\pi) \to J^l(\pi')$ be the corresponding mapping of jet manifolds. Then, as it is easily seen, the diagram

(1.4) can be extended to the following commutative diagram:

$$
\begin{array}{ccccccccc}
& & J^k(\pi) & \longleftarrow & \cdots & \longleftarrow & J^{k+l}(\pi) & \xleftarrow{\ \pi_{k+l+1,k+l}\ } & J^{k+l+1}(\pi) & \longleftarrow & \cdots \\
& \pi_k \nearrow & & & & & \downarrow \Phi_\Delta = \Phi_\Delta^0 & & \downarrow \Phi_\Delta^l & & \downarrow \Phi_\Delta^{l+1} \\
M & & & & & & & & & & \\
& \pi' \searrow & & & & & & & & & \\
& & J^0(\pi') & \longleftarrow & \cdots & \longleftarrow & J^l(\pi') & \xleftarrow{\ \pi'_{l+1,l}\ } & J^{l+1}(\pi') & \longleftarrow & \cdots
\end{array}
$$

The above introduced set of the mappings $\Phi_\Delta = \left\{\Phi_\Delta^l\right\}_{l \geq 0}$ determines a homomorphism $\Phi_\Delta^* \colon \mathcal{F}(\pi') \to \mathcal{F}(\pi)$ possessing the property

$$
\deg \Phi_\Delta^*(\varphi) = \deg(\varphi) + k, \qquad \varphi \in \mathcal{F}(\pi').
$$

Thus, Φ_Δ is a smooth mapping of the manifold $J^\infty(\pi)$ to the manifold $J^\infty(\pi')$.

DEFINITION 1.4. The operator Δ_l is called the *l-th prolongation* of the nonlinear differential operator Δ.

As we shall see below, this notion plays quite an important role in studying differential equations.

Let $\mathcal{U} \subset M$ be a coordinate neighborhood in M, $x_1, \ldots, x_n, u^1, \ldots, u^m, \ldots,$ $p_\sigma^j, \ldots$ be the corresponding coordinates in $\pi_\infty|_\mathcal{U}$, and $x_1, \ldots, x_n, v^1, \ldots, v^{m'}, \ldots,$ $q_\sigma^{j'}, \ldots$ be the coordinates in $\pi'_\infty|_\mathcal{U}$. If

$$
(\Delta(s))^{j'} = \varphi^{j'}\left(x_1, \ldots, x_n, s^1(x), \ldots, s^m(x), \ldots, \frac{\partial^{|\sigma|} s^j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \ldots\right), \qquad (1.5)
$$

where $s \in \Gamma(\pi|_\mathcal{U})$, is the coordinate representation of the operator Δ and

$$
j_k(s)(x_1, \ldots, x_n) = \left(x_1, \ldots, x_n, \ldots, \frac{\partial^{|\sigma|} s^j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \ldots\right) \qquad (1.6)
$$

is that of the operator j_k, then, as it is easily checked, the first prolongation of the operator Δ is described by equality (1.5) together with the relations

$$
(\Delta(s))_i^{j'} = \frac{\partial}{\partial x_i} \varphi^{j'}\left(x_1, \ldots, x_n, \ldots, \frac{\partial^{|\sigma|} s^j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \ldots\right)
$$

$$
= \frac{\partial \varphi^{j'}}{\partial x_i} + \sum_{j,\sigma} \frac{\partial \varphi^{j'}}{\partial p_\sigma^j} \frac{\partial}{\partial x_i} \frac{\partial^{|\sigma|} s^j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}},
$$

$i = 1, \ldots, n$, or, which is equivalent, by the relations

$$
\begin{cases}
v^{j'} = \varphi^{j'}(x, \ldots, p_\sigma^j, \ldots), \\
q_i^{j'} = D_i \varphi^{j'}, \quad i = 1, \ldots, n,
\end{cases}
$$

where $D_i = \partial/\partial x_i + \sum_{|\sigma|=0}^k \sum_{j=1}^m p_{\sigma+1_i}^j \partial/\partial p_\sigma^j$ are the operators of total derivatives.

In a similar manner, the *l*-th prolongation of the operator Δ is described by the system of relations

$$
q_\tau^{j'} = D_\tau \varphi^{j'}, \qquad j' = 1, \ldots, m', \quad |\tau| = 0, \ldots, l, \qquad (1.7)
$$

where $D_\tau = D_1^{l_1} \circ \cdots \circ D_n^{l_n}$, if $\tau = (l_1, \ldots, l_n)$.

EXAMPLE 1.3. Let Δ be the operator determining the Burgers equation (see Ch. 3) and represented in coordinate form by

$$v = p_{(0,1)} - up_{(1,0)} - p_{(2,0)}. \tag{1.8}$$

Then its first prolongation is of the form

$$v = p_{(0,1)} - up_{(1,0)} - p_{(2,0)},$$
$$q_{(1,0)} = p_{(1,1)} - p_{(1,0)}^2 - up_{(2,0)} - p_{(3,0)},$$
$$q_{(0,1)} = p_{(0,2)} - p_{(1,0)}p_{(0,1)} - up_{(1,1)} - p_{(2,1)},$$

while the l-th prolongation is represented as

$$q_{(i,j)} = D_1^i D_2^j \left(p_{(0,1)} - p_{(0,0)}p_{(1,0)} - p_{(2,0)} \right)$$

$$= p_{(i,j+1)} - p_{(i+2,j)} - \sum_{\alpha=0}^{i} \binom{i}{\alpha} \sum_{\beta=0}^{j} \binom{j}{\beta} p_{(\alpha,\beta)} p_{(i+1-\alpha,j-\beta)},$$

where $i + j \leq l$.

EXERCISE 1.1. Describe the l-th prolongation of the operator j_k.

When one considers matrix differential operators, mappings of the objects of a more general nature than the algebras $\mathcal{F}(\pi)$ arise. Let us recall that any differential operator $\nabla \colon \Gamma(\pi) \to \Gamma(\xi)$ of order k is identified with the section φ_∇ of the bundle $\pi_k^*(\xi)$. Denote the set of such sections by $\mathcal{F}_k(\pi, \xi)$. Similar to the case of the algebras $\mathcal{F}(\pi)$, the following chain of embeddings takes place:

$$\mathcal{F}_{-\infty}(\pi, \xi) = \Gamma(\xi) \xrightarrow{\nu} \mathcal{F}_0(\pi, \xi) \longrightarrow \cdots$$

$$\longrightarrow \mathcal{F}_k(\pi, \xi) \xrightarrow{\nu_{k+1,k}} \mathcal{F}_{k+1}(\pi, \xi) \longrightarrow \cdots$$

and consequently the set $\mathcal{F}(\pi, \xi) = \bigcup_{k=-\infty}^{\infty} \mathcal{F}_k(\pi, \xi)$ filtered by the subsets $\mathcal{F}_k(\pi, \xi)$ is defined. The elements of $\mathcal{F}(\pi, \xi)$ can be added to each other and multiplied by the elements of $\mathcal{F}(\pi)$, and these operations are compatible with the filtration, i.e., $\mathcal{F}(\pi, \xi)$ is a *filtered module* over the filtered algebra $\mathcal{F}(\pi)$. If now one considers an operator $\Delta \colon \Gamma(\pi) \to \Gamma(\pi')$ of order l and its composition with another operator $\nabla' \colon \Gamma(\pi') \to \Gamma(\xi)$ of order k, then one will obtain the operator of order $k + l$ acting from $\Gamma(\pi)$ to $\Gamma(\xi)$. Thus, we have the system of mappings $\Phi_{\Delta,\xi}^k \colon \mathcal{F}_k(\pi', \xi) \to \mathcal{F}_{k+l}(\pi, \xi)$ or, which is the same, the mapping $\Phi_{\Delta,\xi}^* \colon \mathcal{F}(\pi', \xi) \to \mathcal{F}(\pi, \xi)$, raising filtration degree by l. The following equalities are valid:

$$\Phi_{\Delta,\xi}^*(\varphi_1 + \varphi_2) = \Phi_{\Delta,\xi}^*(\varphi_1) + \Phi_{\Delta,\xi}^*(\varphi_2),$$
$$\Phi_{\Delta,\xi}^*(\varphi\varphi_1) = \Phi_\Delta^*(\varphi)\Phi_{\Delta,\xi}^*(\varphi_1),$$

where $\varphi_1, \varphi_2 \in \mathcal{F}(\pi', \xi)$, $\varphi \in \mathcal{F}(\pi)$. In other words, for any fiber bundle ξ the mapping $\Phi_{\Delta,\xi}^*$ is a homomorphism of filtered modules acting over the homomorphism Φ_Δ^* of filtered algebras.

EXERCISE 1.2. Construct the homomorphism $\Phi_{\Delta,\xi}^*$ starting from the smooth mapping $\Phi_\Delta \colon J^\infty(\pi) \to J^\infty(\pi')$.
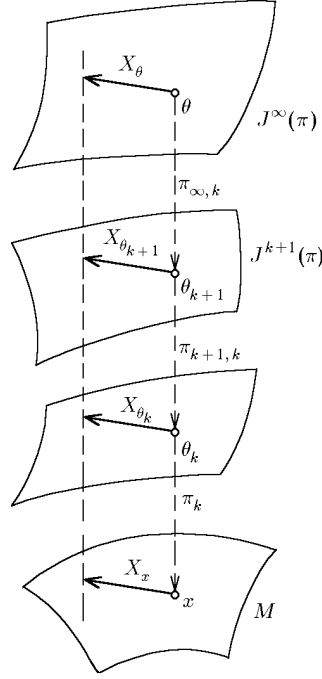
FIGURE 4.2. A tangent vector to the manifold $J^\infty(\pi)$

1.4. Vector fields on $J^\infty(\pi)$. Let us now analyze the concept of a vector field on the manifold $J^\infty(\pi)$. Let X_θ be a tangent vector at a point $\theta \in J^\infty(\pi)$. Then, by natural reasons, the projections $\pi_{\infty,k}$ and π_∞ are to determine the sequence of tangent vectors $(\pi_{\infty,k})_*(X_\theta)$ at the points $\theta_k = \pi_{\infty,k}(\theta)$ together with the tangent vector $\pi_{\infty,*}(X_\theta) \in T_x(M)$, $x = \pi_\infty(\theta)$. Now, using the already familiar reasons, we shall define a tangent vector X_θ of the manifold $J^\infty(\pi)$ at the point θ as the set $\{X_x, X_{\theta_k}\}$ of the tangent vectors to the manifolds M and $J^k(\pi)$ at the points $x = \pi_\infty(\theta)$ and $\theta_k = \pi_{\infty,k}(\theta)$ respectively, such that $(\pi_{k+1,k})_*(X_{\theta_{k+1}}) = X_{\theta_k}$ and $(\pi_k)_*(X_{\theta_k}) = X_x$ (see Figure 4.2).

If $\mathcal{U} \subset M$ is a coordinate neighborhood of the point $x \in M$ and $x_1, \ldots, x_n$, $\ldots, p^j_\sigma, \ldots$ are canonical coordinates in $\pi_\infty^{-1}(\mathcal{U})$, then any tangent to the manifold $J^\infty(\pi)$ vector X_θ at the point θ is represented in the form of infinite sum

$$X_\theta = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + \sum_{|\sigma| \geq 0} \sum_{j=1}^m b^j_\sigma \frac{\partial}{\partial p^j_\sigma}, \tag{1.9}$$

in which the coefficients a_i and b^j_σ are real numbers. One also has

$$(\pi_{\infty,k})_*(X_\theta) = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + \sum_{|\sigma|=0}^k \sum_{j=1}^m b^j_\sigma \frac{\partial}{\partial p^j_\sigma}, \quad (\pi_\infty)_*(X_\theta) = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}.$$

Let X_θ be a tangent vector to $J^\infty(\pi)$ at θ and X_{θ_k} be its projection to $T_{\theta_k}(J^k(\pi))$. Then the vector X_{θ_k} can be understood as a derivation of the algebra of smooth functions on $J^k(\pi)$ with values in the field of constants $\mathbb{R}$. In other words, $X_{\theta_k} : \mathcal{F}_k \to$

$\mathbb{R}$ is a mapping possessing the property

$$X_{\theta_k}(\varphi_1\varphi_2) = \varphi_1(\theta_k)X_{\theta_k}(\varphi_2) + \varphi_2(\theta_k)X_{\theta_k}(\varphi_1),$$

where φ_1, $\varphi_2 \in \mathcal{F}_k$. The condition of compatibility of the vectors X_{θ_k}, X_x with the projections $\pi_{\infty,k}$ and π_k means that $X_{\theta_{k+1}} \circ \nu_{k+1,k} = X_{\theta_k}$ and $X_{\theta_0} \circ \nu = X_x$, i.e., that the diagram

is commutative. In other words, similar to the case of finite-dimensional manifolds, a tangent vector X_θ is interpreted as a derivation of the algebra $\mathcal{F}$ with the values in the field of real numbers:

$$X_\theta(\varphi_1\varphi_2) = \varphi_1(\theta)X_\theta(\varphi_2) + \varphi_2(\theta)X_\theta(\varphi_1), \tag{1.10}$$

where φ_1, $\varphi_2 \in \mathcal{F}$.

If we now take a family $X = \{X_\theta\}$ of tangent vectors on $J^\infty(\pi)$ parametrized by points of $\theta \in J^\infty(\pi)$, i.e., a vector field on $J^\infty(\pi)$, then the formula (1.10) will transform to the relation

$$X(\varphi_1\varphi_2) = \varphi_1 X(\varphi_2) + \varphi_2 X(\varphi_1) \tag{1.11}$$

valid for all smooth functions φ_1, φ_2 on $J^\infty(\pi)$. To complete our arguments, it remains to recall that $\mathcal{F}(\pi)$ is a filtered and to define a vector field on the manifold $J^\infty(\pi)$ as a derivation of the algebra $\mathcal{F}(\pi)$ (i.e., an $\mathbb{R}$-linear mapping $X\colon \mathcal{F} \to \mathcal{F}$ satisfying the Leibniz rule (1.11)), such that

$$\deg X(\varphi) = \deg(\varphi) + k, \quad \varphi \in \mathcal{F},$$

where k is an integer independent of φ and denoted by $\deg(X)$. Denote by $\mathrm{D}(\pi)$ the set of all vector fields on $J^\infty(\pi)$.

DEFINITION 1.5. A field $X \in \mathrm{D}(\pi)$ is called *vertical* (or π-*vertical*), if the equality $X(\pi_\infty^*(\varphi)) = 0$ holds for any function $\varphi \in C^\infty(M) \subset \mathcal{F}$.

The set of all vertical fields is denoted by $\mathrm{D}^v(\pi)$.

EXERCISE 1.3. Show that if $X, Y \in \mathrm{D}(\pi)$ and $\varphi \in \mathcal{F}$, then $X + Y$, φX and $[X, Y] = X \circ Y - Y \circ X$ are also vector fields on $J^\infty(\pi)$ and the following identities fulfill:

$$[X, Y] + [Y, X] = 0,$$
$$[X, \varphi Y] = X(\varphi)Y + \varphi[X, Y],$$
$$[X, Y + Z] = [X, Y] + [X, Z],$$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

where $Z \in \mathrm{D}(\pi)$, i.e., the set of vector fields on $J^\infty(\pi)$ possesses the same algebraic properties as vector fields on finite-dimensional manifolds. In particular, they form a Lie algebra over $\mathbb{R}$. Note that $\mathrm{D}^v(\pi)$ is a subalgebra of this Lie algebra.

EXAMPLE 1.4 (liftings of vector fields). Let X be a vector field on the manifold M, φ be a smooth function on $J^\infty(\pi)$ with $\deg(\varphi) = k$, and $\Delta = \Delta_\varphi \colon \Gamma(\pi) \to C^\infty(M)$ be the differential operator associated to this function. Then, since X is a first order differential operator acting from $C^\infty(M)$ to $C^\infty(M)$, the composition $X \circ \Delta \colon \Gamma(\pi) \to C^\infty(M)$ is defined and the latter is a differential operator of order $k+1$. Denote by $\widehat{X}(\varphi)$ the function on $J^\infty(\pi)$ corresponding to the operator $X \circ \Delta$.

EXERCISE 1.4. Show that the correspondence $\widehat{X} \colon \varphi \mapsto \widehat{X}(\varphi)$ is a vector field on $J^\infty(\pi)$ with $\deg(\widehat{X}) = 1$.

Let us give a geometric version for the definition of the field $\widehat{X}$. Let $x \in M$ be a point in M and θ_k be a point of $J^k(\pi)$ lying in the fiber of the bundle π_k over x. Then one has $\theta_k = [s]_x^k$ for a section $s \in \Gamma(\pi)$. If φ is a smooth function on $J^k(\pi)$ in a neighborhood of the point θ_k, then we set $\widehat{X}_{\theta_k}(\varphi) = X_x(s^*(\varphi))$. Clearly, the right-hand side of the last expression is independent of the representative s of the point θ_k. Consequently, X_{θ_k} is a tangent vector to $J^k(\pi)$ at the point θ_k. If now $\{x, \theta_k\}$ is the sequence of points representing the point $\theta \in J^\infty(\pi)$, then it is easily seen that $(\pi_{k+1,k})_*(\widehat{X}_{\theta_{k+1}}) = \widehat{X}_{\theta_k}$ and $(\pi_k)_*(\widehat{X}_{\theta_k}) = X_x$, i.e., the sequence of vectors $\left\{X_x, \widehat{X}_{\theta_k}\right\}$ determines a tangent vector to the manifold $J^\infty(\pi)$ at the point θ.

DEFINITION 1.6. The field $\widehat{X}$ on $J^\infty(\pi)$ is called the *lifting of the vector field* $X \in \mathrm{D}(M)$ to the space of infinite jets.

The following properties of the lifting operation are directly deduced from the definition:

$$f\widehat{X + g}Y = f\widehat{X} + g\widehat{Y}, \tag{1.12a}$$

$$\widehat{[X,Y]} = [\widehat{X}, \widehat{Y}], \tag{1.12b}$$

$$\widehat{X}(f\varphi) = X(f)\varphi + f\widehat{X}(\varphi), \tag{1.12c}$$

where $f, g \in C^\infty(M)$, $\varphi \in \mathcal{F}$, $X, Y \in D(M)$. Equalities (1.12a)–(1.12b) mean that the lifting operation is an homomorphism of the Lie algebra of vector fields over M to that on $\mathrm{D}(\pi)$, while from equality (1.12c) it follows that the projection of the lifting to $J^\infty(\pi)$ of any field $X \in \mathrm{D}(M)$ to M coincides with the initial field. In other words, the correspondence $X \mapsto \widehat{X}$ is a flat (integrable) connection in the bundle π_∞. This fact plays a fundamental role in geometry of manifolds $J^\infty(\pi)$ and we shall frequently refer to it in the sequel. By the reasons which will become clear in §2, this connection is called the *Cartan connection*.

If $x_1, \ldots, x_n, u^1, \ldots, u^m, \ldots, p_\sigma^j, \ldots$ are coordinates in $J^\infty(\pi)$ over a neighborhood $\mathcal{U} \subset M$, then any field $X \in \mathrm{D}(M)$ is representable as the infinite sum

$$X = X_1 \frac{\partial}{\partial x_1} + \cdots + X_n \frac{\partial}{\partial x_n} + \cdots + X_\sigma^j \frac{\partial}{\partial p_\sigma^j} + \cdots, \tag{1.13}$$

where X_i, $X_\sigma^j \in \mathcal{F}\big(\pi|_{\mathcal{U}}\big)$, and the equalities $\deg(X_\sigma^j) = |\sigma| + k$, $k = \deg(X)$, fulfil for all j and σ such that $|\sigma| \geq k_0 \geq 0$. Vertical fields are characterized by the fact that all their coefficients at $\partial/\partial x_i$ vanish. Note that infinite number of summands in the right-hand side of (1.13) does not cause computational difficulties (such as check of convergence, etc.), since, by definition of the algebra $\mathcal{F}(\pi)$, any function $\varphi \in \mathcal{F}(\pi)$ may depend on finite number of variables only and consequently the number of

summands in the expression for $X(\varphi)$ for a particular function φ is always finite as well.

Let us now derive a coordinate representation for the liftings of vector fields lying in $\mathrm{D}(M)$. As equality (1.12a) shows, it is sufficient to be done for the fields of the form $\partial/\partial x_i$. Let $\varphi = \varphi(x_1, \ldots, x_n, \ldots, p^j_\sigma, \ldots) \in \mathcal{F}$ and $s = (s^1, \ldots, s^m) \in \Gamma(\pi)$. Then

$$\left(\frac{\partial}{\partial x_i} \circ \Delta_\varphi \right)(s) = \frac{\partial}{\partial x_i} \varphi\left(x_1, \ldots, x_n, \ldots, \frac{\partial^{|\sigma|} s^j}{\partial x^\sigma}, \ldots \right)$$
$$= \frac{\partial \varphi}{\partial x_i} + \sum_{j,\sigma} \frac{\partial^{|\sigma|+1} s^j}{\partial x_i \partial x^\sigma} \frac{\partial \varphi}{\partial p^j_\sigma}.$$

Hence

$$\widehat{\frac{\partial}{\partial x_i}} = D_i = \frac{\partial}{\partial x_i} + \sum_{|\sigma|=0}^{\infty} \sum_{j=1}^{m} p^j_{\sigma+1_i} \frac{\partial}{\partial p^j_\sigma}. \qquad (1.14)$$

Thus liftings of the basic vector fields $\partial/\partial x_i$ coincide with the operators D_i of the total derivatives along x_i. Equality (1.12b) shows that $[D_{i_1}, D_{i_2}] = 0$ for all $i_1, i_2 = 1, \ldots, n$. Note that the fields D_i take functions on the k-th jets to those on $(k+1)$-st jets and thus determine no vector field on manifolds of finite jets. It means that treating "truncated" total derivatives as vector fields (see the preceding chapter) is incorrect (though a deceptive appearance of their coordinate representation may cause opposite conclusions). This is one of illustrations why it is useful to pass to infinite jet manifolds.

REMARK 1.1. The lifting construction takes place in a more general setting. Let $\nabla \colon \Gamma(\xi) \to \Gamma(\xi')$ be a linear differential operator and $\varphi \in \mathcal{F}(\pi, \xi)$. Consider the operator $\Delta = \Delta_\varphi \colon \Gamma(\pi) \to \Gamma(\xi)$ and the composition $\nabla \circ \Delta \colon \Gamma(\pi) \to \Gamma(\xi')$. Then, setting $\widehat{\nabla}(\varphi) = \varphi_{\nabla \circ \Delta}$, one obtains the map $\widehat{\nabla} = \widehat{\nabla}_\pi \colon \mathcal{F}(\pi, \xi) \to \mathcal{F}(\pi, \xi')$ which is an $\mathbb{R}$-linear operator.

EXERCISE 1.5. Give a point-wise definition of the operator $\widehat{\nabla}$ similar to the one given above for liftings of vector fields. Deduce a coordinate representation of the operator $\widehat{\nabla}$.

1.5. Differential forms on $J^\infty(\pi)$. Let us now discuss the concept of a differential form on $J^\infty(\pi)$. Let $\Lambda^i(\pi_k) \stackrel{\mathrm{def}}{=} \Lambda^i(J^k(\pi))$ be the module of i-forms on $J^k(\pi)$. The projections π and $\pi_{k+1,k}$ generate the infinite sequence of embeddings

$$\Lambda^i(M) \xrightarrow{\nu} \Lambda^i(\pi_0) \xrightarrow{\nu_{1,0}} \Lambda^i(\pi_1) \longrightarrow \cdots \longrightarrow \Lambda^i(\pi_k) \xrightarrow{\nu_{k+1,k}} \Lambda^i(\pi_{k+1}) \longrightarrow \cdots$$

Skipping the already familiar motivations, we shall define the module $\Lambda^i(\pi)$ of i-forms on $J^\infty(\pi)$ by setting $\Lambda^i(\pi) = \bigcup_{k=0}^{\infty} \Lambda^i(\pi_k)$. In particular, $\Lambda^0(\pi) = \mathcal{F}(\pi)$. Let us also set $\Lambda^*(\pi) = big \oplus_{i=0}^{\infty} \Lambda^i(\pi)$. As above, the modules $\Lambda^i(\pi)$ and $\Lambda^*(\pi)$ are filtered by their submodules $\Lambda^i(\pi_k)$ and $\Lambda^*(\pi_k)$ respectively[2]. By the above given definition, any element of $\Lambda^*(\pi)$ is in fact a form on some manifold of finite jets.

---

[2]To be more exact, the embeddings $\nu_{k+1,k}$ are module homomorphisms over the corresponding algebra embeddings $\mathcal{F}_k \to \mathcal{F}_{k+1}$. To obtain a filtration of the module $\Lambda^*(\pi)$ by a system of $\mathcal{F}$-submodules, it needs to consider the submodules in $\Lambda^*(\pi)$ generated by the elements of $\Lambda^*(\pi_k)$, i.e., consisting of differential forms on $J^k(\pi)$ whose coefficients may be arbitrary functions belonging to $\mathcal{F}(\pi)$.

Hence, the operation $\wedge$ of the wedge product and the differential d are defined for these elements and possess their usual properties.

Let $X \in \mathrm{D}(\pi)$ be a vector field and $\omega \in \Lambda^i(\pi)$ be a differential form on $J^\infty(\pi)$. Let us define the *inner product* operation $\lrcorner$ which puts into correspondence to these two objects the form $X \lrcorner \omega \in \Lambda^{i-1}(\pi)$. Let $\theta = \{x, \theta_k\} \in J^\infty(\pi)$ be a point of $J^\infty(\pi)$ and $X_\theta = \{X_x, X_{\theta_k}\}$ be the vector of the field X at this point. Consider a number k such that $\omega \in \Lambda^i(\pi_k)$. This number always exists by the definition of the module $\Lambda^i(\pi)$. Let us set $(X \lrcorner \omega)_\theta = X_{\theta_k} \lrcorner \omega_{\theta_k}$. For any $k' \geq k$ the equality $(\pi_{k',k})_*(X_{\theta_{k'}}) = X_{\theta_k}$ is valid; hence, one has $X_{\theta_{k'}} \lrcorner (\pi^*_{k'k}\omega)_{\theta_{k'}} = X_{\theta_k} \lrcorner \omega_{\theta_k}$ which means that the operation $\lrcorner$ is well defined. If in addition $\deg(X) = l$, then the vector X_{θ_k} is determined by the point $\theta_{k+l} \in J^{k+l}(\pi)$, i.e., $X \lrcorner \omega \in \Lambda^{i-1}(\pi_{k+l}) \subset \Lambda^{i-1}(\pi)$. In particular, inner product of a vector field with a 1-form determines the isomorphism

$$\mathrm{D}(\pi) \simeq \mathrm{Hom}^{\mathbf{0}}_{\mathcal{F}(\pi)}(\Lambda^1(\pi), \mathcal{F}(\pi)), \tag{1.15}$$

where $\mathrm{Hom}^{\mathbf{0}}_{\mathcal{F}(\pi)}$ denotes the set of homomorphism preserving filtration. This isomorphism puts into correspondence to any field $X \in \mathrm{D}(\pi)$ the $\mathcal{F}(\pi)$-homomorphism $f_X \colon \Lambda^1(\pi) \to \mathcal{F}(\pi)$ acting as $f_X(d\varphi) = X(\varphi)$, $\varphi \in \mathcal{F}(\pi)$.

Now we can define the *Lie derivative* $L_X\omega$ of a form $\omega \in \Lambda^*(\pi)$ along a vector field $X \in \mathrm{D}(\pi)$. To do this, taking into account the infinitesimal Stokes formula, let us set

$$L_X\omega = X \lrcorner d\omega + d(X \lrcorner \omega). \tag{1.16}$$

Obviously, if $\omega \in \Lambda^i(\pi_k)$ and $\deg(X) = l$, then $L_X\omega \in \Lambda^i(\pi_{k+l})$. We shall also use the notation $X(\omega)$ for the Lie derivative of a form ω along X.

The above introduced operations on the set $\Lambda^*(\pi)$ (wedge product, the de Rham differential, inner product, and Lie derivative) possess all properties they possess in the finite-dimensional case. The same is valid for their coordinate representation. If $\mathcal{U}$ is a coordinate neighborhood in the manifold M and $(x_1, \ldots, x_n, \ldots, p^j_\sigma, \ldots)$ are the corresponding coordinates in $\pi_\infty^{-1}(\mathcal{U})$, then any form $\omega \in \Lambda^i(\pi)$ can be represented as

$$\omega = \sum_{\alpha+\beta=i} \varphi^{\sigma_1,\ldots,\sigma_\beta}_{i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta} dx_{i_1} \wedge \cdots \wedge dx_{i_\alpha} \wedge dp^{j_1}_{\sigma_1} \wedge \cdots \wedge dp^{j_\beta}_{\sigma_\beta}, \tag{1.17}$$

where $|\sigma_1|, \ldots, |\sigma_\beta| \leq k$ while $\varphi^{\sigma_1,\ldots,\sigma_\beta}_{i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta}$ are smooth functions on $\pi_\infty^{-1}(\mathcal{U})$. Then the coordinate representation of the de Rham differential is of the form

$$d\omega = \sum_{\alpha+\beta=i} d\varphi^{\sigma_1,\ldots,\sigma_\beta}_{i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\alpha} \wedge dp^{j_1}_{\sigma_1} \wedge \cdots \wedge dp^{j_\beta}_{\sigma_\beta},$$

where

$$d\varphi^{\sigma_1,\ldots,\sigma_\beta}_{i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta} = \sum_\gamma \frac{\partial \varphi^{\sigma_1,\ldots,\sigma_\beta}_{i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta}}{\partial x_\gamma} \, dx_\gamma + \sum_{\delta,\tau} \frac{\partial \varphi^{\sigma_1,\ldots,\sigma_\beta}_{i_1,\ldots,i_\alpha,j_1,\ldots,j_\beta}}{\partial p^\delta_\tau} \, dp^\delta_\tau.$$

EXERCISE 1.6. Write down coordinate formulas for the operation of inner product and for the Lie derivative on $J^\infty(\pi)$.

1.6. The horizontal de Rham complex. We shall now describe a special class of forms on $J^\infty(\pi)$ which will be denoted by $\Lambda_0^*(\pi)$. Elements of this class will be called *horizontal forms*.

DEFINITION 1.7. A form $\omega \in \Lambda^*(\pi)$ is called *horizontal*, if $X \lrcorner \omega = 0$ for any vertical field $X \in \mathrm{D}^v(\pi)$.

Since any vertical field is locally of the form

$$X = \sum_{j,\sigma} X_\sigma^j \frac{\partial}{\partial p_\sigma^j},$$

from the representation (1.17) it follows that a form ω is horizontal if and only if it is representable in coordinates as

$$\omega = \sum \varphi_{i_1,\dots,i_\alpha} dx_{i_1} \wedge \cdots \wedge dx_{i_\alpha}, \qquad \varphi_{i_1,\dots,i_\alpha} \in \mathcal{F}(\pi). \tag{1.18}$$

Hence locally any horizontal form is a linear combination of forms on the manifold M with coefficients in $\mathcal{F}$: $\omega = \varphi^1 \omega_1 + \cdots + \varphi^l \omega_l$, where $\varphi^1, \dots, \varphi^l \in \mathcal{F}(\pi)$, $\omega_1, \dots, \omega_l \in \Lambda^*(M)$. Let $\Delta^1, \dots, \Delta^l$ be the nonlinear differential operators corresponding to the functions $\varphi^1, \dots, \varphi^l$ and acting from sections of the bundle π to $C^\infty(M)$. Then one can put into correspondence to the form ω the operator Δ_ω acting as

$$\Delta_\omega(s) = \Delta^1(s)\omega_1 + \cdots + \Delta^l(s)\omega_l, \qquad s \in \Gamma(\pi),$$

and taking sections of the bundle π to differential forms on the manifold M. Conversely, to any such an operator there corresponds a form (1.18). Consequently, horizontal forms on $J^\infty(\pi)$ coincide with nonlinear differential operators on the bundle π with the values in differential forms on the manifold M. The module of horizontal i-forms on $J^\infty(\pi)$ is denoted by $\Lambda_0^i(\pi)$.

If $t^* \colon T^*(M) \to M$ denotes the cotangent bundle of the manifold M and $\Lambda^*(t^*)$ denotes the bundle $\bigoplus_{i=1}^n \Lambda^i(t^*)$, then the following identifications are a corollary of the above said:

$$\Lambda_0^i(\pi) = \mathcal{F}(\pi, \textstyle\bigwedge^i(t^*)), \qquad \Lambda_0^*(\pi) = \mathcal{F}(\pi, \textstyle\bigwedge^*(t^*)).$$

Note now that the operator $d = d_M$ of exterior differentiation (the de Rham differential) on the manifold M is a linear differential operator of first order acting from sections of the bundle $\bigwedge^i(t^*)$ to those of the bundle $\bigwedge^{i+1}(t^*)$, $i = 0, \dots, n$. Using the definition of the lifting of differential operators (see Remark 1.1), we shall obtain the operator $\widehat{d} \colon \mathcal{F}(\pi, \Lambda^i(t^*)) \to \mathcal{F}(\pi, \Lambda^{i+1}(t^*))$. Since $d \circ d = 0$, one has $\widehat{d}^2(\omega) = 0$ for any horizontal form $\omega \in \Lambda_0^*(\pi)$ as well. In fact, $\widehat{d}^2(\omega) = (\widehat{d} \circ \widehat{d})(\omega) = (\widehat{d \circ d})(\omega) = 0$. Hence the sequence

$$0 \longrightarrow \mathcal{F}(\pi) \xrightarrow{\widehat{d}} \Lambda_0^1(\pi) \longrightarrow \cdots$$
$$\longrightarrow \Lambda_0^i(\pi) \xrightarrow{\widehat{d}} \Lambda_0^{i+1}(\pi) \longrightarrow \cdots \longrightarrow \Lambda_0^n(\pi) \longrightarrow 0 \tag{1.19}$$

is defined in which the composition of any two adjacent operators is trivial.

DEFINITION 1.8. The sequence (1.19) is called the *horizontal de Rham complex* of the bundle π.

Let us write down the operator $\widehat{d}$ in local coordinates. It is sufficient to be done for horizontal 0-forms, i.e., for functions on $J^\infty(\pi)$. Let us consider a function $\varphi = \varphi(x_1, \ldots, x_n, \ldots, p^j_\sigma, \ldots) \in \mathcal{F}(\pi|_\mathcal{U})$, where $\mathcal{U}$ is a coordinate neighborhood in M, and a section $s \in \Gamma(\pi)$. Then

$$(\widehat{d}\varphi)(s) = d(\varphi(s)) = d\varphi\left(x_1, \ldots, x_n, \ldots, \frac{\partial^{|\sigma|}s^j}{\partial x^\sigma}, \ldots\right)$$

$$= \sum_{\alpha=1}^n \left(\frac{\partial\varphi}{\partial x_\alpha} + \sum_{\sigma,j} \frac{\partial^{|\sigma|+1}s^j}{\partial x_\alpha \partial x^\sigma}\frac{\partial\varphi}{\partial p^j_\sigma}\right) dx_\alpha = \sum_{\alpha=1}^n D_\alpha(\varphi)\,dx_\alpha.$$

Hence,

$$\widehat{d}(\varphi) = \sum_{\alpha=1}^n D_\alpha(\varphi)\,dx_\alpha, \tag{1.20}$$

where D_α, $\alpha = 1, \ldots, n$, are the above introduced operators of total derivatives.

1.7. Distributions on $J^\infty(\pi)$ and their automorphisms. The last problem we intend to discuss in this section is the definition of distributions on the manifolds $J^\infty(\pi)$. Similar to the finite-dimensional case (see §3 of Ch. 1), a distribution $\mathcal{P}$ on $J^\infty(\pi)$ is to be understood as a correspondence $\mathcal{P}\colon \theta \mapsto \mathcal{P}_\theta \subset T_\theta(J^\infty(\pi))$, which takes any point $\theta \in J^\infty(\pi)$ to a subspace of the tangent space "smoothly depending" on θ. But by specifics of the manifold $J^\infty(\pi)$ caused by its infinite dimension, this definition needs to be clarified. Assume that for any $k \geq 0$ a distribution $\mathcal{P}^k\colon \theta_k \mapsto \mathcal{P}^k_{\theta_k} \subset T_{\theta_k}(J^k(\pi))$ is given on the manifold $J^k(\pi)$ and

$$(\pi_{k+1,k})_*(\mathcal{P}^{k+1}_{\theta_{k+1}}) \subset \mathcal{P}^k_{\pi_{k+1,k}(\theta_{k+1})} \tag{1.21}$$

for all $k \geq 0$ and all points $\theta_{k+1} \in J^{k+1}(\pi)$.

DEFINITION 1.9. A system of distributions $\mathcal{P} = \left\{\mathcal{P}^k\right\}$ satisfying the condition (1.21) will be called a *predistribution on $J^\infty(\pi)$*. We shall say that two distributions $\mathcal{P}$ and $\bar{\mathcal{P}}$ are *equivalent*, if there exists a $k_0 \geq 0$ such that $\mathcal{P}^k = \bar{\mathcal{P}}^k$ for all $k \geq k_0$. A class of equivalent predistributions will be called a *distribution on $J^\infty(\pi)$*.

We shall say that a vector $X_\theta \in T_\theta(J^\infty(\pi))$ lies in a distribution, if $(\pi_{\infty,k})_*(X_\theta) \in \mathcal{P}^k_{\pi_{\infty,k}(\theta)}$ for all k, θ and some representative $\mathcal{P}$ of this distribution.

EXERCISE 1.7. Show that if a vector lies in some representative of a distribution, then it lies in any representative of the same distribution.

Consider the subset $\mathcal{P}^k\Lambda^1$ in the module $\Lambda^1(\pi_k)$ consisted of all 1-forms annihilated by vectors of the distribution $\mathcal{P}^k$; let us also set $\mathcal{P}^k\Lambda^* = \Lambda^*(\pi_k) \wedge \mathcal{P}^k\Lambda^1$. The elements of the last set are of the form $\omega = \sum_\alpha \Omega_\alpha \wedge \omega_\alpha$, where $\Omega_\alpha \in \Lambda^*(\pi_k)$ and $\omega_\alpha \in \mathcal{P}^k\Lambda^1$. The set $\mathcal{P}^k\Lambda^*$ is closed with respect to adding and to multiplication by an arbitrary form from $\Lambda^*(\pi_k)$ and is called the *ideal of the distribution $\mathcal{P}^k$*. From the embeddings (1.21) it follows that the chain of embeddings

$$\mathcal{P}^0\Lambda^* \subset \mathcal{P}^1\Lambda^* \subset \cdots \subset \mathcal{P}^k\Lambda^* \subset \mathcal{P}^{k+1}\Lambda^* \subset \cdots,$$

takes place and this chain determines the ideal $\mathcal{P}\Lambda^* = \bigcup_{k\geq 0}\mathcal{P}^k\Lambda^*$ in $\Lambda^*(\pi)$. This ideal is called the ideal of the distribution $\mathcal{P}$. Vice versa, any ideal in $\Lambda^*(\pi)$ of the above form determines a distribution on $J^\infty(\pi)$. Using now the isomorphism (1.15), we can give a dual definition for a distribution on $J^\infty(\pi)$. Namely, a distribution

$\mathcal{P}\,\mathrm{D}(\pi)$ consists of vector fields X on $J^\infty(\pi)$ such that $X \lrcorner\, \omega = 0$ for any form $\omega \in \mathcal{P}\Lambda^1(\pi)$.

EXERCISE 1.8. Prove that $\mathcal{P}\Lambda^*$ and $\mathcal{P}\,\mathrm{D}(\pi)$ do not depend on choice of a representative in the distribution $\mathcal{P}$.

Let us recall that a submanifold N of a finite-dimensional manifold M with a distribution $\mathcal{P}'$ is called *integral*, if for any point $x \in N$ one has an embedding $T_x(N) \subset \mathcal{P}'_x$. It is called locally maximal, if it is not contained in any other integral manifold of greater dimension. A distribution $\mathcal{P}'$ is called completely integrable, if for any point there exists a unique locally maximal integral manifolds passing through this point. The classical Frobenius theorem [**115**] says that a distribution $\mathcal{P}'$ is *completely integrable*, if for any form $\omega \in \mathcal{P}'\Lambda^1(N)$ the equality $d\omega = \Omega \wedge \omega'$ is valid for some $\omega' \in \mathcal{P}'\Lambda^1(N)$ and a form Ω on the manifold M. In other words, the operator d takes the ideal $\mathcal{P}'\Lambda^*$ to itself, i.e., this ideal is differentially closed. Using these considerations, we shall say that a distribution $\mathcal{P}$ on $J^\infty(\pi)$ is *completely integrable*, if its ideal $\mathcal{P}\Lambda^*$ is differentially closed.

EXERCISE 1.9. Show that a distribution $\mathcal{P}$ on $J^\infty(\pi)$ is completely integrable if and only if the set $\mathcal{P}\,\mathrm{D}(\pi)$ is closed with respect to the commutator of vector fields, i.e. is a Lie subalgebra in $\mathrm{D}(\pi)$.

EXAMPLE 1.5. Let $\mathcal{P}'$ be a distribution on M. Let us consider the submodule $\widehat{\mathcal{P}'}\,\mathrm{D}(\pi)$ in $\mathrm{D}(\pi)$ consisting of linear combinations (with coefficients in $\mathcal{F}(\pi)$) of the fields $\widehat{X}$, $X \in \mathcal{P}'\,\mathrm{D}(M)$. Then the set $\mathcal{P}'\,\mathrm{D}(\pi)$ determines a distribution $\mathcal{P} = \widehat{\mathcal{P}'}$ on $J^\infty(\pi)$. If $\mathcal{P}'$ is a completely integrable distribution, then $\mathcal{P}$ is completely integrable as well. In fact,

$$[\varphi\widehat{X}, \psi\widehat{Y}] = \varphi\widehat{X}(\psi)\widehat{Y} - \psi\widehat{Y}(\varphi)\widehat{X} + \varphi\psi[\widehat{X}, \widehat{Y}].$$

But $[\widehat{X}, \widehat{Y}] = \widehat{[X,Y]}$ by (1.12) and consequently the right-hand side of the last equality lies in $\mathcal{P}\,\mathrm{D}(\pi)$, if $X, Y \in \mathcal{P}'\,\mathrm{D}(M)$ and $\varphi, \psi \in \mathcal{F}$.

Let now $\mathcal{P}$ be a completely integrable distribution on $J^\infty(\pi)$. As in §3 of Ch. 1, a vector field $X \in \mathrm{D}(\pi)$ will be called an (*infinitesimal*) *automorphism* of the distribution $\mathcal{P}$, if

$$L_X \mathcal{P}\Lambda^*(\pi) \subset \mathcal{P}\Lambda^*(\pi).$$

In a dual way, automorphisms of the distribution $\mathcal{P}$ can be defined by the condition

$$[X, \mathcal{P}\,\mathrm{D}(\pi)] \subset \mathcal{P}\,\mathrm{D}(\pi). \tag{1.22}$$

Denote by $\mathrm{D}_\mathcal{P}(\pi)$ the set of automorphisms of the distribution $\mathcal{P}$.

EXERCISE 1.10. Prove that $\mathrm{D}_\mathcal{P}(\pi)$ is a Lie algebra while $\mathcal{P}\,\mathrm{D}(\pi) \subset \mathrm{D}_\mathcal{P}(\pi)$ is its ideal.

The elements of the quotient Lie algebra

$$\mathrm{sym}(\mathcal{P}) = \mathrm{D}_\mathcal{P}(\pi)/\mathcal{P}\,\mathrm{D}(\pi)$$

are called (*infinitesimal*) *symmetries* of the completely integrable distribution $\mathcal{P}$.
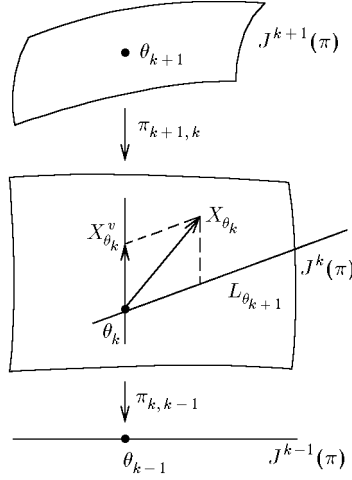
FIGURE 4.3. Construction of the element U_1

2. The Cartan distribution on $J^\infty(\pi)$ and its infinitesimal automorphisms

In this section, we shall define the Cartan distribution on $J^\infty(\pi)$ and study its maximal integral manifolds, infinitesimal automorphisms, and symmetries. This study will lead us to the concept of an evolutionary derivation on $J^\infty(\pi)$, which plays a key role in the theory of symmetries of nonlinear differential equations.

2.1. The Cartan distribution. Let, as above, $\pi \colon P \to M$ be a smooth vector bundle and $J^\infty(\pi)$ be the manifold of its infinite jets. Consider a point $\theta = \{x, \theta_k\}$ in $J^\infty(\pi)$ and the Cartan planes $\mathcal{C}^k_{\theta_k}$ at every point $\theta_k \in J^k(\pi)$. Then, if the point θ is represented as the infinite jet $[s]^\infty_x$ of a section $s \in \Gamma(\pi)$ at the point $x \in M$, the plane $\mathcal{C}^{k+1}_{\theta_{k+1}}$ projects to the R-plane $L_{\theta_{k+1}} \subset \mathcal{C}^k_{\theta_k} \subset T_{\theta_k}(J^k(\pi))$ under the mapping $\pi_{k+1,k}$, where $L_{\theta_{k+1}}$ is the tangent plane to the graph of the jet $j_k(s)$ at the point θ_k. Thus we see that for all k the embeddings $(\pi_{k+1,k})_*(\mathcal{C}^{k+1}_{\theta_{k+1}}) \subset \mathcal{C}^k_{\theta_k}$ take place, i.e., the system of distributions $\mathcal{C}^k$ determines the distribution $\mathcal{C} = \mathcal{C}(\pi)$ on $J^\infty(\pi)$.

DEFINITION 2.1. The distribution $\mathcal{C}(\pi)$ is called the *Cartan distribution* on the manifold of infinite jets.

From the above said it follows that a vector X_θ tangent to the manifold $J^\infty(\pi)$ lies in the plane $\mathcal{C}_\theta$ if and only if its projection $X_{\theta_k} = (\pi_{\infty,k})_*(X_\theta)$ lies in the space L_{θ_k}. Using this remark, we shall give a more efficient description of the Cartan distribution on $J^\infty(\pi)$. Namely, if $X_\theta = \{X_x, X_{\theta_k}\}$ is an arbitrary tangent vector and $\varphi \in \mathcal{F}(\pi)$ is a smooth function on $J^\infty(\pi)$, we shall consider the projections $X^v_{\theta_k}$ of every vector X_{θ_k} along the plane $L_{\theta_{k+1}}$ to the fiber of the bundle $\pi_{k,k-1} \colon J^k(\pi) \to J^{k-1}(\pi)$ over the point θ_{k-1} (see Figure 4.3). Then $X^v_\theta = \{0, X^v_{\theta_k}\}$ is also a tangent vector at the point θ and applying it to the function φ one will obtain a real number. If now $X \in D(\pi)$ is a vector field, one can repeat the described construction at every point $\theta \in J^\infty(\pi)$ and obtain a vertical field $X^v \in D^v(\pi)$, which can be applied to the function $\varphi \in \mathcal{F}$. Thus for any function $\varphi \in \mathcal{F}$ we obtain a 1-form on $J^\infty(\pi)$

denoted by $U_1(\varphi)$ and defined by the equality

$$X \lrcorner U_1(\varphi) = X^v(\varphi), \qquad X \in D(\pi), \quad \varphi \in \mathcal{F}(\pi). \qquad (2.1)$$

From (2.1) it is seen that the correspondence $U_1 \colon \mathcal{F}(\pi) \to \Lambda^1(\pi)$ is a derivation of the algebra $\mathcal{F}$ with the values in the $\mathcal{F}$-module $\Lambda^1(\pi)$, i.e., $U_1(\varphi_1\varphi_2) = \varphi_1 U_1(\varphi_2) + \varphi_2 U_1(\varphi_1)$ for any functions $\varphi_1, \varphi_2 \in \mathcal{F}$.

Coming back to the starting point of our arguments, we see that the following statement is valid.

PROPOSITION 2.1. *A field $X \in D(\pi)$ lies in the Cartan distribution $\mathcal{C}$ on $J^\infty(\pi)$ if and only if*

$$X \lrcorner U_1(\varphi) = 0$$

for all functions $\varphi \in \mathcal{F}(\pi)$. In other words, the ideal $\mathcal{C}\Lambda^*(\pi) \subset \Lambda^*(\pi)$ of the Cartan distribution is generated by the forms $U_1(\varphi)$.

The forms $U_1(\varphi)$ will be called the *Cartan forms*[3].

In what follows, we shall need a more convenient, from the computational point of view, description of the operator U_1 and, as a consequence, of Cartan forms. To obtain such a description, note the following. From the above constructions it follows that any vector field $X \in \mathrm{D}(\pi)$ admits the canonical representation

$$X = X^v + X^h, \qquad (2.2)$$

where, as it was already said, the field X^v is vertical while the field $X^h \stackrel{\text{def}}{=} X - X^v$ is to all manifolds of the form $j_\infty(s)(M)$, i.e., to the graphs of infinite jets of sections s of the bundle π. By this reason, equality (2.1) can be rewritten in the form $X \lrcorner U_1(\varphi) = (X - X^h)(\varphi) = X(\varphi) - X^h(\varphi)$. Note that the first summand in the right-hand side of the last equality is $X \lrcorner d\varphi$. To rewrite the second summand in the form we need, let us compare the expression for $X^h(\varphi)$ with the definition of the form $\widehat{d\varphi}$. By the point-wise definition of the lifting (see §1.4), the restriction of the form $\widehat{d\varphi}$ to the graph of the infinite jet of a section $s \in \Gamma(\pi)$ is represented as $d\Delta_\varphi(s)$. Since the form $\widehat{d\varphi}$ is horizontal, the equality $X \lrcorner \widehat{d\varphi} = X^h \lrcorner \widehat{d\varphi}$ is valid. Finally, since the field X^h is tangent to every submanifold $j_\infty(s)(M)$ and at least one such a manifold passes through any point $\theta \in J^\infty(\pi)$[4], the equality $X^h(\widehat{d\varphi}) = X^h(\varphi)$ holds. Thus we arrive to the following important equality:

$$U_1 = d - \widehat{d}. \qquad (2.3)$$

Consequently, the operator U_1 "measures" to what extent the de Rham differential of a function $\varphi \in \mathcal{F}$ differs from the horizontal one.

We shall now describe the set $\mathcal{C}\mathrm{D}(\pi)$, i.e., vector fields $X \in \mathrm{D}(\pi)$ annihilating the Cartan forms on $J^\infty(\pi)$ or, in other words, the ones which satisfy the equalities $X(\varphi) = X \lrcorner \widehat{d\varphi}$ for all smooth functions $\varphi \in \mathcal{F}$. Consider a field $X \in D(\pi)$ and its restriction $X_M = X|_{\mathcal{F}_{-\infty}(\pi)} \colon C^\infty(M) \to \mathcal{F}_k(\pi)$, which is a derivation of the ring $C^\infty(M)$ with the values in the algebra $\mathcal{F}_k(\pi)$ (such a k exists, since X is a derivation compatible with filtration of $\mathcal{F}(\pi)$). Informally, the derivation X_M may

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[3]Below we shall see that this definition generalizes the coordinate representation of Cartan forms considered in preceding chapters.

[4]It is a consequence of the Borel lemma [**80**].

be understood as a vector field on M whose coefficients are functions from $\mathcal{F}_k$. More exactly, the field X_M, at least locally, can be represented in the form

$$X_M = \varphi_1 X_1 + \cdots + \varphi_l X_l, \tag{2.4}$$

where $X_1, \ldots, X_l \in \mathrm{D}(M)$ are "real" vector fields on the manifold M and the functions $\varphi_1, \ldots, \varphi_l$ lie in $\mathcal{F}_k(\pi)$. Using the representation (2.4), let us define the field $\widehat{X}_M \in \mathrm{D}(\pi)$ by setting $\widehat{X}_M = \varphi_1 \widehat{X}_1 + \cdots + \varphi_l \widehat{X}_l$. Since, by the point-wise definition of the lifting operation, the vectors $(\widehat{X}_l)_\theta$ are tangent to the graph of the infinite jet of a section $s \in \Gamma(\pi)$ at any point $\theta = [s]_x^\infty \in J^\infty(\pi)$, the same is true for the vector $(\widehat{X}_M)_\theta = \varphi_1(\theta)(\widehat{X}_1)_\theta + \cdots + \varphi_l(\theta)(\widehat{X}_l)_\theta$. Therefore, $\widehat{X}_M^v = 0$ and consequently $\widehat{X}_M \lrcorner U_1(\varphi) = \widehat{X}_M^v(\varphi) = 0$ for any function $\varphi \in \mathcal{F}$.

Let us consider (also locally) the field $X' = X - \widehat{X}_M$. Since, by definition, $\widehat{X}_M\big|_{C^\infty(M)} = X_M = X\big|_{\mathcal{F}_{-\infty}(\pi)}$, the field X' is vertical and consequently $X' \lrcorner U_1(\varphi) = X'(\varphi)$ for any function $\varphi \in \mathcal{F}$. On the other hand, as we established just now, $\widehat{X}_M \lrcorner U_1(\varphi) = 0$ and if $X \in \mathcal{C}\mathrm{D}(\pi)$, then $X \lrcorner U_1(\varphi) = 0$ as well. Hence, $X'(\varphi) = 0$ for any function $\varphi \in \mathcal{F}$. Therefore, $X' = 0$ and consequently $X = \widehat{X}_M$. Combining together the above said, we obtain the following result.

PROPOSITION 2.2. *The set $\mathcal{C}\mathrm{D}(\pi)$ of vector fields lying in the Cartan distribution is generated by the liftings to $J^\infty(\pi)$ of the fields on the manifold M:*

$$\mathcal{C}\mathrm{D}(\pi) = \left\{ \varphi_1 \widehat{X}_1 + \cdots + \varphi_l \widehat{X}_l \mid \varphi_i \in \mathcal{F}(\pi), \ X_i \in D(M), \ l = 1, 2, \ldots \right\}.$$

2.2. Integral manifolds. From Proposition 2.2 and Example 1.5 it follows that the Cartan distribution on $J^\infty(\pi)$ is completely integrable. Let study its maximal integral manifolds. To do this, let us first stress two facts: (a) the Cartan plane $\mathcal{C}_\theta$ at a point $\theta \in J^\infty(\pi)$ does not contain nonzero vertical vectors—this fact follows from the constructions used in the beginning of this section to define the forms $U_1(\varphi)$; (b) dimension of the space $\mathcal{C}_\theta$ coincides with that of the manifold M, and this is a consequence of the proposition just proved. Let $\mathcal{R}^\infty$ be a maximal integral manifold of the Cartan distribution. Then from (a) it follows that the projections $\pi_{\infty,k}\big|_{\mathcal{R}^\infty} : \mathcal{R}^\infty \to \mathcal{R}^k \overset{\mathrm{def}}{=} \pi_{\infty,k}(\mathcal{R}^\infty) \subset J^k(\pi)$, $k = 0, 1, \ldots$, and $\pi_\infty\big|_{\mathcal{R}^\infty} : \mathcal{R}^\infty \to \mathcal{R} \overset{\mathrm{def}}{=} \pi_\infty(\mathcal{R}^\infty) \subset M$ are local diffeomorphisms. On the other hand, as it follows from (b), one has $\dim \mathcal{R}^k = \dim \mathcal{R}^\infty \leq n$. Consider the projection $\pi\big|_{\mathcal{R}^0} : \mathcal{R}^0 \to \mathcal{R}$. Since it is a local diffeomorphism, there exists a mapping $s' : \mathcal{R} \to \mathcal{R}^0$ such that the composition $\pi\big|_{\mathcal{R}^0} \circ s'$ is the identity on $\mathcal{R}$. In other words, s' is a partially defined section of the bundle π. Obviously, there exists a section $s \in \Gamma(\pi)$ such that $s\big|_{\mathcal{R}} = s'$. Assume now that the embedding $\mathcal{R} \subset M$ is strict, i.e., that $\dim \mathcal{R}^k < n$. Then the conditions that the manifolds $\Gamma_s^k = j_k(s)(M) \subset J^k(\pi)$, $k = 0, 1, \ldots, \infty$, contain the manifolds $\mathcal{R}^k$ for the corresponding values of k are the conditions for the normal (with respect to $\mathcal{R}$) derivatives of the section s. By the Whitney theorem on extension for smooth functions [**80**], it follows that one can always construct a section satisfying these conditions. This result contradicts to maximality of $\mathcal{R}^\infty$. Hence, the following statement is valid:

PROPOSITION 2.3. *Any maximal integral manifold of the Cartan distribution on $J^\infty(\pi)$ is locally the graph of the infinite jet of a section of the bundle π.*

Thus, at least one maximal integral manifold of the Cartan distribution passes through any point $\theta \in J^\infty(\pi)$ and we have a complete description of these manifolds.
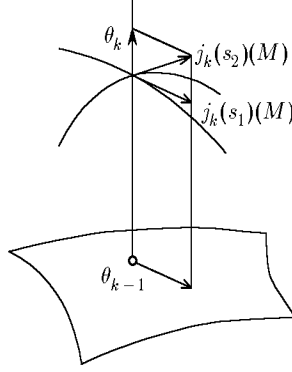
FIGURE 4.4. "Blowing-up" of an R-plane

Of course, such a manifold is not uniquely defined. For example, if $\pi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the trivial bundle, the graph of infinite jet of the function $u = \exp(-1/x^2)$ passes through the origin in $J^\infty(\pi)$ together with the graph of the infinite jet of the zero function. In general, if π is an arbitrary bundle and $\theta = [s]_x^\infty \in J^\infty(\pi)$, then the graphs of the infinite jets of all sections of the bundle π pass through the point θ differing from s by a flat at the point $x \in M$ section. It is useful to compare the results obtained here with what we know about maximal integral manifolds in the spaces of finite jets. As it was noted in the previous chapter, the Cartan distribution on $J^k(\pi)$, $k < \infty$, contains vertical vectors. Namely, any vector tangent to the fiber of the bundle $\pi_{k,k-1}$ lies in the Cartan plane $\mathcal{C}_{\theta_k}^k$. This leads to the fact that the set of integral manifolds passing through the point $\theta_k \in J^k(\pi)$ can be classified by types and the type of a manifold describes the degree of its degeneracy under the projection to $J^{k-1}(\pi)$ (or dimension of the space of vertical vectors tangent the integral manifold under consideration at the given point). In particular, the graphs of jets of sections of the bundle π are the manifolds of type 0 and are characterized by the fact that they project to $J^{k-1}(\pi)$ locally diffeomorphically. The reasons why the cases of finite and infinite jets are different become clear, if one stands on the analytical point of view. In fact, fixing a point $\theta_k \in J^k(\pi)$ for $k < \infty$, we obtain information on all partial derivatives of sections $s \in \Gamma(\pi)$ at the point $x = \pi_k(\theta_k) \in M$ up to order k. Drawing through the point θ_k graphs of different sections, we have freedom of choice which is measured infinitesimally by vectors tangent to the fiber of the projection $\pi_{k,k-1}$ (see Figure 4.4). This leads to the fact that an R-plane at the point θ_k "blows-up" by the set $\pi_{k,k-1}$-vertical vectors. Unlike the situation described, the choice of a point $\theta \in J^\infty(\pi)$ fixes all partial derivatives of a section $s \in \Gamma(\pi)$ at the point x and, by obvious reasons, our freedom is limited now by the class of flat functions. Note also that the picture we observe on $J^\infty(\pi)$ is related to existence of the Cartan connection on π_∞ and to integrability of this connection.

We now start to study automorphisms and symmetries of the Cartan distribution on the manifolds $J^\infty(\pi)$. Before starting to discuss the facts of general nature, we shall fulfil some coordinate computations useful to get a kind of intuition on the results to be obtained.

2.3. A computational experiment. We shall need here (as well as everywhere below) a coordinate representation of the Cartan forms $U_1(\varphi)$. Let $x \in M$ and $\mathcal{U} \ni x$ be a coordinate neighborhood. Consider, as usual, local coordinates $x_1, \ldots, x_n, \ldots, p_\sigma^j, \ldots$ in the neighborhood $\pi_\infty^{-1}(\mathcal{U})$. Then, using formulas (1.20) and (2.3) together with the fact that the operator $U_1 \colon \mathcal{F}(\pi) \to \Lambda^1(\pi)$ is a derivation, it is easily seen that the Cartan module $\mathcal{C}\Lambda^*(\pi)|_{\pi^{-1}(\mathcal{U})}$ is generated by the forms

$$\omega_\sigma^j = U_1(p_\sigma^j) = dp_\sigma^j - \sum_{\alpha=1}^n p_{\sigma+1_\alpha} \, dx_\alpha, \tag{2.5}$$

where $j = 1, \ldots, m$, $|\sigma| = 0, 1, \ldots$, i.e., by the Cartan forms defined in Ch. 3. In particular,

$$\omega_\varnothing^j = U_1(u^j) = du^j - \sum_{\alpha=1}^n p_{1_\alpha} \, dx_\alpha. \tag{2.6}$$

Consider the one-dimensional bundle $\pi \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ over the real line. Let $x = x_1$ be the coordinate in the base and set $p_{(0)} = u, \ldots, p_{(k)} = p_k, \ldots$ to be the coordinates in $J^\infty(\pi)$. By (2.5), the Cartan module $\mathcal{C}\Lambda^*(\pi)$ is generated by the forms

$$\omega_k = dp_k - p_{k+1} \, dx \tag{2.7}$$

in this case, where $k = 0, 1, \ldots$. Let the field $X = a\partial/\partial x + \sum_{k=0}^\infty b_k \partial/\partial p_k \in \mathrm{D}(\pi)$, $a, b_k \in \mathcal{F}(\pi)$, be an automorphism of the Cartan distribution. This is equivalent to the fact that $X(\omega_k) = \sum_i \lambda_k^i \omega_i$ for all k or, which is the same, to

$$db_k - b_{k+1} \, dx - p_{k+1} \, da = \sum_i \lambda_k^i (dp_i - p_{i+1} \, dx). \tag{2.8}$$

Writing down the differential at the left-hand side of (2.8) in an explicit form and collecting similar terms, we obtain the following system of equations

$$\left(\frac{\partial b_k}{\partial x} - b_{k+1} - p_{k+1} \frac{\partial a}{\partial x} \right) dx + \sum_i \left(\frac{\partial b_k}{\partial p_i} - p_{k+1} \frac{\partial a}{\partial p_i} \right) dp_i$$

$$= \sum_i \lambda_k^i (dp_i - p_{i+1} \, dx),$$

which is equivalent to the system

$$\begin{cases} \dfrac{\partial b_k}{\partial x} - b_{k+1} - p_{k+1} \dfrac{\partial a}{\partial x} = -\sum_i \lambda_k^i p_{i+1}, \\[2mm] \dfrac{\partial b_k}{\partial p_i} - p_{k+1} \dfrac{\partial a}{\partial p_i} = \lambda_k^i, \quad i, k \geq 0. \end{cases} \tag{2.9}$$

Substituting the expressions obtained for λ_k^i to the first equation in (2.9), we see that

$$b_{k+1} = \frac{\partial b_k}{\partial x} - p_{k+1} \frac{\partial a}{\partial x} + \sum_i p_{i+1} \left(\frac{\partial b_k}{\partial p_i} - p_{k+1} \frac{\partial a}{\partial p_i} \right) = D(b_k) - p_{k+1} D(a), \tag{2.10}$$

where $D = D_1$ is the operator of the total derivative along x. As above, let us split the field X into the horizontal and vertical components by setting $X^h = a\widehat{\partial/\partial x} = aD$ and $X^v = X - X^h$. Let $X^v = \sum_k b_k^v \partial/\partial p_k$; then $b_k = b_k^v + ap_{k+1}$ for all

$k \geq 0$. Substituting the expressions obtained to (2.10) and keeping in mind that $D(p_k) = p_{k+1}$, we obtain the following expressions for the coefficients b_k^v:

$$b_{k+1}^v = D(b_k^v), \qquad k = 0, 1, \dots,$$

or

$$b_k^v = D^k(b_0^v), \qquad k = 0, 1, \dots$$

Thus any infinitesimal automorphism of the Cartan distribution on $J^\infty(\pi)$ is representable in the form

$$X = aD + \sum_{k=0}^\infty D^k(b_0^v) \frac{\partial}{\partial p_k}, \tag{2.11}$$

where $b_0^v = b_0 - ap_1$, and consequently is uniquely determined by its restriction to the subalgebra $C^\infty(M) \subset \mathcal{F}(\pi)$ (the first summand in the representation (2.11)) and by some function $b_0^v \in \mathcal{F}(\pi)$ (the second summand). It turns out that the result obtained is of a general nature.

2.4. Evolutionary derivations. Let us again return back to an arbitrary vector bundle $\pi \colon P \to M$ and consider an infinitesimal automorphism $X \in \mathrm{D}_{\mathcal{C}}(\pi)$ of the Cartan distribution on $J^\infty(\pi)$.

PROPOSITION 2.4. *Any vector field $X \in \mathrm{D}_{\mathcal{C}}(\pi)$ is uniquely determined by its restriction to the subalgebra $\mathcal{F}_0(\pi) = C^\infty(J^0(\pi)) \subset \mathcal{F}(\pi)$.*

PROOF. Since for any two fields $X, X' \in \mathrm{D}_{\mathcal{C}}(\pi)$ their difference lies in $\mathrm{D}_{\mathcal{C}}(\pi)$ as well, it is sufficient to show that if $X|_{\mathcal{F}_0} = 0$, then $X|_{\mathcal{F}_k} = 0$ for all $k \geq 0$. Let us prove this fact by induction on k. The first step of induction is $(k = 0)$ is the assumptions of the proposition.

Let now $k > 0$ and $X|_{\mathcal{F}_k} = 0$. We must prove that under this condition the derivation $X|_{\mathcal{F}_{k+1}}$ is also trivial. Take $\varphi \in \mathcal{F}_k$. Then

$$X(U_1(\varphi)) = X(d\varphi - \widehat{d}\varphi) = dX(\varphi) - X(\widehat{d}\varphi),$$

or, by the induction hypothesis,

$$X(U_1(\varphi)) = -X(\widehat{d}\varphi).$$

On the other hand, since $X \in \mathrm{D}_{\mathcal{C}}(\pi)$, there exist functions $f_\alpha, g_\alpha \in \mathcal{F}$, $\alpha = 1, \dots, l$, such that $X(U_1(\varphi)) = \sum_\alpha f_\alpha U_1(g_\alpha)$, i.e.,

$$X(\widehat{d}\varphi) = -\sum_\alpha f_\alpha U_1(g_\alpha). \tag{2.12}$$

The form at the right-hand side of equation (2.12) belongs to the Cartan module and consequently its restriction to the graph of the jet of any section $s \in \Gamma(\pi)$ vanishes.

Let us use now the fact that for any function $\varphi \in \mathcal{F}(\pi)$ the form $\widehat{d}\varphi$ is horizontal, i.e., can be represented as $\widehat{d}\varphi = \sum_i \varphi_i \, dh_i$, where $\varphi_i \in \mathcal{F}$ and $h_i \in C^\infty(M) \subset \mathcal{F}_0$. Consequently,

$$X(\widehat{d}\varphi) = \sum_i \left(X(\varphi_i) \, dh_i + \varphi_i \, dX(h_i) \right) = \sum_i X(\varphi_i) \, dh_i,$$

since by the induction hypothesis one has $X(h_i) = 0$. Thus $X(\widehat{d}\varphi)$ is also a horizontal form. From (2.12) and from the above said it follows that its restriction to
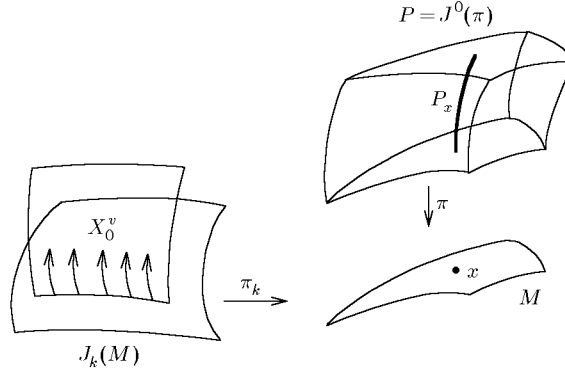
FIGURE 4.5. Vertical field X_0^v

the graphs of jets of sections are trivial. Since horizontal forms are determined by their restrictions on the graphs of jets (see §1), we obtain that

$$X(\widehat{d}\varphi) = 0 \tag{2.13}$$

for any function $\varphi \in \mathcal{F}_k$.

Consider an arbitrary coordinate neighborhood in M and the corresponding local coordinates $(x_1, \ldots, x_n, \ldots, p_\sigma^j, \ldots)$ in $J^\infty(\pi)$. Let us choose one of the coordinate functions p_σ^j, where $|\sigma| \leq k$, for φ. Then by (2.13) one has

$$X(\widehat{d}p_\sigma^j) = X\left(\sum_{i=1}^n p_{\sigma+1_i}^j \, dx_i\right) = \sum_{i=1}^n X(p_{\sigma+1_i}^j) \, dx_i = 0.$$

Hence, $X(p_{\sigma+1_i}^j) = 0$ for all σ, $|\sigma| \leq k$, $i = 1, \ldots, n$, and $j = 1, \ldots, m$. In other words, the action of the field X on all coordinate functions on the manifold $J^{k+1}(\pi)$ is trivial, i.e., $X|_{\mathcal{F}_{k+1}} = 0$. This proves the induction step and consequently the statement as a whole. $\qquad\square$

Let still $X \in \mathrm{D}_\mathcal{C}(\pi)$ be an automorphism of the Cartan distribution on $J^\infty(\pi)$. Consider the restriction X_M of the field X to the subalgebra $C^\infty(M) \subset \mathcal{F}$ and the corresponding lifting $\widehat{X}_M \in \mathrm{D}(\pi)$. Since, as we already know, $\widehat{X}_M \in \mathcal{C}\mathrm{D}(\pi) \subset \mathrm{D}_\mathcal{C}(\pi)$, the field $X^v = X - \widehat{X}_M$ is also an automorphism and, moreover, is vertical. Then the restriction $X_0^v = X^v|_{\mathcal{F}_0}$ is a derivation of the algebra $\mathcal{F}_0$ with values in the algebra $\mathcal{F}_k$ for some finite k. Since the field X^v is vertical, the derivation X_0^v can be identified with a family of vectors parametrized by points of the space $J^k(\pi)$ and directed along the fibers of the pullback $\pi_k^*(\pi)$ (see Figure 4.5). On the other hand, since π is a linear bundle, tangent vectors to its fibers can be identified with the points of these fibers. Combining together the above constructions, we obtain the mapping

$$\mathrm{D}_\mathcal{C}(\pi) \longrightarrow \mathcal{F}(\pi, \pi), \tag{2.14}$$

and, as Propositions 2.2 and 2.4 show, a field X maps to zero under this correspondence if and only if $X \in \mathcal{C}\mathrm{D}(\pi)$. Let us study the image of the mapping (2.14).

Let φ be a section of the bundle $\pi_k^*(\pi)$. Consider a coordinate neighborhood $\mathcal{U} \subset M$ and the neighborhood $\mathcal{U}^k = \pi_k^{-1}(\mathcal{U}) \subset J^k(\pi)$. Similar to the results

obtained in the computational experiment above, let us define in the neighborhood $\mathcal{U}^k$ the field[5]

$$\Theta_{\varphi,\mathcal{U}} = \sum_{j,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial p_\sigma^j}, \tag{2.15}$$

where φ^j is the j-th component of the restriction of the section φ to the neighborhood $\mathcal{U}^k$ while D_σ is the composition of the total derivatives corresponding to the multi-index σ. Let us show that the derivation (2.15) in the neighborhood under consideration is an automorphism of the Cartan distribution. In fact, let $\psi \in \mathcal{F}$. Then

$$\Theta_{\varphi,\mathcal{U}}(U_1(\psi)) = \Theta_{\varphi,\mathcal{U}}(d\psi - \widehat{d}\psi) = d(\Theta_{\varphi,\mathcal{U}}(\psi)) - \Theta_{\varphi,\mathcal{U}}(\widehat{d}\psi). \tag{2.16}$$

By relations (1.20) and (2.15), we have

$$\Theta\varphi,\mathcal{U}(d\widehat{\psi}) = \sum_{i,j,\sigma} D_\sigma(\psi^j) \frac{\partial}{\partial p_\sigma^j} D_i(\psi)\, dx_i.$$

But as it is easily seen,

$$\frac{\partial}{\partial p_\sigma^j} D_i(\psi) = \delta(i,\sigma) \frac{\partial \psi}{\partial p_{\sigma-1_i}^j} + D_i\left(\frac{\partial \psi}{\partial p_\sigma^j}\right), \tag{2.17}$$

where

$$\delta(i,\sigma) = \begin{cases} 1 & \text{if the } i\text{-th component of } \sigma \text{ differs from } 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\Theta_{\varphi,\mathcal{U}}(\widehat{d}\psi) = \sum_{i,j,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial p_\sigma^j} D_i(\psi)\, dx_i$$

$$= \sum_{i,j,\sigma} D_\sigma(\varphi^j) \left(\delta(i,\sigma) \frac{\partial \psi}{\partial p_{\sigma-1_i}^j} + D_i\left(\frac{\partial \psi}{\partial p_\sigma^j}\right) \right) dx_i$$

$$= \sum_{i,j,\sigma} \left(D_\sigma(\varphi^i)\delta(i,\sigma) \frac{\partial \psi}{\partial p_{\sigma-1_i}^j} + D_i\left(D_\sigma(\varphi^j) \frac{\partial \psi}{\partial p_\sigma^j}\right) - D_{\sigma+1_i}(\varphi^j) \frac{\partial \psi}{\partial p_\sigma^j} \right) dx^i.$$

In the expression obtained, the first and the last summands mutually annihilate and thus

$$\Theta\varphi,\mathcal{U}(\widehat{d}\psi) = \sum_{i,j,\sigma} D_i\left(D_\sigma(\varphi^j) \frac{\partial \psi}{\partial p_\sigma^j}\right) dx_i = \widehat{d}(\Theta\varphi,\mathcal{U}(\psi)).$$

Coming back to equality (2.16), we see that

$$\Theta_{\varphi,\mathcal{U}}(U_1(\psi)) = d(\Theta_{\varphi,\mathcal{U}}(\psi)) - \widehat{d}(\Theta_{\varphi,\mathcal{U}}(\psi)) = U_1 \Theta_{\varphi,\mathcal{U}}(\psi). \tag{2.18}$$

Equality (2.18) shows that the derivations of the form $\Theta_{\varphi,\mathcal{U}}$ commute with the operator U_1 and consequently are infinitesimal automorphisms of the Cartan distribution. The restriction $\Theta_{\varphi,\mathcal{U}}$ to the subalgebra $\mathcal{F}_0(\pi|_\mathcal{U})$ is identified with the section φ. Thus we have a procedure (so far, local) to construct an automorphism of the Cartan distribution determined by a section φ. Let now $\mathcal{U}, \mathcal{U}' \subset M$ be two coordinate neighborhoods in M. Then the restrictions of the fields $\Theta_{\varphi,\mathcal{U}}$ and $\Theta_{\varphi,\mathcal{U}'}$ to the algebra $\mathcal{F}_0(\pi|_{\mathcal{U}\cap\mathcal{U}'})$ coincide, from where, by Proposition 2.4, it follows that

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[5]In the notation below, we use the Cyrillic letter Θ, which is pronounced like "e" in "*ten*".

the fields coincide over the intersection $\mathcal{U} \cap \mathcal{U}'$ as well. Hence our constructions in a well-defined way determine a vertical vector field ϑ_φ on the entire space $J^\infty(\pi)$ and this field is an infinitesimal automorphism of the Cartan distribution. In other words, the mapping (2.14) is epimorphic and therefore the following theorem is proved.

THEOREM 2.5. *Any infinitesimal automorphism X of the Cartan distribution on $J^\infty(\pi)$ is uniquely represented in the form*

$$X = \vartheta\varphi + \widehat{X}_M,$$

where φ is a section from $\mathcal{F}(\pi, \pi)$. The algebra $\operatorname{sym}\mathcal{C}(\pi) = \mathrm{D}_\mathcal{C}(\pi)/\mathcal{C}\mathrm{D}(\pi)$ of symmetries of the Cartan distribution is identified with the module $\mathcal{F}(\pi, \pi)$ of the sections of the pullback $\pi_\infty^*(\pi)$,

$$\operatorname{sym}\mathcal{C}(\pi) \simeq \mathcal{F}(\pi, \pi).$$

Let us consider the fields ϑ_φ in more details. In the preceding chapter we saw that Lie fields, being automorphisms of the Cartan distribution on the manifolds $J^k(\pi)$ of finite jets, generate flows on the set of sections of the bundle π. The same is valid for the fields of the form ϑ_φ, though in general they do not possess one-parameter groups of transformations (see below) on the infinite-dimensional manifold $J^\infty(\pi)$. In fact, let $s \in \Gamma(\pi)$ be a section and $\varphi \in \mathcal{F}_k(\pi, \pi)$. Let us restrict the field ϑ_φ to the graph of the k-th jet of the section s. Then, under identification of the tangent spaces to the fibers of π with the fibers themselves, the components of this restriction will be of the form $(j_k(s))^*(\varphi^\alpha)$, $\alpha = 1, \ldots, m$. Therefore, the motion of the section s along the trajectories of the field ϑ_φ (provided these trajectories exist) should be governed by the equations

$$\frac{\partial s^j}{\partial t} = \varphi^j\left(x, \ldots, \frac{\partial^{|\sigma|}s^l}{\partial x^\sigma}, \ldots\right), \qquad j, l = 1, \ldots, m, \quad |\sigma| \leq k,$$

where t is the parameter along the trajectory. In other words, the derivation ϑ_φ determines evolution of sections of the bundle π, while the "flow" corresponding to ϑ_φ is determined by evolution equations of the form

$$\frac{\partial u^j}{\partial t} = \varphi^j\left(x, \ldots, \frac{\partial^{|\sigma|}u^l}{\partial x^\sigma}, \ldots\right), \qquad j, l = 1, \ldots, m, \quad |\sigma| \leq k. \tag{2.19}$$

DEFINITION 2.2. The fields ϑ_φ are called *evolutionary derivations*; the corresponding section $\varphi \in \mathcal{F}(\pi, \pi)$ is called the *generating section* of the evolutionary derivation.

Note that a special form of coefficients of evolutionary derivations (see equation (2.15)) guarantees mutual compatibility for evolution of derivatives of the functions u^j:

$$\frac{\partial}{\partial t}\frac{\partial^{|\tau|}u}{\partial x^\tau} = (D_\tau\varphi^j)\left(x, \ldots, \frac{\partial^{|\sigma|}u}{\partial x^\sigma}, \ldots\right). \tag{2.20}$$

Equation (2.20) is a coordinate expression of the fact that ϑ_φ is an automorphism of the Cartan distribution.

It is interesting to note that if one treats evolutionary derivations as vector fields on the "manifold" $\Gamma(\pi)$ of sections of the bundle π, the operator $U_1: \mathcal{F}(\pi) \to \Lambda^1(\pi)$ plays the role similar to that of the external differential $d: C^\infty(M) \to \Lambda^1(M)$ in classical differential geometry. In fact, if X is an evolutionary derivation, then one

has (see the splitting (2.2)) $X^v = X$ and therefore, by the definition of the operator U_1,

$$X \lrcorner U_1(\varphi) = X(\varphi). \tag{2.21}$$

Equation (2.21) is an exact counterpart of the equation

$$Y \lrcorner df = Y(f),$$

valid for fields Y and functions f on some finite-dimensional manifold. Therefore, the operator U_1 can be called the *universal evolutionary differential*.

2.5. Jacobi brackets. Above we showed that the mapping

$$\eth \colon \mathcal{F}(\pi, \pi) \longrightarrow \mathrm{sym}(\pi) = \mathrm{D}_{\mathcal{C}}(\pi)/\mathcal{C}\mathrm{D}(\pi),$$

putting into correspondence to any section $\varphi \in \mathcal{F}(\pi, \pi)$ the evolutionary derivation $\eth_\varphi$, is a bijection. On the other hand, since the set of evolutionary derivations is identified with the Lie quotient algebra $\mathrm{sym}(\pi)$, the commutator of two evolutionary derivations is an evolutionary derivation again. Therefore, for any two elements $\varphi, \psi \in \mathcal{F}(\pi, \pi)$ one can define their commutator $\{\varphi, \psi\} \in \mathcal{F}(\pi, \pi)$ by setting

$$\{\varphi, \psi\} = \eth^{-1}\left([\eth\varphi, \eth\psi]\right), \tag{2.22}$$

or

$$\eth_{\{\varphi, \psi\}} = [\eth\varphi, \eth\psi]. \tag{2.23}$$

DEFINITION 2.3. The element $\{\varphi, \psi\}$ defined by equation (2.22) is called the *higher Jacobi bracket* of the sections φ and ψ.

Since evolutionary derivations are vertical and form a Lie algebra with respect to the commutator of vector fields, the module $\mathcal{F}(\pi, \pi)$ is a Lie $\mathbb{R}$-algebra with respect to the Jacobi bracket. In fact, let us show for example that the bracket $\{\cdot, \cdot\}$ satisfies the Jacobi identity. By (2.23) we have

$$\eth_{\{\varphi, \{\psi, \chi\}\}} = [\eth\varphi, \eth_{\{\psi, \chi\}}] = [\eth\varphi, [\eth\psi, \eth\chi]] = [[\eth\varphi, \eth\psi], \eth\chi] + [\eth\psi, [\eth\varphi, \eth\chi]]$$
$$= \eth_{\{\{\varphi, \psi\}, \chi\}} + \eth_{\{\psi, \{\varphi, \chi\}\}} = \eth_{\{\{\varphi, \psi\}, \chi\} + \{\psi, \{\varphi, \chi\}\}},$$

i.e.,

$$\{\varphi, \{\psi, \chi\}\} + \{\psi, \{\chi, \varphi\}\} + \{\chi, \{\varphi, \psi\}\} = 0$$

for any sections $\varphi, \psi, \chi \in \mathcal{F}(\pi, \pi)$. Other properties of Lie algebras can be checked in a similar way.

In local coordinates, the Jacobi bracket of two sections φ and ψ, as it follows from (2.15) and (2.23), is of the form

$$\{\varphi, \psi\}^j = \sum_{\sigma, \alpha} \left(D_\sigma(\varphi^\alpha) \frac{\partial \psi^j}{\partial p_\sigma^\alpha} - D_\sigma(\psi^\alpha) \frac{\partial \varphi^j}{\partial p_\sigma^\alpha} \right), \qquad j = 1, \dots, m. \tag{2.24}$$

2.6. Comparison with Lie fields. In the preceding chapter, we defined Lie fields as derivations $Y \in D(J^k(\pi))$ preserving the Cartan distribution on $J^k(\pi)$. Since any Lie field can be lifted to any space $J^{k+l}(\pi)$ in a canonical way and this lifting is also a Lie field, we obtain a field $Y^* \in D(\pi)$. The condition that all the liftings preserve the Cartan distribution on the corresponding spaces of finite jets means that $Y^* \in D_\mathcal{C}(\pi)$, i.e., the infinite lifting of a Lie field is an automorphism of the Cartan distribution on $J^\infty(\pi)$. A characteristic feature of the derivation Y^* is that it preserves filtration degree of elements from $\mathcal{F}(\pi)$: $\deg(Y^*) = 0$. Conversely, let an automorphism $X \in D_\mathcal{C}(\pi)$ possess this property. Then there exists a k_0 such that for all $k \geq k_0$ the restrictions $X_k = X|_{\mathcal{F}_k}$ are derivations of the algebras $\mathcal{F}_k$ to themselves and consequently these restrictions determine Lie fields on $J^k(\pi)$. For these fields one has

$$X_{k+1} \circ \pi^*_{k+1,k} = \pi^*_{k+1,k} \circ X_k, \tag{2.25}$$

i.e., they are compatible with the projections $\pi_{k+1,k}\colon J^{k+1}(\pi) \to J^k(\pi)$. By the theorem describing Lie fields and proved in the preceding chapter, for any field X_k there exists a uniquely determined Lie field $X_k^\varepsilon \in D(J^\varepsilon(\pi))$ such that X_k is its $(k-\varepsilon)$-lifting[6]. Using equations (2.25), we see that X_{k+1} is the lifting of the field X_k to $J^{k+1}(\pi)$ and consequently X is of the form Y^*, where Y is a Lie field. Hence, the following statement is valid.

PROPOSITION 2.6. *An automorphism $X \in D_\mathcal{C}(\pi)$ is of the form $X = Y^*$, where Y is a Lie field, if and only if $\deg(X) = 0$.*

DEFINITION 2.4. Automorphisms of the Cartan distribution satisfying the assumptions of Proposition 2.6 will be called *Lie fields* on the manifold $J^\infty(\pi)$.

Let X be such a field. Then, by Theorem 2.5, one has the canonical decomposition

$$X = \vartheta_\varphi + \widehat{X}_M. \tag{2.26}$$

Note that due to coordinate representation of evolutionary derivations (see equation (2.15)), the field ϑ_φ raises filtration of elements in $\mathcal{F}(\pi)$ by $\deg(\varphi)$. On the other hand, $\deg(\widehat{X}_M) = 1$ for any field $\widehat{X}_M \in \mathcal{C}D(\pi)$. Therefore, from Proposition 2.6 it follows that if X is a Lie field, then the section φ from the decomposition (2.26) lies in the module $\mathcal{F}_1(\pi, \pi)$, i.e., $\deg(\varphi) = 1$. Recall now that any Lie field on $J^k(\pi)$ is determined by its generating section and let us compare the expression for Lie fields in terms of generating sections (see §3 of Ch. 3) with the definition of the derivation ϑ_φ (see (2.15)). Then it is easily seen that equation (2.26) can be written as

$$X_\varphi^* = \vartheta_\varphi - \sum_{i=1}^n \frac{\partial \varphi^j}{\partial p_{1_i}^j} D_i, \tag{2.27}$$

where X_φ^* is the lifting of the Lie field X_φ with the generating section $\varphi \in \mathcal{F}(\pi, \pi)$ to $J^\infty(\pi)$ while $\varphi^1, \ldots, \varphi^m$ are the components of this section. As we know, if $\dim \pi = 1$, then φ is an arbitrary section, while for $\dim \pi > 1$ it is to be of the form

$$\varphi^j = a_0 + \sum_{i=1}^n p_{1_i}^j a_i, \qquad j = 1, \ldots, m, \quad a_0, \ldots, a_n \in \mathcal{F}_0(\pi).$$

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[6]Recall that $\varepsilon = 0$ for $\dim \pi > 1$ and $\varepsilon = 1$ for $\dim \pi = 1$; in the first case, X_k^0 is an arbitrary vector field on $J^0(\pi)$ while in the second one X_k^1 is a contact field on $J^1(\pi)$.

Hence, generating functions of Lie fields defined in Ch. 3 coincide with those considered here.

EXERCISE 2.1. Show that if X_φ and X_ψ are two Lie fields, then

$$[X_\varphi^*, X_\psi^*] = [X_\varphi, X_\psi]^* = \eth_{\{\varphi,\psi\}} - \sum_{i=1}^n \frac{\partial\{\varphi,\psi\}^j}{\partial p_{1_i}^j} D_i.$$

In other words, if φ and ψ are generating sections of Lie fields, then the Jacobi bracket of these two sections defined earlier coincides with the Jacobi bracket introduced in this section. Taking into account that by (2.27) one has $X_\varphi = 0$ if and only if the corresponding evolutionary derivation is trivial, we can formulate the following statement.

PROPOSITION 2.7. *The mapping taking Lie fields to the corresponding evolutionary derivations is a Lie algebra monomorphism. The set of generating sections of Lie fields is a Lie subalgebra in the Lie algebra $\mathcal{F}(\pi, \pi)$ with respect to the Jacobi bracket.*

Thus, the theory of symmetries of the Cartan distribution on $J^\infty(\pi)$ is a natural generalization of the theory of Lie fields on the manifolds of finite jets.

Let X be a Lie field on $J^\infty(\pi)$. Then, considering one-parameter groups of transformations corresponding to the fields $X_k = X|_{J^k(\pi)}$, we shall see that a one-parameter group of transformations also corresponds to the field X and it acts on the manifold $J^\infty(\pi)$. In fact, any automorphism of the Cartan distribution on $J^\infty(\pi)$ possessing a one-parameter group of transformations can be represented as a Lie field, possibly in some new bundle [**17**]. Let us explain the above said using somewhat informal reasoning.

Let $\pi\colon P \to M$ be a fiber bundle, $X \in D_\mathcal{C}(\pi)$ be an integrable vector field, and A_t be the corresponding one-parameter group of transformations. Then

$$A_t = \exp(tX) = \mathrm{id} + tX + \cdots + \frac{t^k}{k!}X^k + \cdots \tag{2.28}$$

From (2.28) it follows that there exists an l such that $X^k(\mathcal{F}_0) \subset \mathcal{F}_l$ for all k. In fact, if it were not the case, the transformation A_t could not transform the manifold $J^0(\pi)$ to any finite manifold $J^k(\pi)$ and it would contradict to the definition of a smooth map of the manifold $J^\infty(\pi)$.

Consider in the ring $\mathcal{F}_l$ the subring $\mathcal{F}(X)$ generated by the elements of the form $X(\varphi)$, $\varphi \in \mathcal{F}_0$, i.e., we consider the set of sums of the form $\sum X(\varphi_1) \cdot \ldots \cdot X(\varphi_r)$, $\varphi_i \in \mathcal{F}_0$. From the above said it follows that the subring $\mathcal{F}(X)$ is invariant with respect to the action of the field X: $X(\mathcal{F}(X)) \subset \mathcal{F}(X)$. Let us pass to the dual viewpoint and consider the maximal $\mathbb{R}$-spectrum of the ring $\mathcal{F}(X)$, i.e., the space whose points are the kernels of homomorphisms $\mathcal{F}(X) \to \mathbb{R}$. Denote this space by P_X. Then locally, in a neighborhood of a generic point, the commutative diagram takes place:

$$J^l(\pi) \xrightarrow{\;\;\varphi_X\;\;} P_X$$
$$\pi_l \searrow \qquad \swarrow \pi_X$$
$$M$$

where φ_X is the mapping dual to the embedding $\mathcal{F}(X) \hookrightarrow \mathcal{F}$. By invariance of $\mathcal{F}(X)$ with respect to X, the field X determines on P_X a field $\widetilde{X}_0$. The set

of all possible prolongations of the mapping φ_X determines a smooth mapping $\varphi_X^*\colon J^\infty(\pi) \to J^\infty(\pi_X)$, for which the diagram

$$
\begin{array}{ccc}
J^\infty(\pi) & \xrightarrow{\ \varphi_X^*\ } & J^\infty(\pi_X) \\
& & \\
\pi_\infty \searrow & & \swarrow (\pi_X)_\infty \\
& M &
\end{array}
$$

is also commutative. Let $\widetilde{X}$ be the lifting of the field $\widetilde{X}_0$ to $J^\infty(\pi_X)$. Then by the construction of lifting and due to the fact that X is an integrable automorphism of the Cartan distribution on $J^\infty(\pi)$, the mapping φ_X^* takes X to the restriction of the field $\widetilde{X}$ to the submanifold $\mathcal{E}_X = \varphi_X^*(J^\infty(\pi)) \subset J^\infty(\pi_X)$. Hence, the structure of the fields X and $\widetilde{X}\big|_{\mathcal{E}_X}$ is locally the same, and this is what we wanted to show.

2.7. Linearizations. Let us note that the correspondence between evolutionary derivations and generating section makes it possible, having a pair (φ, ψ), $\varphi \in \mathcal{F}(\pi, \pi)$, $\psi \in \mathcal{F}(\pi)$, to construct a new section $\mathfrak{D}_\varphi(\psi) \in \mathcal{F}(\pi)$. When φ is fixed while ψ is an arbitrary section, we obtain an evolutionary derivation in the algebra $\mathcal{F}(\pi)$. Passing to the "adjoint" point of view, i.e., by fixing a section ψ and making φ "free", we shall obtain a new operator ℓ_ψ acting from the module $\mathcal{F}(\pi, \pi)$ to the algebra $\mathcal{F}(\pi)$ by the rule

$$
\ell_\psi(\varphi) = \mathfrak{D}_\varphi(\psi). \tag{2.29}
$$

To understand the meaning of this operator, let us consider equations (2.15) and using them rewrite the definition (2.29) in the form

$$
\ell_\psi(\varphi) = \sum_{j,\sigma} \frac{\partial \psi}{\partial p_\sigma^j} D_\sigma(\varphi^j),
$$

or

$$
\ell_\psi = \sum_{j,\sigma} \frac{\partial \psi}{\partial p_\sigma^j} D_\sigma^{(j)}, \tag{2.30}
$$

where the symbol $D_\sigma^{(j)}$ means that the operator D_σ is applied to the j-th component of the corresponding section. Let us study equation (2.30) in more details.

Let s be a section of the bundle π. Since every field D_i is tangent to all maximal integral manifolds of the Cartan distribution on $J^\infty(\pi)$ and consequently admits restrictions to the graphs of infinite jets, the operator ℓ_ψ, by (2.30), admits such a restriction as well. To be more precise, one has the commutative diagram

$$
\begin{array}{ccc}
\mathcal{F}(\pi, \pi) & \xrightarrow{\ \ell_\psi\ } & \mathcal{F}(\pi) \\
j_\infty(s)^* \downarrow & & \downarrow j_\infty(s)^* \\
\mathcal{F}(\pi) & \xrightarrow{\ \ell_\psi|_s\ } & \Gamma(\pi)
\end{array} \tag{2.31}
$$

where $\ell_\psi\big|_s$ is the restriction under consideration. Combining the diagram (2.31) with the equality (2.30), we see that for any section $\varphi \in \Gamma(\pi)$ the equality

$$
\ell_\psi\big|_s(\varphi) = \sum_{j,\sigma} \frac{\partial \psi}{\partial p_\sigma^j}\bigg|_s \frac{\partial^{|\sigma|} \varphi^j}{\partial x^\sigma}. \tag{2.32}
$$

holds. Formula (2.32) shows that the operator $\ell_\psi\big|_s$ is the linearization of the operator Δ_ψ corresponding to the function ψ taken at the section $s \in \Gamma(\pi)$. All

such linearizations are obtained by restricting the operator ℓ_ψ to the corresponding section.

DEFINITION 2.5. The operator ℓ_ψ is called the *operator of the universal linearization* of the nonlinear differential operator Δ_ψ.

Let us list the main properties of universal linearizations:
1. The operator $\ell_\psi\colon \mathcal{F}(\pi, \pi) \to \mathcal{F}(\pi)$, $\psi \in \mathcal{F}(\pi)$, is linear.
2. It admits restriction to the graphs of infinite jets.
3. The operator ℓ_ψ is dual to evolutionary derivations: $\ell_\psi(\varphi) = \partial_\varphi(\psi)$.

Note also that if ψ is linear with respect to all variables p_σ^j (i.e., the operator Δ_ψ is linear), then the equality

$$\ell_\psi = \widehat{\Delta}_\psi \tag{2.33}$$

holds.

The above constructed operator ℓ_ψ was defined for nonlinear differential operators of the form $\Delta_\psi\colon \Gamma(\pi) \to C^\infty(M)$. Let us generalize the construction of the universal linearization to arbitrary operators $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$, where both π' and π are locally trivial vector bundles over the manifold M, $\dim(\pi') = m'$. Let $\psi \in \mathcal{F}(\pi, \pi')$ be the representing section of the operator Δ. Consider an arbitrary element $\varphi \in \mathcal{F}(\pi, \pi)$ and a family of elements $\varphi_t \in \mathcal{F}(\pi, \pi)$ smoothly depending on $t \in \mathbb{R}$, such that $\partial\varphi_t/\partial t|_{t=0} = \varphi$. Let $\nabla_t = \nabla_{\varphi_t}\colon \Gamma(\pi) \to \Gamma(\pi)$ be the corresponding family of operators and $\bar\varphi_t = \varphi_{\Delta \circ \nabla_t}$. Let us set

$$\ell_\psi(\varphi) = \left(\frac{\partial}{\partial t}\bar\varphi_t\right)\bigg|_{t=0}. \tag{2.34}$$

EXERCISE 2.2. Show that $\ell_\psi(\varphi)$ is well defined by (2.34), i.e., does not depend on the choice of the family φ_t satisfying the condition $\partial\varphi_t/\partial t|_{t=0} = \varphi$.

The operator $\ell_\psi\colon \mathcal{F}(\pi, \pi) \to \mathcal{F}(\pi, \pi')$ is called the *operator of the universal linearization* of the nonlinear differential operator Δ_ψ and in local coordinates is represented in the form $\ell_\psi = \|\ell_\psi^{\alpha\beta}\|$, where $\|\ell_\psi^{\alpha\beta}\|$ is the $m' \times m$ matrix whose elements are

$$\ell_\psi^{\alpha\beta} = \sum_\sigma \frac{\partial\psi^\alpha}{\partial p_\sigma^\beta} D_\sigma, \qquad \alpha = 1, \dots, m', \quad \beta = 1, \dots, m. \tag{2.35}$$

Obviously, in the case $m' = 1$ this operator coincides with the operator introduced above. Moreover, it possesses the above listed properties 1–3, with the only difference that the operator $\partial_\varphi^{\pi'}$ defined by the equality $\partial_\varphi^{\pi'}(\psi) = \ell_\psi(\varphi)$ acts now not in the algebra $\mathcal{F}(\pi)$, but in the module $\mathcal{F}(\pi, \pi')$. If $\psi \in \mathcal{F}(\pi, \pi')$ and $f \in \mathcal{F}(\pi)$, then the operators $\partial_\varphi^{\pi'}$ and ∂_φ are related to each other by the equalities

$$\partial_\varphi^{\pi'}(f\psi) = \partial_\varphi(f)\psi + f\partial_\varphi^{\pi'}(\psi). \tag{2.36}$$

Thus we see that every section $\varphi \in \mathcal{F}(\pi, \pi)$ determines a family of derivations $\partial_\varphi^{\pi'}\colon \mathcal{F}(\pi, \pi') \to \mathcal{F}(\pi, \pi')$ and, as it follows from (2.36), every operator $\partial_\varphi^{\pi'}$ is a derivation of the module $\mathcal{F}(\pi, \pi)$ over the derivation ∂_φ of the algebra $\mathcal{F}(\pi)$.

REMARK 2.1. Relation between evolutionary derivations and linearizations becomes more clear, if we again use a parallel with classical differential geometry. In fact, if M and M' are smooth finite-dimensional manifolds and $G\colon M \to M'$ is a smooth mapping, then the linearization (i.e., the differential) G_* of the mapping

G at a point $x \in M$ is constructed as follows: one takes a curve x_t in M passing through the point x, $x_0 = x$, considers the tangent vector $v = dx_t/dt|_{t=0}$ to this curve at the point x and shows that the equality $G_*(v) = dG(x_t)/dt$ determines a tangent vector to M' at the point $G(x)$ depending on the choice of v only. If X is a vector field on M, then, in general, this construction does not allow to obtain the corresponding vector field $G_*(X)$ on M'. If nevertheless one understands vector fields as sections of the tangent bundle, one can put into correspondence, in a canonical way, to any field $X \in D(M)$ a section $G_*(X)$ of the bundle over M, which is the pullback of the tangent bundle over M' by the mapping G.

In our situation, the role of points of "manifolds" under consideration is played by sections of the bundles π and π' respectively while a nonlinear differential operator $\Delta = \Delta_\psi$ (or, more exactly, the mapping $\Phi_\Delta \colon J^\infty(\pi) \to J^\infty(\pi')$, see §1.3) plays the role of G. Taking a curve in $\Gamma(\pi)$, we have to consider a family of sections $s(t) \in \Gamma(\pi)$. As we know, "tangent vectors" to such curves are determined by evolutionary derivations in $\mathcal{F}(\pi)$, which play the role of vector fields on $\Gamma(\pi)$, i.e., by sections $\varphi \in \mathcal{F}(\pi, \pi)$. If ∇ is the operator corresponding to the section φ, then the curve $s(t)$ is determined by the evolution equation

$$\frac{\partial s}{\partial t} = \nabla(s),$$

while the "tangent vector" to the image of this curve can be found from the equality

$$\frac{\partial}{\partial t}\Delta(s) = j_\infty(s(0))^*(\ell_\psi(\varphi)).$$

Thus, similar to the finite-dimensional, the linearization is defined on vector fields, i.e., on evolutionary derivations. Arising ambiguity can be eliminated, if one considers the pullback of the bundle $(\pi'_\infty)^*(\pi')$ (i.e., of a tangent bundle analog) to $J^\infty(\pi)$ by the mapping $G = \Phi_\Delta \colon J^\infty(\pi) \to J^\infty(\pi')$ (defined by all prolongations of the operator Δ); see Figure 4.6. One can easily see that

$$(\Phi_\Delta)^* (\pi'_\infty)^*(\pi') = (\pi'_\infty \circ \Phi_\Delta)^* (\pi') = \pi_\infty^*(\pi').$$

The operators ℓ_ψ and evolutionary derivations of the form $\eth_\varphi^{\pi'}$ make it possible to rewrite in a convenient form some relations obtained in this section. First of all, let us note that if $\mathrm{id} \colon \Gamma(\pi) \to \Gamma(\pi)$ is the identical operator, then its linearization $\ell_{\mathrm{id}} \colon \mathcal{F}(\pi, \pi) \to \mathcal{F}(\pi, \pi)$ is the identity as well. Hence, for any section $\varphi \in \mathcal{F}(\pi, \pi)$ the equality

$$\eth_\varphi^\pi(\varphi_{\mathrm{id}}) = \ell_{\mathrm{id}}(\varphi) = \varphi. \tag{2.37}$$

fulfills. Comparing (2.37) with the definition of the Jacobi bracket, we see that

$$\left[\eth_\varphi^\pi, \eth_\psi^\pi\right](\varphi_{\mathrm{id}}) = \eth_\varphi^\pi\left(\eth_\psi^\pi(\varphi_{\mathrm{id}})\right) - \eth_\psi^\pi\left(\eth_\varphi^\pi(\varphi_{\mathrm{id}})\right) = \eth_\varphi^\pi(\psi) - \eth_\psi^\pi(\varphi),$$

i.e.,

$$\{\varphi, \psi\} = \eth_\varphi^\pi(\psi) - \eth_\psi^\pi(\varphi). \tag{2.38}$$

Another expression for the Jacobi bracket immediately follows from (2.38) and is of the form

$$\{\varphi, \psi\} = \eth_\varphi^\pi(\psi) - \ell_\varphi(\psi),$$

or

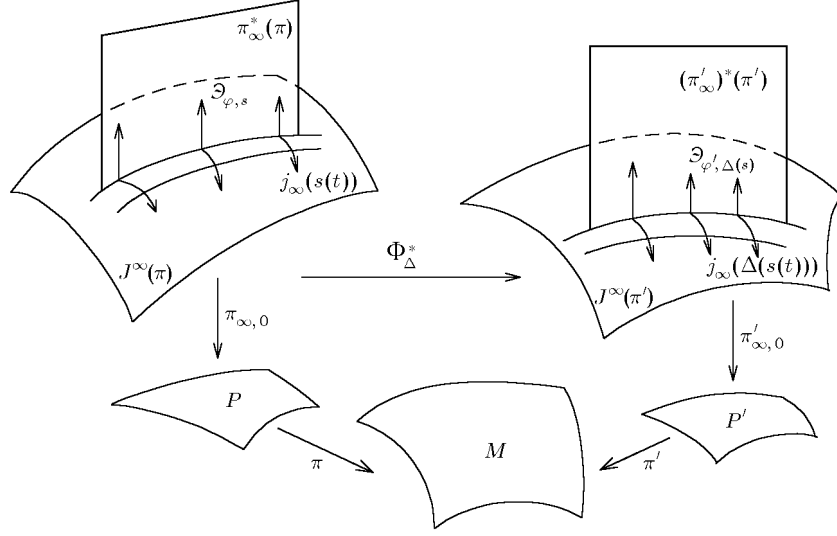$$\{\varphi, \cdot\} = \eth_\varphi^\pi - \ell_\varphi. \tag{2.39}$$

FIGURE 4.6. Action of the linearization on evolutionary vector fields

Comparing equation (2.27) with the coordinate representation of the universal linearization, one can also see that if π is a trivial bundle, then the Lie field X_φ^* on $J^\infty(\pi)$ can be represented in the form

$$X_\varphi^* = \eth_\varphi - \ell_\varphi + \ell_\varphi(\mathbf{1}) = \{\varphi, \cdot\} + \ell_\varphi(\mathbf{1}), \qquad (2.40)$$

where $\mathbf{1}$ is the section of the bundle $\pi_\infty^*(\pi)\colon J^\infty(\pi) \times \mathbb{R} \to J^\infty(\pi)$ identically equal to 1 at any point of the manifold $J^\infty(\pi)$.

Evolutionary derivations and universal linearizations play an important role in the theory of higher symmetries for nonlinear differential equation. We now begin to study this theory.

3. Infinitely prolonged equations and the theory of higher symmetries

In the preceding section, studying infinitesimal automorphisms of the Cartan distribution on $J^\infty(\pi)$, we arrived to the concept of evolutionary derivations. These derivations can be informally understood as vector fields on the "manifold" of sections $\Gamma(\pi)$. Our next step consists of extending this approach to arbitrary differential equations $\mathcal{E} \subset J^k(\pi)$. Trying to understand what a vector field on the "manifold" $\mathrm{Sol}(\mathcal{E})$ of solutions to $\mathcal{E}$ is, we shall construct the theory of higher symmetries for nonlinear differential equations. This theory in a natural way generalizes the theory of classical symmetries constructed before.

Since solutions of the equation $\mathcal{E}$ are sections $s \in \Gamma(\pi)$ satisfying the condition $\Gamma_s^k = j_k(s)(M) \subset \mathcal{E}$, it seems natural to assume that at least some of the fields we seek for on $\mathrm{Sol}(\mathcal{E})$ are to be obtained by restricting the fields from the enveloping space $\Gamma(\pi)$, i.e., by restricting evolutionary derivations. Nevertheless, if we take an arbitrary evolutionary derivation $\eth_\varphi$ on $J^\infty(\pi)$ and try restrict it to $\mathcal{E}$ (i.e., to apply it to functions from $C^\infty(\mathcal{E})$), then, in the case $\deg(\varphi) \neq 0$, our efforts will be unsuccessful: the function $\eth_\varphi(\psi)$, $\psi \in C^\infty(\mathcal{E})$, will cease to be an element of the algebra $C^\infty(\mathcal{E})$. Using corrections to $\eth_\varphi$ by the fields of the form $\widehat{X}_M$, leads to

the situation when all "fields" on $\mathrm{Sol}(\mathcal{E})$ obtained in this way are exhausted by the already known classical symmetries of the equation $\mathcal{E}$. The reasons of this are clear: it suffices to remember that evolutionary derivations aroused, when we passed from manifolds $J^k(\pi)$ of finite jets to the tower

$$\cdots \longleftarrow J^k(\pi) \longleftarrow J^{k+1}(\pi) \longleftarrow \cdots$$

and considered the Cartan distribution on $J^\infty(\pi)$. An analog of this construction for the equation $\mathcal{E}$ is the chain of its *prolongations* $\mathcal{E}^{(l)}$, $l = 0, 1 \ldots$, leading to the *infinitely prolonged equation* $\mathcal{E}^\infty \subset J^\infty(\pi)$. Note that the concept of prolongation plays an important role in many aspects of differential equations, such as the theory of formal integrability, singularities of solutions (discontinuities, shock waves), etc.

3.1. Prolongations. Let $\mathcal{E} \subset J^k(\pi)$ be an equation of order k locally given by the conditions

$$\mathcal{E} = \left\{ \theta_k \in J^k(\pi) \mid F_1(\theta_k) = \cdots = F_r(\theta_k) = 0, \ F_1, \cdots, F_r \in \mathcal{F}_k(\pi) \right\}.$$

If $s \in \Gamma(\pi)$ is a solution of the equation $\mathcal{E}$, i.e., if

$$F_\alpha\left(x, \ldots, \frac{\partial^{|\sigma|} s^j}{\partial x^\sigma}, \ldots\right) = 0, \qquad \alpha = 1, \ldots, r, \quad j = 1, \ldots, m, \quad |\sigma| \le k, \quad (3.1)$$

then s must satisfy all differential consequences of the system $(3.1)^7$. In particular,

$$\frac{\partial F_\alpha}{\partial x_i} + \sum \frac{\partial F_\alpha}{\partial p_\sigma^j} \frac{\partial^{|\sigma|+1} s^j}{\partial x_i \partial x^\sigma} = 0, \qquad \alpha = 1, \ldots, r, \quad j = 1, \ldots, m, \quad |\sigma| \le k, \quad (3.2)$$

for all $i = 1, \ldots, n$, or

$$(j_{k+1}(s))^* D_i(F_\alpha) = 0, \qquad i = 1, \ldots, n, \quad \alpha = 1, \ldots, r, \qquad (3.3)$$

where D_i are the operators of total derivatives. Combining the systems (3.1) and (3.2), we obtain a differential consequence of (3.1) of order ≤ 1 and thus an equation $\mathcal{E}^{(1)} \subset J^{k+1}(\pi)$ of order $k + 1$ which is called the *first prolongation* of $\mathcal{E}$.

To define the concept of the first prolongation in a coordinate-free manner, let us find out what are the conditions for the point $\theta_{k+1} \in J^{k+1}(\pi)$ to belong to the set $\mathcal{E}^{(1)}$. First, it is obvious that the point $\theta_k = \pi_{k+1,k}(\theta_{k+1})$ needs to lie on the equation $\mathcal{E}$. Further, let us represent the point $\theta_{k+1} \in \mathcal{E}^{(1)}$ in the form $\theta_{k+1} = [s]_x^{k+1}$, $s \in \Gamma(\pi)$, and substitute the Taylor expansion of length $k + 1$ of the section s to equation (3.1). Expanding the result to the Taylor series and comparing the expression obtained with equations (3.2), we see that the point θ_{k+1} lies on $\mathcal{E}^{(1)}$ if and only if the corresponding section $s \in \Gamma(\pi)$ satisfies the equation $\mathcal{E}$ at the point θ_k up to infinitesimals of second order. In other words, the point $\theta_{k+1} = [s]_x^{k+1}$ lies on $\mathcal{E}^{(1)}$ if and only if the manifold $j_k(s)(M)$ is tangent to $\mathcal{E}$ at the point $\theta_k = [s]_x^k$. This is the desired invariant definition of the first prolongation. Generalizing it in a natural way, we arrive to the following definition:

DEFINITION 3.1. The set $\mathcal{E}^{(l)} \subset J^{k+l}(\pi)$, consisting of the points $\theta_{k+l} = [s]_x^{k+l}$, such that the graph $j_k(s)(M)$ of the k-th jet of the section $s \in \Gamma(\pi)$ is tangent to the equation $\mathcal{E}$ at the point $\theta_k = [s]_x^k$ with order $\ge l$, is called the *l-th prolongation of the equation* $\mathcal{E} \subset J^k(\pi)$.

---

[7]I.e., they satisfy all equations obtained by differentiation of the system at hand.

Obviously, the conditions describing the l-th prolongation of the equation $\mathcal{E}$ are differential consequences of the conditions (3.1) up to order l, i.e., they are of the form

$$D_\tau(F_\alpha) = 0, \qquad |\tau| \leq l, \quad \alpha = 1, \ldots, r, \tag{3.4}$$

where τ is a multi-index and D_τ is the corresponding composition of total derivatives.

Let now ξ be a bundle of dimension r over M and $\Delta \colon \Gamma(\pi) \to \Gamma(\xi)$ be an operator of order k defining the equation, i.e., possessing the property $\theta_k \in \mathcal{E}$ if and only if $\varphi_\Delta(\theta_k) = 0$. Comparing expressions (1.7) obtained in §1 for prolongations of nonlinear differential operators with equations (3.4), we get the following result:

PROPOSITION 3.1. *If the equation $\mathcal{E} \subset J^k(\pi)$ is given by a differential operator $\Delta \colon \Gamma(\pi) \to \Gamma(\xi)$, then its l-th prolongation $\mathcal{E}^{(l)} \subset J^{k+l}(\pi)$ is given by the operator $\Delta_l = j_l \circ \Delta \colon \Gamma(\pi) \to \Gamma(\xi_l)$.*

EXERCISE 3.1. Let the equation $\mathcal{E}$ be such that its l-th prolongation is a smooth submanifold in $J^{k+l}(\pi)$. Show that in this case the equality $(\mathcal{E}^{(l)})^{(t)} = \mathcal{E}^{(l+t)}$ holds for all $t \geq 0$. In particular, this means that in this situation the $(l+1)$-st prolongation can be defined by induction: $\mathcal{E}^{(l+1)} \stackrel{\text{def}}{=} (\mathcal{E}^{(l)})^{(1)}$.

EXERCISE 3.2. Construct examples when $\mathcal{E}^{(1)}$ is not a smooth submanifold in $J^{k+1}(\pi)$.

3.2. Infinitely prolonged equations. Thus, for any $k \geq 0$ we constructed the set $\mathcal{E}^{(l)} \subset J^{k+l}(\pi)$, the l-th prolongation of the equation $\mathcal{E}$. Since tangency of order $l+1$ leads to tangency with all less orders, the natural mappings $\mathcal{E}^{(l+1)} \to \mathcal{E}^{(l)}$ compatible with the projections of the corresponding jet spaces take place,

$$
\begin{array}{ccc}
J^{k+l+1}(\pi) & \supset & \mathcal{E}^{(l+1)} \\
{\scriptstyle \pi_{k+l+1,k+l}} \downarrow & & \downarrow \\
J^{k+l}(\pi) & \supset & \mathcal{E}^{(l)}
\end{array}
\tag{3.5}
$$

also denoted by $\pi_{k+l+1,k+l}$. It should be noted that unlike the projections of jet spaces, the mappings $\pi_{k+l+1,k+l} \colon \mathcal{E}^{(l+1)} \to \mathcal{E}^{(l)}$ may not be surjective.

EXERCISE 3.3. Construct examples of differential equations for which the mapping $\pi_{k+1,k} \colon \mathcal{E}^{(1)} \to \mathcal{E}$ is not surjective.

The chain of mappings

$$\mathcal{E} = \mathcal{E}^{(0)} \xleftarrow{\pi_{k+1,k}} \mathcal{E}^{(1)} \longleftarrow \cdots \longleftarrow \mathcal{E}^{(l)} \xleftarrow{\pi_{k+l+1,k+l}} \mathcal{E}^{(l+1)} \longleftarrow \cdots \tag{3.6}$$

allows one to define the inverse limit $\mathcal{E}^\infty = \operatorname{proj lim}_{l \to \infty} \mathcal{E}^{(l)}$ of prolongations of the equation $\mathcal{E}$ which is called the *infinite prolongation* of this equation. The set $\mathcal{E}^\infty$ lies in the manifold $J^\infty(\pi)$ of infinite jets and its points can be interpreted in a quite obvious manner.

As we know, a point on $\mathcal{E}^{(l)}$ is the Taylor expansion of length $k+l$ of a section of the bundle π satisfying $\mathcal{E}$ up to infinitesimals of order $l+1$. On the other hand, the points of the manifold $J^\infty(\pi)$ are total Taylor series of sections of the bundle π. Therefore, a point $\theta = [s]_x^\infty \in J^\infty(\pi)$ belongs to the set $\mathcal{E}^\infty$ if and only if the Taylor expansion of the section s at the point $x \in M$ satisfies the equation $\mathcal{E}$. In other words, the points of $\mathcal{E}^\infty$ are formal solutions of the equation $\mathcal{E}$. In particular,

from here it follows that a necessary condition for solvability of the equation $\mathcal{E}$ at a point $x \in M$ is that set $\mathcal{E}^\infty \cap \pi_\infty^{-1}(x)$ is not empty.

Let us define the algebra of smooth functions on $\mathcal{E}^{(l)}$ as the set of restrictions of smooth functions from the manifold $J^{k+l}(\pi)$:

$$\mathcal{F}_l(\mathcal{E}) = \left\{ \varphi : \mathcal{E}^{(l)} \to \mathbb{R} \mid \exists \bar\varphi \in \mathcal{F}_{k+l}(\pi): \ \bar\varphi|_{\mathcal{E}^{(l)}} = \varphi \right\}.$$

In the case when $\mathcal{E}^{(l)}$ is a smooth submanifold in $J^{k+l}(\pi)$, the equality $\mathcal{F}_l(\mathcal{E}) = C^\infty(\mathcal{E}^{(l)})$ holds. The commutativity of the diagram (3.5) makes it possible, using the chain of mappings (3.6), to construct the chain of commutative algebra homomorphisms

$$\mathcal{F}_0(\mathcal{E}) = C^\infty(\mathcal{E}) \xrightarrow{\pi_{k+1,k}^*} \mathcal{F}_1(\mathcal{E}) \longrightarrow \cdots \longrightarrow \mathcal{F}_l(\mathcal{E}) \xrightarrow{\pi_{k+l+1,k+l}^*} \mathcal{F}_{l+1}(\mathcal{E}) \longrightarrow \cdots,$$

whose direct limit $\mathcal{F}(\mathcal{E}) = \mathrm{inj}\lim_{l\to\infty} \mathcal{F}_l(\mathcal{E})$ is called the *algebra of smooth functions* on the infinite prolongation $\mathcal{E}^\infty$. For any $l \geq 0$, the natural homomorphisms of algebras $\pi_{\infty,k+l}^* : \mathcal{F}_l(\mathcal{E}) \to \mathcal{F}(\mathcal{E})$ are defined; since $\mathrm{im}(\pi_{\infty,k+l}^*) \subset \mathrm{im}\,\pi_{\infty,k+l+1}^*$, the algebra is filtered by the images of these homomorphisms. The image of the homomorphism $\pi_{\infty,k+l}^*$ in $\mathcal{F}(\mathcal{E})$ can be identified with the ring of smooth functions on the set $\mathcal{E}_l = \pi_{\infty,k+l}(\mathcal{E}^\infty) \subset \mathcal{E}^{(l)} \subset J^{k+l}(\pi)$. Therefore, if all the mappings $\mathcal{E}^\infty \to \mathcal{E}^{(l)}$ are surjective, one can consider the algebra $\mathcal{F}(\mathcal{E})$ to be filtered by the subalgebras $\mathcal{F}_l(\mathcal{E})$. As it is easily seen, the equation for which the above said is valid possess the following important property: any solution of such an equation constructed up to infinitesimals of order l can be continued up to a formal solution.

If $\pi' \colon P' \to M$ is another bundle over M, one can introduce the $\mathcal{F}_l(\mathcal{E})$-modules

$$\mathcal{F}_l(\mathcal{E}, \pi') = \left\{ \varphi \in \Gamma(\varepsilon_l^*(\pi')) \mid \exists \bar\varphi \in \Gamma(\pi_{k+l}^*(\pi')): \ \bar\varphi|_{\mathcal{E}^{(l)}} = \varphi \right\},$$

where $\varepsilon_l \colon \mathcal{E}^{(l)} \hookrightarrow J^{k+l}(\pi)$ is the canonical embedding, and to construct the filtered $\mathcal{F}(\mathcal{E})$-module $\mathcal{F}(\mathcal{E}, \pi') = \mathrm{inj}\lim_{l\to\infty} \mathcal{F}_l(\mathcal{E}, \pi')$.

From the definition of the algebras $\mathcal{F}_l(\mathcal{E})$ it follows that for all $l \geq 0$ one has the epimorphisms $\varepsilon_l^* \colon \mathcal{F}_{k+l}(\pi) \to \mathcal{F}_l(\mathcal{E})$. Setting formally $\mathcal{E}^{(l)} \stackrel{\mathrm{def}}{=} \mathcal{E}_l = \pi_{k,k+l}(\mathcal{E})$ for $l < 0$ and defining the algebras $\mathcal{F}_l(\mathcal{E})$ in an appropriate way, we arrive to the commutative diagram

$$
\begin{array}{ccccccccc}
\mathcal{F}_{-\infty}(\mathcal{E}) & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_{k-1}(\pi) & \longrightarrow & \mathcal{F}_k(\pi) & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_{k+l}(\pi) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \varepsilon_{-\infty}^*} & & & & \downarrow{\scriptstyle \varepsilon_{-1}^*} & & \downarrow{\scriptstyle \varepsilon_0^*} & & & & \downarrow{\scriptstyle \varepsilon_l^*} & & \\
\mathcal{F}_{-\infty}(\mathcal{E}) & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_{-1}(\pi) & \longrightarrow & \mathcal{F}_0(\pi) & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_l(\pi) & \longrightarrow & \cdots
\end{array}
\tag{3.7}
$$

where $\mathcal{F}_{-\infty}(\mathcal{E}) \stackrel{\mathrm{def}}{=} C^\infty(\pi_\infty(\mathcal{E}^\infty))$. Let us denote by $I_l(\mathcal{E})$ the kernel of the epimorphism ε_l^*. Then $I_l(\mathcal{E})$ is the ideal of the algebra $\mathcal{F}_{k+l}(\mathcal{E})$ possessing the property that the function φ lies in this ideal if and only if $\varphi(\theta) = 0$ for all points $\theta \in \mathcal{E}^{(l)}$. From commutativity of the diagram (3.7) it follows that for all $l \in \mathbb{Z}$ the embeddings $I_l(\mathcal{E}) \subset I_{l+1}(\mathcal{E})$ take place. Therefore, the system of ideals $\{I_l(\mathcal{E})\}$ determines the ideal $I(\mathcal{E}) = \bigcup_{l\in\mathbb{Z}} I_l(\mathcal{E})$ of the filtered algebra $\mathcal{F}(\mathcal{E})$ called the *ideal of the equation* $\mathcal{E}$.

Let $X \in \mathrm{D}(\pi)$ be a vector field on $J^\infty(\pi)$ possessing the property that the ideal $I(\mathcal{E})$ is closed with respect to the derivation X: $X(I(\mathcal{E})) \subset I(\mathcal{E})$. Then X generates the derivation $X|_{\mathcal{E}}$ of the quotient algebra $\mathcal{F}(\pi)/I(\mathcal{E})$, i.e., a vector field on $\mathcal{E}^\infty$. We say in this case that the field X is tangent to the manifold $\mathcal{E}^\infty$ or admits restriction to this manifold.

Note that without loss of generality we may assume that the homomorphism $\varepsilon^*_{-\infty}$ is an isomorphism, i.e., $\mathcal{E}^{(-\infty)} \stackrel{\text{def}}{=} \pi_\infty(\mathcal{E}^\infty) = M$. In fact, if the projection $\pi_\infty|_{\mathcal{E}^\infty} : \mathcal{E}^\infty \to M$ is not surjective, we can restrict the bundle π to the submanifold $\pi_\infty(\mathcal{E}^\infty) \subset M$ (or, if necessary, to its nonsingular part) and later on consider all constructions on this restriction. Moreover, by similar reasons one may assume that $\mathcal{E}^{(-k)} = \pi_{\infty,0}(\mathcal{E}^\infty) = J^0(\pi)$. Thus, in what follows we consider $\mathcal{E}^\infty$ to be surjectively projected to the manifold $J^0(\pi)$.

Let $F_\alpha = 0$, $\alpha = 1, \ldots, r$, be the relations defining the equation $\mathcal{E}$. Then the ideal $I_0(\mathcal{E})$ is generated by the elements $F_\alpha \in \mathcal{F}_k(\pi)$, i.e., any element $\varphi \in I_0(\mathcal{E})$ is of the form $\varphi = \varphi_1 F_1 + \cdots + \varphi_r F_r$, $\varphi_\alpha \in \mathcal{F}_k(\mathcal{E})$. Adding to these generators the elements of the form $D_i F_\alpha$, $i = 1, \ldots, n$, we obtain, as it easily follows from (3.2), the ideal $I_1(\mathcal{E})$, etc. Finally, the ideal $I(\mathcal{E})$ is generated by all elements of the form $D_\tau(F_\alpha)$, $|\tau| \geq 0$, $\alpha = 1, \ldots, r$, and $I_l(\mathcal{E}) = I(\mathcal{E}) \cap \mathcal{F}_{k+l}(\pi)$. In particular, the ideal $I_0(\mathcal{E})$ is trivial (and its triviality is equivalent to surjectivity of the mapping $\pi_{\infty,0}|_{\mathcal{E}^\infty}$) if and only if there are no equations of the form $f(x, u) = 0$ in consequences of equations $D_\tau(F_\alpha) = 0$. This means that the equation $\mathcal{E}$ does not contain (even in implicit form) functional relations. If such relations exist, we can locally choose a subsystem of maximal rank among them and express other variables x_i, u^j via the chosen ones. This procedure is a coordinate formulation of the above described procedure of reduction an arbitrary equation $\mathcal{E}$ to an equation for which $\pi_{\infty,0}(\mathcal{E}^\infty) = J^0(\pi)$.

The above given description of the ideals $I_l(\mathcal{E})$ shows that the ideal $I(\mathcal{E})$ of the equation $\mathcal{E}$ possesses the following two important properties:

(a) It is an ideal of the filtered algebra, i.e.,

$$I(\mathcal{E}) \cap \mathcal{F}_{k+l}(\mathcal{E}) = I_l(\mathcal{E}), \qquad I(\mathcal{E}) = \bigcup_{l \in \mathbb{Z}} I_l(\mathcal{E}).$$

(b) For any vector field $X \in D(M)$, the ideal $I(\mathcal{E})$ is closed with respect to the lifting $\widehat{X} \in D(\pi)$: $\widehat{X} I(\mathcal{E}) \subset I(\mathcal{E})$.[8]

Conversely, let in the algebra $\mathcal{F}(\pi)$ a filtered differentially closed ideal I be given, which is differentially generated[9] by a finite number of elements $F_1, \ldots, F_r \in \mathcal{F}_k(\pi)$, where F_α are functions in variables $x_1, \ldots, x_n, \ldots, p^j_\sigma, \ldots$, $|\sigma| \leq k$. Let us define the set $\mathcal{E}_{I,l} \subset J^{k+l}(\pi)$ as the zero manifold of the ideal $I_l = I \cap \mathcal{F}_{k+l}(\pi)$:

$$\mathcal{E}_{I,l} = \left\{ \theta \in J^{k+l}(\pi) \mid \varphi(\theta) = 0 \ \forall \varphi \in I_l \right\}.$$

Then, as it is easily seen, $\mathcal{E}_{I,l} = \mathcal{E}_I^{(l)}$, where $l \geq 0$ and $\mathcal{E}_I = \mathcal{E}_{I,0}$, while $I = I(\mathcal{E}_I)$. Thus there exists a one-to-one correspondence between submanifolds of the form $\mathcal{E}^\infty$ in $J^\infty(\pi)$ and finitely generated differentially closed filtered ideals of the algebra $\mathcal{F}(\pi)$. Note a parallel between this correspondence and duality between algebraic manifolds and ideals of commutative algebras known in algebraic geometry (see, e.g., [**108**]).

3.3. Higher symmetries. Let $\mathcal{E} \subset J^k(\pi)$ be an equation, $\pi_{\infty,0}(\mathcal{E}^\infty) = J^0(\pi)$, and $I(\mathcal{E}) \subset \mathcal{F}(\pi)$ be its ideal. Then the quotient algebra $\mathcal{F}(\pi)/I(\mathcal{E}) = \mathcal{F}(\mathcal{E})$ is a filtered algebra and is identified with the algebra of smooth functions on the manifold $\mathcal{E}^\infty$. The theory of "differential objects" (vector fields, forms, etc.) is

---

[8]When this property holds, the ideal $I(\mathcal{E})$ is called *differentially closed*.

[9]We say that an ideal I is differentially generated by elements $F_1, \ldots, F_r$, if it is algebraically generated by the elements $D_\tau F_j$, $|\tau| \geq 0$, $j = 1, \ldots, r$.

constructed over $\mathcal{E}^\infty$ exactly in the same way as it was done in §1 for the case $J^\infty(\pi)$. For example, a vector field on $\mathcal{E}^\infty$ is a derivation of the algebra $\mathcal{F}(\mathcal{E})$ compatible with filtration, the module $\Lambda^i(\mathcal{E}) = \Lambda^i(\mathcal{E}^\infty)$ of differential i-forms is the direct limit of the modules $\Lambda^i(\mathcal{E}^{(l)})$, etc.

Let us define the *Cartan distribution* $\mathcal{C}(\mathcal{E})$ on $\mathcal{E}^\infty$ by setting[10]

$$\mathcal{C}_\theta(\mathcal{E}) = T_\theta(\mathcal{E}^\infty) \cap \mathcal{C}_\theta, \quad \theta \in \mathcal{E}^\infty,$$

where $\mathcal{C}_\theta$ is the corresponding element of the Cartan distribution on the manifold $J^\infty(\pi)$. By the definition of the infinite prolongation, the distribution $\mathcal{C}(\mathcal{E})$ is nontrivial. From the description of integral manifolds of the Cartan distribution on $J^\infty(\pi)$ given in §2 and from the definition of the distribution $\mathcal{C}(\mathcal{E})$ it follows that the maximal integral manifolds of the latter are manifolds of the form $\Gamma_s^\infty = j_\infty(s)(M)$, $s \in \Gamma(\pi)$, lying in $\mathcal{E}^\infty$, and they only. Note that if $\Gamma_s^\infty \subset \mathcal{E}^\infty$, then the manifold $\pi_{\infty,k}(\Gamma_s^\infty) = \Gamma_s^k$ lies in $\mathcal{E}$, i.e., s is a solution of the equation $\mathcal{E}$. Conversely, for any solution s of the equation $\mathcal{E}$ the corresponding manifold Γ_s^∞ lies in $\mathcal{E}^\infty$ and is naturally a maximal integral manifold of the distribution $\mathcal{C}(\mathcal{E})$. Thus maximal integral manifolds of the Cartan distribution on $\mathcal{E}^\infty$ are solutions of the equation $\mathcal{E}$.

REMARK 3.1. It may seem more natural to define the Cartan plane $\mathcal{C}_\theta(\mathcal{E})$ at the point $\theta \in \mathcal{E}^\infty$ as the span of tangent planes to solutions of the equation $\mathcal{E}$ passing through this point. But this approach is seriously deficient: first, to use it, one needs to know solutions of the equation $\mathcal{E}$, and secondly, the theory obtained becomes much more poor than exposed here. Our approach, if one may say so, consists of constructing the Cartan distribution on $\mathcal{E}^\infty$ using "tangent planes to formal solutions of the equation".

Let us define the module of Cartan forms on $\mathcal{E}^\infty$ as the set $\mathcal{C}\Lambda^1(\mathcal{E}^\infty) \subset \Lambda^1(\mathcal{E}^\infty)$ of one-forms such that at every point $\theta \in \mathcal{E}^\infty$ they are annihilated by vectors of the distribution $\mathcal{C}(\mathcal{E})$.

EXERCISE 3.4. Show that the equality $\mathcal{C}\Lambda^1(\mathcal{E}^\infty) = \mathcal{C}\Lambda^1(\pi)\big|_{\mathcal{E}^\infty}$ holds, i.e., any form $\omega \in \mathcal{C}\Lambda^1(\mathcal{E}^\infty)$ may be represented as a restriction to $\mathcal{E}^\infty$ of a Cartan form from $J^\infty(\pi)$.

From §2 and from the above said it follows that the ideal $\mathcal{C}\Lambda^1(\mathcal{E}^\infty) \wedge \Lambda^*(\mathcal{E}^\infty)$ is differentially closed with respect to the de Rham differential $d\colon \Lambda^*(\mathcal{E}^\infty) \to \Lambda^*(\mathcal{E}^\infty)$, i.e.,

$$d\left(\mathcal{C}\Lambda^1(\mathcal{E}^\infty) \wedge \Lambda^*(\mathcal{E}^\infty)\right) \subset \mathcal{C}\Lambda^1(\mathcal{E}^\infty) \wedge \Lambda^*(\mathcal{E}^\infty). \tag{3.8}$$

Further, following the already known motivations of the preceding section, let us introduce the sets

$$\mathcal{C}D(\mathcal{E}^\infty) = \left\{ X \in D(\mathcal{E}^\infty) \mid X \lrcorner \omega = 0, \ \forall \omega \in \mathcal{C}\Lambda^1(\mathcal{E}^\infty) \right\}$$

and

$$D_\mathcal{C}(\mathcal{E}^\infty) = \left\{ X \in D(\mathcal{E}^\infty) \mid [X, \mathcal{C}D(\mathcal{E}^\infty)] \subset \mathcal{C}D(\mathcal{E}^\infty) \right\}.$$

---

[10]We use the same notation for the Cartan distribution on $\mathcal{E}^\infty$ as was used for that on $\mathcal{E}$. It will not cause ambiguity, because everywhere below we deal with infinite prolongations only.

From the definition it follows that the set $\mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)$ is a Lie subalgebra of the Lie algebra of vector fields on $\mathcal{E}^\infty$ while $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is an ideal in $\mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)$. Literally repeating the reasoning from §2, we shall introduce the Lie $\mathbb{R}$-algebra

$$\operatorname{sym}\mathcal{E} = \mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)/\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$$

of symmetries of the Cartan distribution on $\mathcal{E}^\infty$ and informally identify elements of this algebra with vector fields on the set of maximal integral manifolds of this distribution, i.e., on the "manifold" $\mathrm{Sol}(\mathcal{E})$ of solutions of the equation $\mathcal{E}$.

DEFINITION 3.2. Elements of the Lie algebra $\operatorname{sym}\mathcal{E}$ are called *higher (infinitesimal) symmetries* of the equation $\mathcal{E}$.

Our nearest goal is to describe the algebra $\operatorname{sym}\mathcal{E}$. To do this, let us first note the following.

Let $X \in \mathrm{D}(M)$ be a vector field on the manifold M. Then, since the ideal $I(\mathcal{E})$ of the equation $\mathcal{E}$ is differentially closed, i.e., the embedding $\widehat{X}(I(\mathcal{E})) \subset I(\mathcal{E})$ holds, the derivation $\widehat{X}\colon \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ determines a derivation $\widehat{X}\big|_{\mathcal{E}^\infty}$ of the filtered algebra $\mathcal{F}(\mathcal{E}) = \mathcal{F}(\pi)/I(\mathcal{E})$, i.e., a vector field on $\mathcal{E}^\infty$. In other words, any field of the form $\widehat{X}$, $X \in \mathrm{D}(M)$ admits restriction to $\mathcal{E}^\infty$. The above said is obviously valid for all operators generated by such fields, i.e., for operators of the form $\Delta = \sum \varphi_{i_1,\dots,i_h} \widehat{X}_{i_1} \circ \cdots \circ \widehat{X}_{i_h}\colon \mathcal{F}(\pi) \to \mathcal{F}(\pi)$, $\varphi_{i_1,\dots,i_h} \in \mathcal{F}(\pi)$. In particular, as it is seen from the geometric definition of the fields $\widehat{X}$, this is valid for the fields from $\mathcal{C}\mathrm{D}(\pi)$. Thus, the following homomorphism of Lie algebras takes place:

$$\mathcal{C}\mathrm{D}(\pi) \longrightarrow \mathcal{C}\mathrm{D}(\mathcal{E}^\infty). \tag{3.9}$$

Below we shall need the following lemma.

LEMMA 3.2. *Assume that the equation $\mathcal{E} \subset J^k(\pi)$ is such that $\mathcal{E}^\infty$ is surjectively projected to some manifold $J^{l_0}(\pi)$, $l_0 < k$. Then for any $l \leq l_0$ and any derivation $X\colon \mathcal{F}_{l-k}(\mathcal{E}) \to \mathcal{F}(\mathcal{E})$ there exists a derivation $X'\colon \mathcal{F}_l(\pi) \to \mathcal{F}(\pi)$ such that the diagram*

$$
\begin{array}{ccc}
\mathcal{F}_l(\pi) & \xrightarrow{\ X'\ } & \mathcal{F}(\pi) \\
{\scriptstyle \varepsilon_{l-k}^*}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon^*} \\
\mathcal{F}_{l-k}(\mathcal{E}) & \xrightarrow{\ X\ } & \mathcal{F}(\mathcal{E})
\end{array}
$$

is commutative.

PROOF. The lemma assumptions mean that for $l \leq l_0$ the mapping ε_{k-l}^* is an isomorphism, i.e., $\mathcal{F}_l(\pi) = \mathcal{F}_{l-k}(\mathcal{E})$. Let us (locally) represent X in the form $X = \sum_{i=1}^n \varphi_i \partial/\partial x_i + \sum_{j=1}^m \sum_{|\sigma| \leq l} \varphi_\sigma^j \partial/\partial p_\sigma^j$. In this representation, $\varphi_i, \varphi_\sigma^j$ are functions on some finite prolongation $\mathcal{E}^{(r)}$ of the equation $\mathcal{E}$ and they can be continued to smooth functions on the enveloping manifold $J^{k+r}(\pi)$. $\qquad\square$

Let us recall that we reduced study of arbitrary equations $\mathcal{E} \subset J^k(\pi)$ to equations such that $\mathcal{E}^\infty$ is surjectively projected to $J^0(\pi)$. In this situation, from the lemma proved it follows that the mapping (3.9) is epimorphic, i.e., any field $X \in \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is a restriction to $\mathcal{E}^\infty$ of some field $X' \in \mathcal{C}\mathrm{D}(\pi)$. Using Proposition 2.2, we see that X is represented in the form $X = \sum_i \varphi_i \widehat{X}_i$ in this case, where $\varphi_i \in \mathcal{F}(\mathcal{E})$, while $X_i \in \mathrm{D}(M)$.

Let now $X \in \mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)$. Restricting X to $\pi_\infty(\mathcal{E}^\infty) = M$, we shall obtain the derivation $X_M \colon C^\infty(M) \to \mathcal{F}(\mathcal{E})$, which can be continued to a derivation $X'_M \colon C^\infty(M) \to \mathcal{F}(\pi)$, since the mapping (3.9) is epimorphic. Considering its lifting $\widehat{X}'_M \colon \mathcal{F}(\pi) \to \mathcal{F}(\pi)$, $\widehat{X}'_M \in \mathcal{C}\mathrm{D}(\pi)$ and restricting the latter to $\mathcal{E}^\infty$, we shall obtain the derivation $\mathcal{C}X$, which lies in $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$.

EXERCISE 3.5. Show that $\mathcal{C}X$ is uniquely determined by the field $X \in \mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)$, i.e., is independent of extension of the derivation X_M up to X'_M.

Let us denote by $\mathrm{D}^v_{\mathcal{C}}(\mathcal{E}^\infty) \subset \mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)$ the set of vertical automorphisms of the Cartan distribution on $\mathcal{E}^\infty$, i.e., of elements $X \in \mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)$ such that $X|_{C^\infty(M)} = 0$. From the construction of the field $\mathcal{C}X$ it follows that the correspondence $v \colon X \mapsto X^v = X - \mathcal{C}X$ determines the mapping

$$v \colon \mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty) \longrightarrow \mathrm{D}^v_{\mathcal{C}}(\mathcal{E}^\infty). \tag{3.10}$$

LEMMA 3.3. *The mapping* (3.10) *is a projector, i.e.,* $X^v = X$ *for all* $X \in \mathrm{D}^v_{\mathcal{C}}(\mathcal{E}^\infty)$.

PROOF. In fact, if X is a vertical field, then the derivation $X_M = X|_{C^\infty(M)}$ is trivial. Hence, from the definition of the field $\mathcal{C}X$ it follows that it is also trivial. $\square$

From the lemma proved it follows that the splitting

$$\mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty) = \mathrm{D}^v_{\mathcal{C}}(\mathcal{E}^\infty) \oplus \ker(v)$$

takes place. It is also obvious that $\ker(v) = \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$. Therefore, the following statement is valid.

PROPOSITION 3.4. *If the projection of the manifold* $\mathcal{E}^\infty$ *to* M *is surjective, then the Lie algebra* $\mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)$ *splits to the semidirect product of the Lie subalgebra* $\mathrm{D}^v_{\mathcal{C}}(\mathcal{E}^\infty)$ *of vertical fields and of the ideal* $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$:

$$\mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty) = \mathrm{D}^v_{\mathcal{C}}(\mathcal{E}^\infty) \,\tilde{\oplus}\, \mathcal{C}\mathrm{D}(\mathcal{E}^\infty). \tag{3.11}$$

COROLLARY 3.5. *The splitting* (3.11) *induces an isomorphism of Lie algebras*

$$\mathrm{sym}\,\mathcal{E} \simeq \mathrm{D}^v_{\mathcal{C}}(\mathcal{E}^\infty).$$

3.4. Exterior and interior higher symmetries. Let us return back to the algebra $\mathrm{sym}\,\mathcal{C}(\pi) = \mathrm{D}_{\mathcal{C}}(\pi)/\mathcal{C}\mathrm{D}(\pi)$ of symmetries of the Cartan distribution on $J^\infty(\pi)$ and let us note the following. Since elements of a coset $\chi \in \mathrm{sym}\,\mathcal{C}(\pi)$ differ by derivations from $\mathcal{C}\mathrm{D}(\pi)$, then either all of them are tangent to the manifold $\mathcal{E}^\infty$, or, on the contrary, neither of them is tangent to this manifold. In the first case, the element χ generates a symmetry of the equation $\mathcal{E}$ and is called an *exterior (higher) symmetry* of this equation. Therefore, like in the case of classical symmetries (§7 of Ch. 3) the problem of comparison of exterior and interior approaches to definition of higher symmetries arises.

Denote the set of exterior symmetries by $\mathrm{sym}_e\,\mathcal{E}$. Obviously, there is a Lie algebra homomorphism

$$\mathrm{sym}_e\,\mathcal{E} \longrightarrow \mathrm{sym}\,\mathcal{E}, \tag{3.12}$$

determined by the restriction operation. From the results of §2 we know that in every coset $\chi \in \mathrm{sym}\,\mathcal{C}(\pi)$ there is a canonical representative, i.e., the vertical field which is an evolutionary derivation. Therefore, to check whether an element $\chi \in \mathrm{sym}\,\mathcal{C}(\pi)$ is an exterior symmetry of the equation $\mathcal{E}$, it suffices to verify whether

the corresponding representative is tangent to $\mathcal{E}^\infty$. Using these remarks, we shall now show that the mapping (3.12) is epimorphic. To do this, we shall need the following lemma.

LEMMA 3.6. *Higher symmetries commute with all fields of the form $\widehat{Y}$, $Y \in \mathrm{D}(M)$. More exactly, for any element $X \in \mathrm{D}_\mathcal{C}^v(\mathcal{E}^\infty)$ the equality $[X, \widehat{Y}] = 0$ is valid.*

PROOF. Since the field $\widehat{Y}$ (in fact, its restriction to $\mathcal{E}^\infty$) lies in the set $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$, which is an ideal of the Lie algebra $\mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)$, the commutator $[X, \widehat{Y}]$ also lies in $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$. Let now $f \in C^\infty(M)$ be a function on the manifold M. Then $\widehat{Y}(f) = Y(f)$ also lies in $C^\infty(M)$. Consequently, by verticality of the field X, the equalities $X(f) = 0$ and $X(Y(f)) = 0$ hold. This means that $[X, \widehat{Y}](f) = X(\widehat{Y}(f)) - \widehat{Y}(X(f)) = 0$, i.e., the field $[X, \widehat{Y}]$ is vertical. Using the decomposition (3.11), we can state now that $[X, \widehat{Y}] = 0$. □

Consider a symmetry $X \in \mathrm{D}_\mathcal{C}^v(\mathcal{E}^\infty)$ of the equation $\mathcal{E}$ and let us restrict it to the manifold $\pi_{\infty,0}(\mathcal{E}^\infty) = J^0(\pi)$. By Lemma 3.6, thus obtained derivation $X_0 \colon \mathcal{F}_0(\pi) \to \mathcal{F}(\mathcal{E})$ can be continued to a derivation $X_0' \colon \mathcal{F}_0(\pi) \to \mathcal{F}(\pi)$. Further, from the results obtained in the preceding section it follows that there exists a uniquely defined vertical automorphism $X' \in \mathrm{D}_\mathcal{C}^v(\pi)$ of the Cartan distribution on $J^\infty(\pi)$ such that $X'|_{\mathcal{F}_0(\pi)} = X_0'$. Moreover, one has $X'|_{\mathcal{F}_0(\pi)} = X|_{\mathcal{F}_0(\pi)}$, i.e., $X'(\varphi) = X(\varphi)$ for any function $\varphi \in \mathcal{F}_0(\pi)$.

Let $Y_1, \ldots, Y_l$ be arbitrary vector fields on M and $\widehat{Y}_* = \widehat{Y}_1 \circ \cdots \circ \widehat{Y}_l$ be the composition of their liftings to $J^\infty(\pi)$. Then, by Lemma 3.6, one has the equality

$$[X, \widehat{Y}_*] = \sum_{i=1}^l \widehat{Y}_1 \circ \cdots \circ [X, \widehat{Y}_i] \circ \cdots \circ \widehat{Y}_l = 0,$$

from where it follows that

$$X(\widehat{Y}_*(\varphi)) = \widehat{Y}_*(X(\varphi)) = \widehat{Y}_*(X'(\varphi)) = X'(\widehat{Y}_*(\varphi)). \tag{3.13}$$

Thus we showed that $X'(\psi) = X(\psi)$ for any function ψ of the form $\psi = \widehat{Y}_*(\varphi)$, $\varphi \in \mathcal{F}_0(\pi)$. Let us locally choose for φ the coordinate functions $p_{\mathbf{0}}^j = u^j$ and for Y_* the composition of total derivatives D_σ. Since $D_\sigma(u^j) = p_\sigma^j$, from (3.13) it follows that $X'(p_\sigma^j) = X(p_\sigma^j|_{\mathcal{E}^\infty})$ for all σ and $j = 1, \ldots, m$. Since any smooth function ψ on $J^\infty(\pi)$ is locally a function in the variables $x_1, \ldots, x_n, p_\sigma^j$, we obtain the equality $X'(\psi) = X(\psi|_{\mathcal{E}^\infty})$ for all $\psi \in \mathcal{F}(\pi)$. In other words, we proved the following theorem.

THEOREM 3.7. *If the equation $\mathcal{E} \subset J^k(\pi)$ is such that the mapping (3.12) is epimorphic, then any interior higher symmetry of the equation $\mathcal{E}$ can be represented as a restriction to $\mathcal{E}^\infty$ of some exterior symmetry. More exactly, for any field $X \in \mathrm{D}_\mathcal{C}^v(\mathcal{E}^\infty)$ there exists a field $X' \in \mathrm{D}_\mathcal{C}^v(\pi)$ such that $X'|_{\mathcal{E}^\infty} = X$.*

Recall that in the classical theory a similar statement is not valid in general (see §7 of Ch. 3).

3.5. Defining equations for higher symmetries. Now, using Theorem 3.7, we shall give an analytical description of the algebra sym $\mathcal{E}$ needed in practical computational applications. Let $\mathcal{E} \subset J^k(\pi)$ be an arbitrary equation of order k whose infinite prolongation surjectively projects to $J^0(\pi)$. By the above obtained

results, any higher symmetry of the equation $\mathcal{E}$ may be obtained by restricting to $\mathcal{E}^\infty$ some evolutionary derivation $\vartheta_\varphi \in D^v_\mathcal{C}(\pi)$, $\varphi \in \mathcal{F}(\pi, \pi)$. In turn, all evolutionary derivations admitting restriction to $\mathcal{E}^\infty$ are determined by the condition

$$\vartheta_\varphi(I(\mathcal{E})) \subset I(\mathcal{E}), \tag{3.14}$$

where $I(\mathcal{E}) \subset \mathcal{F}(\pi)$ is the ideal of the equation $\mathcal{E}$. Let the equation $\mathcal{E}$ be given by the relations $F_\alpha = 0$, $F_\alpha \in \mathcal{F}_k(\pi)$, $\alpha = 1, \dots, r$, and assume that at any point $\theta \in \mathcal{E}$ the differentials $d_\theta F_\alpha$ are linear independent. This means that the set $\{F_1, \dots, F_r\}$ is a system of differential generators for the ideal $I(\mathcal{E})$:

$$I(\mathcal{E}) = \left\{ \psi \in \mathcal{F}(\pi) \mid \psi = \sum_{\alpha, \sigma} \psi_{\alpha, \sigma} D_\sigma(F_\alpha), \ \psi_\alpha \in \mathcal{F}(\pi) \right\}.$$

Hence, condition (3.14) are equivalent to the fact that for any smooth functions $\psi_1, \dots, \psi_r \in \mathcal{F}(\pi)$ there exist functions $\psi'_1, \dots, \psi'_r \in \mathcal{F}(\pi)$, such that

$$\vartheta_\varphi \left(\sum_{\alpha, \sigma} \psi_{\alpha, \sigma} D_\sigma(F_\alpha) \right) = \sum_{\alpha, \sigma} \psi'_{\alpha, \sigma} D_\sigma(F_\alpha). \tag{3.15}$$

But ϑ_φ is a derivation commuting with the operators of total derivatives and consequently

$$\vartheta_\varphi \left(\sum_{\alpha, \sigma} \psi_{\alpha, \sigma} D_\sigma(F_\alpha) \right) = \sum_{\alpha, \sigma} \vartheta_\varphi(\psi_{\alpha, \sigma}) D_\sigma(F_\alpha) + \sum_{\alpha, \sigma} \psi_{\alpha, \sigma} D_\sigma(\vartheta_\varphi(F_\alpha)).$$

Hence, the conditions (3.15) are sufficient to be checked for the generators of the ideal $I(\mathcal{E})$ only, i.e., equation (3.14) is equivalent to the system

$$\begin{cases} \vartheta_\varphi(F_1) = \sum_{\alpha, \sigma} \psi^1_{\alpha, \sigma} D_\sigma(F_\alpha), \\ \vartheta_\varphi(F_2) = \sum_{\alpha, \sigma} \psi^2_{\alpha, \sigma} D_\sigma(F_\alpha), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \vartheta_\varphi(F_r) = \sum_{\alpha, \sigma} \psi^r_{\alpha, \sigma} D_\sigma(F_\alpha), \end{cases} \tag{3.16}$$

where $\varphi \in \mathcal{F}(\pi, \pi)$ is the unknown section and

$$\psi^\beta_{\alpha, \sigma} \in \mathcal{F}(\pi), \qquad \alpha, \beta = 1, \dots, r, \quad |\sigma| \le k + \deg(\varphi).$$

Applying the definition of the universal linearization operator given in §2, let us rewrite the system (3.16) in the form

$$\ell_{F_\beta}(\varphi) = \sum_{\alpha, \sigma} \psi^\beta_{\alpha, \sigma} D_\sigma(F_\alpha), \quad \beta = 1, \dots, r. \tag{3.17}$$

Among solutions of the system (3.17) there are trivial ones corresponding to the kernel of the epimorphism (3.12) and characterized by the condition that the restriction of the derivation ϑ_φ to $\mathcal{E}^\infty$ for these solutions leads to the trivial vector field on $\mathcal{E}^\infty$.

EXERCISE 3.6. Show that the set formed by the components φ^j of trivial solutions of the system (3.17) coincides with the ideal $I(\mathcal{E})$ of the equation $\mathcal{E}$.

To eliminate trivial solutions, let us recall that by the representation (2.23) the operator of the universal linearization is expressed in total derivatives and consequently admits restriction to manifolds of the form $\mathcal{E}^\infty$. Set $\ell_F|_{\mathcal{E}^\infty} \stackrel{\text{def}}{=} \ell_F^{\mathcal{E}}$ and let us restrict equation (3.17) to $\mathcal{E}^\infty$. Then, since the right-hand side consists of the elements of the ideal $I(\mathcal{E})$, we arrive to the system of equations

$$\ell_{F_\alpha}^{\mathcal{E}}(\bar{\varphi}) = 0, \qquad \bar{\varphi} = \varphi|_{\mathcal{E}^\infty},$$

whose solutions are in one-to-one correspondence with higher symmetries of the equation $\mathcal{E}$. Moreover, under this correspondence the commutator of two symmetries is taken to the element

$$\{\bar{\varphi}, \bar{\psi}\}_{\mathcal{E}} \stackrel{\text{def}}{=} \ell_\psi^{\mathcal{E}}(\bar{\varphi}) - \ell_\varphi^{\mathcal{E}}(\bar{\psi}) = \{\varphi, \psi\}|_{\mathcal{E}^\infty}, \qquad \bar{\psi} = \psi|_{\mathcal{E}^\infty}.$$

Combining together the results obtained, we see that the following theorem takes place:

THEOREM 3.8. *If $\mathcal{E} \subset J^k(\pi)$ is an equation such that $\pi_{\infty,0}(\mathcal{E}^\infty) = J^0(\pi)$, and $F_1, \ldots, F_r$ are the generators of the ideal $I(\mathcal{E})$, then the Lie algebra $\mathrm{sym}\,\mathcal{E}$ is isomorphic to the Lie algebra of solutions of the system of equations*

$$\ell_{F_\alpha}^{\mathcal{E}}(\varphi) = 0, \qquad \alpha = 1, \ldots, r, \quad \varphi \in \mathcal{F}(\mathcal{E}, \pi), \tag{3.18}$$

where the Lie algebra structure is given by the bracket $\{\cdot, \cdot\}_{\mathcal{E}}$.

It is easily seen that if $\Delta \colon \Gamma(\pi) \to \Gamma(\pi')$ is the operator determining the equation $\mathcal{E}$ and chosen in such a way that the corresponding section $F = \varphi_\Delta \in \mathcal{F}(\pi, \pi')$ is transversal to the base of the bundle $\pi^*(\pi')$ (more exactly, such that the images of the zero section and of F are transversal at the points of their intersection), then the system (3.18) can be rewritten in the form

$$\ell_F^{\mathcal{E}}(\varphi) = 0, \qquad \varphi \in \mathcal{F}(\mathcal{E}, \pi), \quad F \in \mathcal{F}(\pi, \pi'). \tag{3.19}$$

Equations (3.18) and (3.19) are called the *defining equations* for higher symmetries.

4. Examples of computation

In this section, we illustrate the techniques of higher symmetries computations for some equations of mathematical physics. Other examples one can find, e.g., in the collection [**137**]. A theoretical base for the computations below is given by Theorem 3.8. Both parts of the theorem are essential for the techniques. Representation of symmetries as solutions of the linear system (3.18) makes it possible to obtain "upper and lower estimates" for the Lie algebra $\mathrm{sym}\,\mathcal{E}$ while closure with respect to the higher Jacobi bracket $\{\cdot, \cdot\}_{\mathcal{E}}$ allows one to make these estimates more precise and, in some cases, to obtain a complete description for the symmetry algebra. We also rely on the fact that exterior and interior symmetries coincide and use *internal coordinates* for the manifold $\mathcal{E}^\infty$.

EXERCISE 4.1. To obtain "upper estimate" for the algebra $\mathrm{sym}\,\mathcal{E}$ it is also useful to apply the *commutator relation*

$$[\bar{\partial}_\varphi - \bar{\ell}_\varphi, \ell_F^{\mathcal{E}}] = \mathcal{D} \circ \ell_F^{\mathcal{E}}, \tag{4.1}$$

where φ is a symmetry, $\bar{\partial}_\varphi$ and $\bar{\ell}_\varphi$ are restrictions of the corresponding operators to $\mathcal{E}^\infty$, while $\mathcal{D}$ is an operator of the form $\mathcal{D} = \sum_\sigma a_\sigma \bar{D}_\sigma$. Prove this equality.

Let us start with specifying general constructions for scalar evolution equations of second order.

4.1. Preparatory remarks. Consider an equation of the form

$$u_t = \Phi(x, t, u, u_x, u_{xx}). \tag{4.2}$$

Its infinite prolongation $\mathcal{E}^\infty$ is a submanifold in the space $J^\infty(\pi)$ of infinite jets of the trivial one-dimensional bundle over the plane $M = \mathbb{R}^2$ of independent variables $x = x_1$ and $t = x_2$ while a coordinate in the fiber of the bundle π is the dependent variable u. The standard coordinates $p_{(\alpha,\beta)}$, $\alpha, \beta \geq 0$ arise in $J^\infty(\pi)$ uniquely determined by the equalities

$$p_{(\alpha,\beta)}\big|_{j_\infty(s)} = \frac{\partial^{\alpha+\beta} s}{\partial x^\alpha \partial t^\beta}, \tag{4.3}$$

where $s = s(x, t)$ is an arbitrary section of the bundle π, i.e., a smooth function on $\mathbb{R}^2$. From (4.3), by the definition of the total derivative, it follows that $p_{(\alpha,\beta)} = D_1^\alpha D_2^\beta(u)$, where

$$D_1 = \widehat{\frac{\partial}{\partial x}} = \frac{\partial}{\partial x} + p_{(1,0)}\frac{\partial}{\partial p_{(0,0)}} + \cdots + p_{(\alpha+1,\beta)}\frac{\partial}{\partial p_{(\alpha,\beta)}} + \cdots,$$

$$D_2 = \widehat{\frac{\partial}{\partial t}} = \frac{\partial}{\partial t} + p_{(0,1)}\frac{\partial}{\partial p_{(0,0)}} + \cdots + p_{(\alpha,\beta+1)}\frac{\partial}{\partial p_{(\alpha,\beta)}} + \cdots$$

Therefore, by (4.2), the equalities

$$p_{(\alpha,\beta+1)} = D_1^\alpha D_2^\beta(\Phi), \quad \alpha, \beta \geq 0,$$

hold on $\mathcal{E}^\infty$ and we can take the functions x, t, and $p_{(\alpha,0)}\big|_{\mathcal{E}^\infty} \overset{\text{def}}{=} p_\alpha$ for internal coordinates on $\mathcal{E}^\infty$. In these coordinates, the restrictions of the total derivatives to $\mathcal{E}^\infty$ are of the form

$$D_x \overset{\text{def}}{=} D_1\big|_{\mathcal{E}^\infty} = \frac{\partial}{\partial x} + \sum_{\alpha \geq 0} p_{\alpha+1}\frac{\partial}{\partial p_\alpha},$$

$$D_t \overset{\text{def}}{=} D_2\big|_{\mathcal{E}^\infty} = \frac{\partial}{\partial t} + \sum_{\alpha \geq 0} D_x^\alpha(\Phi)\frac{\partial}{\partial p_\alpha}, \tag{4.4}$$

while the system (3.18) determining higher symmetries of the equation $\mathcal{E}$ reduces to the equation

$$D_t\varphi = \Phi_0\varphi + \Phi_1 D_x\varphi + \Phi_2 D_x^2\varphi, \tag{4.5}$$

where $\varphi = \varphi(x, t, p_0, \ldots, p_k)$ is the restriction of the generating function of the symmetry we seek for to $\mathcal{E}^\infty$, while $\Phi_i \overset{\text{def}}{=} \partial\Phi/\partial p_i$. The maximal k for which $\varphi_k = \partial\varphi/\partial p_k \neq 0$ will be called the order of the symmetry φ and will be denoted by $\deg\varphi$.

Below we shall need:

(a) To determine for which Φ (inside some chosen class) equation (4.5) possesses solutions of arbitrary high order.
(b) To describe such solutions, if possible.

To obtain answers to these questions, it is more convenient technically to pass from equation (4.5) to a new system of equations for functions $\varphi_i \overset{\text{def}}{=} \partial\varphi/\partial p_i$, where φ is a solution of the system (4.5). To do this, let us introduce the operators

$$R_\beta^\alpha \overset{\text{def}}{=} \begin{cases} \dfrac{\partial}{\partial p_\beta} \circ D_x^\alpha & \text{if } \alpha, \beta \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

acting in the ring of functions on $\mathcal{E}^\infty$.

LEMMA 4.1. *For any function $\varphi = \varphi(x, t, p_0, \ldots, p_k)$ and for all α, β the equality*

$$R_\beta^\alpha(\varphi) = \sum_{i=0}^{k} \binom{\alpha}{\alpha - \beta + i} D_x^{\alpha-\beta+i}(\varphi_i), \tag{4.6}$$

holds, where, by definition, $\binom{a}{b} = 0$, if at least one of the numbers a, b is negative.

PROOF. Induction on α. For $\alpha = 0$ the statement is obvious. Let $\alpha > 0$ and assume that for $\alpha - 1$ the identity (4.6) is valid. Note that from (4.4) and from (2.17) it follow that

$$R_\beta^\alpha = R_{\beta-1}^{\alpha-1} + D_x \circ R_\beta^{\alpha-1}.$$

Therefore

$$R_\beta^\alpha(\varphi) = R_{\beta-1}^{\alpha-1}(\varphi) + D_x R_\beta^{\alpha-1}(\varphi)$$

$$= \sum_{i=0}^{k} \binom{\alpha-1}{\alpha-\beta+i} D_x^{\alpha-\beta+i}(\varphi_i) + D_x \sum_{i=0}^{k} \binom{\alpha-1}{\alpha-\beta+i-1} D_x^{\alpha-\beta+i-1}(\varphi_i)$$

$$= \sum_{i=0}^{k} \left[\binom{\alpha-1}{\alpha-\beta+i} + \binom{\alpha-1}{\alpha-\beta+i-1} \right] D_x^{\alpha-\beta+i}(\varphi_i) =$$

$$= \sum_{\alpha=0}^{k} \binom{\alpha}{\alpha-\beta+i} D_x^{\alpha-\beta+i}(\varphi_i).$$

$\square$

REMARK 4.1. From equation (4.6) one can obtain "asymptotic expansion" of $D_x^\alpha \varphi$ with respect to the variables p_i of higher order, useful in particular computations. Namely, since $\deg D_x^{\alpha-\beta+i}(\varphi) \leq \alpha - \beta + i + k$, the order of the right-hand side in (4.6) is not greater than $\alpha - \beta + 2k$. Therefore, for even α the estimate

$$D_x^{2r}\varphi = \sum_{\beta=r+k+1}^{2r+k} p_\beta \sum_{i=0}^{k} \binom{2r}{2r-\beta+i} D_x^{2r-\beta+i}\varphi_i$$

$$+ \frac{1}{2} p_{r+k}^2 \sum_{i=0}^{k} \binom{2r}{r-k+i} D_x^{r-k+i}\varphi_i + O(r+k-1),$$

is valid while for odd ones one has

$$D_x^{2r+1}\varphi = \sum_{\beta=r+k}^{2r+k+1} p_\beta \sum_{i=0}^{k} \binom{2r+1}{2r-\beta+i+1} D_x^{2r-\beta+i+1}\varphi_i + O(r+k-1),$$

where $O(i)$ is a function on $\mathcal{E}^\infty$ independent of p_β for $\beta > i$.

Let us continue the study of the equation $\ell_F(\varphi) = 0$ and apply the operators $\partial/\partial p_\beta$, $\beta > 2$ to this equation. From (4.5) one has

$$D_t\varphi_\beta + \sum_{\alpha \geq 0} R^\alpha_\beta \varphi = \Phi_0 \varphi_\beta + \Phi_1 R^1_\beta \varphi + \Phi_2 R^2_\beta \varphi.$$

Using the results of the lemma proved and taking into consideration the above agreement on binomial coefficients, the last relation can be rewritten as

$$\ell_F(\varphi_\beta) + \sum_{i=\beta}^{k} \left[\binom{i}{i-\beta} D_x^{i-\beta}(\Phi_0) + \binom{i}{i-\beta+1} D_x^{i-\beta+1}(\Phi_1) \right.$$
$$\left. + \binom{i}{i-\beta+2} D_x^{i-\beta+2}(\Phi_2) \right] \varphi_i + (\beta-1)D_x(\Phi_2)\varphi_{\beta-1} = 2\Phi_2 D_x(\varphi_{\beta-1}).$$

Thus, we have the following statement.

PROPOSITION 4.2. *If a function $\varphi = \varphi(x, t, p_0, \ldots, p_k)$, $k \geq 3$, is a solution of the equation $\ell_F(\varphi) \equiv D_t(\varphi) - \Phi_0\varphi - \Phi_1 D_x(\varphi) - \Phi_2 D_x^2(\varphi) = 0$, i.e., is a higher symmetry of order k for the equation $u_t = \Phi(x, t, u, u_x, u_{xx})$, then the functions $\varphi_\beta = \partial\varphi/\partial p_\beta$, $\beta = 2, \ldots, k$, satisfy the following system of equations:*

$$
\begin{cases}
k D_x(\Phi_2)\varphi_k = 2\Phi_2 D_x(\varphi_k), \\[2mm]
\ell_F(\varphi_k) + \left[\Phi_0 + \binom{k}{1} D_x(\Phi_1) + \binom{k}{2} D_x^2(\Phi_2) \right] \varphi_k \\[2mm]
\qquad + (k-1)D_x(\Phi_2)\varphi_{k-1} = 2\Phi_2 D_x(\varphi_{k-1}) \\[1mm]
\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[2mm]
\ell_F(\varphi_\beta) + \sum_{i=\beta}^{k} \left[\binom{i}{i-\beta} D_x^{i-\beta}(\Phi_0) + \binom{i}{i-\beta+1} D_x^{i-\beta+1}(\Phi_1) \right. \\[2mm]
\qquad \left. + \binom{i}{i-\beta+2} D_x^{i-\beta+2}(\Phi_2) \right] \varphi_i + (\beta-1)D_x(\Phi_2)\varphi_{\beta-1} \\[2mm]
\qquad = 2\Phi_2 D_x(\varphi_{\beta-1}), \\[1mm]
\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[2mm]
\ell_F(\varphi_3) + \sum_{i=3}^{k} \left[\binom{i}{i-3} D_x^{i-3}(\Phi_0) + \binom{i}{i-2} D_x^{i-2}(\Phi_1) \right. \\[2mm]
\qquad \left. + \binom{i}{i-1} D_x^{i-1}(\Phi_2) \right] \varphi_i + 2D_x(\Phi_2)\varphi_2 = 2\Phi_2 D_x(\varphi_2).
\end{cases}
\tag{4.7}
$$

System (4.7) is of a diagonal form and is easier to study than the initial equation $\ell_F(\varphi) = 0$. Let us illustrate the procedure using one class of evolution equations.

4.2. The Burgers and heat equations. Let us describe equations of the form

$$u_t = u_{xx} + f(u, u_x), \tag{4.8}$$

possessing symmetries of arbitrary high order. We are also going to compute the corresponding symmetry algebras.

In the case of equation (4.8) the system (4.7) acquires the form

$$
\begin{cases}
D_x(\varphi_k) = 0, \\[2mm]
2D_x(\varphi_{k-1}) = \ell_F(\varphi_k) + \left[f_0 + \binom{k}{1} D_x(f_1) \right] \varphi_k, \\[2mm]
\dotfill \\[2mm]
2D_x(\varphi_{\beta-1}) = \ell_F(\varphi_\beta) \\[1mm]
\qquad + \sum_{i=\beta}^{k} \left[\binom{i}{i-\beta} D_x^{i-\beta}(f_0) + \binom{i}{i-\beta+1} D_x^{i-\beta+1}(f_1) \right] \varphi_i, \\[2mm]
\dotfill \\[2mm]
2D_x(\varphi_2) = \ell_F(\varphi_3) + \sum_{i=3}^{k} \left[\binom{i}{i-3} D_x^{i-3}(f_0) + \binom{i}{i-2} D_x^{i-2}(f_1) \right] \varphi_i.
\end{cases}
\tag{4.9}
$$

Looking at the system (4.9), one can see what can be obstructions for equation (4.8) to possess higher symmetries. Assume that we managed to solve the first i equations of the system (4.9); then the condition for solvability of the next equation is that its right-hand side (which is expressed in terms of the already obtained solutions) belongs to the image of the operator D_x. This condition is determined by the function f. Let us show how this technique works. To simplify computations, we shall make the change of variables $\varphi_\alpha = 2^{\alpha-k}\psi_\alpha$.

From the first equation of the system (4.9) it follows that

$$
\psi_k = a_k(t).
\tag{4.10}
$$

Substituting ψ_k to the second equation, we obtain

$$
D_x(\psi_{k-1}) = \ell_F(\psi_k) + \left[f_0 + \binom{k}{1} D_x(f_1) \right] \psi_k = \dot{a}_k + k D_x(f_1)\psi_k,
$$

where $\dot{a}_k = da_k/dt$. Thus, we obtain the expression

$$
\psi_{k-1} = \dot{a}_k x + k f_1 a_k + a_{k-1}(t).
\tag{4.11}
$$

Substituting expressions (4.10) and (4.11) to the third equation of the system (4.9) and making necessary transformations, we obtain

$$
\begin{aligned}
D_x(\psi_{k-2}) = {} & \ddot{a}_k x + (k-1)\dot{a}_k \left[f_1 + x D_x(f_1) \right] \\
& + k a_k \left[2 D_x(f_0) + (k-2) f_1 D_x(f_1) + (k-2) D_x^2(f_1) + D_t(f_1) \right] \\
& + D_x \left[\frac{1}{2}\ddot{a}_k x^2 + (k-1)\dot{a}_k x f_1 + k a_k \left(2 f_0 + \frac{1}{2}(k-2) f_1^2 + (k-2) D_x(f_1) \right) \right. \\
& \qquad\qquad\qquad\qquad\quad \left. + \dot{a}_{k-1} x + (k-1) a_{k-1} f_1 \right] + k a_k D_t(f_1).
\end{aligned}
$$

Thus, if $\deg(\varphi) = k$, then the third equation is solvable if and only if

$$
D_t(f_1) \in \operatorname{im} D_x
\tag{4.12}
$$

(note that this means that f_1 is a *conservation law* for equation (4.8), see Ch. 5).

Let us describe the functions f for which the condition (4.12) fulfills. We have[11]

$$D_t(f_1) = (p_2 + f)f_{01} + (p_3 + D_x(f))f_{11}$$
$$= (p_2 + f)(f_{01} - D_x(f_{11})) + D_x((p_2 + f)f_{11}).$$

In other words, $D_t(f_1) \in \operatorname{im} D_x$ if and only if

$$(p_2 + f)(f_{01} - D_x(f_{11})) \in \operatorname{im} D_x. \tag{4.13}$$

From (4.4) it is obvious that any element lying in the image of the operator D_x is linear with respect to the variable p_α of highest order; on the other hand, expression (4.13) is of the form $f_{111}p_2^2 + O(1)$. Therefore, the conditions (4.12) hold if and only if the third derivative f_{111} vanishes. Hence,

$$f = Ap_1^2 + Bp_1 + C, \tag{4.14}$$

where A, B, C are functions in p_0, x, and t. Substituting the expression obtained to (4.13) and making similar computations, one can see that the condition (4.12) holds if and only if the functions A, B and C in (4.14) satisfy the equations

$$AB_0 = B_{00}, \qquad CB_0 = \text{const.} \tag{4.15}$$

Note now that any equation

$$u_t = u_{xx} + A(u)u_x^2 + B(u)u_x + C(u)$$

by change of variables $u \mapsto \Psi(u)$, where the function Ψ, $\Psi_u \neq 0$, satisfies the differential equation

$$\Psi_{uu} + A(\Psi)\Psi_u^2 = 0,$$

can be transformed to the form

$$u_t = u_{xx} + \widetilde{B}(u)u_x + \widetilde{C}(u).$$

Therefore, without loss of generality, we may put $A = 0$ in (4.15) and take $B = \beta_1 u + \beta_0$, β_0, $\beta_1 = \text{const}$, and $\beta_1 C = \text{const}$. We have now two options: $\beta_1 \neq 0$ and $\beta_1 = 0$. In the first case the initial equation transforms to the form

$$u_t = u_{xx} + (\beta_1 u + \beta_0)u_x + \gamma, \qquad \gamma = \text{const}, \quad \beta_1 \neq 0,$$

while in the second one it transforms to

$$u_t = u_{xx} + \beta_0 u_x + C(u). \tag{4.16}$$

The first of these equations by change of variables

$$x \longmapsto x - \frac{\gamma t^2}{2}, \quad t \longmapsto t, \quad u \longmapsto \frac{u + \gamma t - \beta_0}{\beta_1}$$

transforms to

$$u_t = u_{xx} + uu_x, \tag{4.17}$$

i.e., is equivalent to the Burgers equation.

If we now come back to system (4.9), we shall see that[12] for (4.17) the fourth equation is solvable, while in the case (4.16) linearity of the function $C(u)$ is necessary. In other words, equation (4.16) has to be of the form

$$u_t = u_{xx} + \beta_0 u_x + \gamma_1 u + \gamma_0.$$

---

[11]Below by f_{ij} the partial derivative $\partial f/\partial p_i \partial p_j$ is denoted. A similar meaning has the notation f_{ijk}.

[12]We omit the corresponding computations which are simple and not instructive already.

The last equation, by change of variables

$$u \mapsto u \exp\left[\left(\gamma_1 - \frac{\beta_0^2}{4}\right)t - \frac{\beta_0}{2}x\right] + u_0,$$

where u_0 is an arbitrary solution of the corresponding homogeneous equation, reduces to the form

$$u_t = u_{xx}.$$

Thus, we proved the following result:

PROPOSITION 4.3. *Any equation*

$$u_t = u_{xx} + f(u, u_x),$$

possessing symmetries of arbitrary high order is equivalent either to the Burgers equation

$$u_t = u_{xx} + uu_x,$$

or to the heat equation

$$u_t = u_{xx}.$$

Our next aim is to show that these equations really possess infinite algebras of higher symmetries and to describe these algebras. In doing this we follow paper [**141**]. First of all, we shall need some information on algebraic structure of symmetries we are looking for.

From equations (4.10) and (4.11) it follows that any symmetry of order k, if it exists, is of the form

$$\varphi_k[a] = ap_k + \left(\frac{1}{2}\dot{a}x + \frac{k}{2}f_1 a + a'\right)p_{k-1} + O(k-2), \tag{4.18}$$

where a, a' are functions in t. It is easily seen that any symmetry is uniquely (up to symmetries of lower order) determined by the function a (here $f_1 = p_0$ for the Burgers equation and $f_1 = 0$ for the heat equation). Let $\varphi_l[b]$ be also a function of the form (4.18). Let us compute the Jacobi bracket of functions $\varphi_k[a]$ and $\varphi_l[b]$. For any functions $\varphi, \psi \in \mathcal{F}(\mathcal{E})$, $\deg(\varphi) = k, \deg(\psi) = l$, one has (see (2.24))

$$\{\varphi, \psi\}_{\mathcal{E}} = \sum_{i=0}^{l} D_x^i(\varphi)\psi_i - \sum_{j=0}^{k} D_x^j(\psi)\varphi_j.$$

Among summands at the right-hand side of this equality, the summands $D_x^l(\varphi)\psi_l$ and $-D_x^k(\psi)\varphi_k$ are of the maximal (being equal to $k + l$) order. But by remark to the above proved lemma, one has

$$D_x^l(\varphi)\psi_l = (\varphi_k p_{k+l} + O(k+l-1))\psi_l, \quad D_x^k(\psi)\varphi_k = (\psi_l p_{k+l} + O(k+l-1))\varphi_k.$$

Therefore, $\deg\{\varphi, \psi\}_{\mathcal{E}} = k + l - 1$. In particular, for functions $\varphi_k[a]$ and $\varphi_l[b]$, up to elements of order $k + l - 3$, one has

$$\{\varphi_k[a], \varphi_l[b]\}_{\mathcal{E}} = \{ap_k, bp_l\}_{\mathcal{E}} + \left\{ap_k, \left(\frac{1}{2}\dot{b}x + \frac{l}{2}f_1 b + b'\right)p_{l-1}\right\}_{\mathcal{E}}$$

$$+ \left\{\left(\frac{1}{2}\dot{a}x + \frac{k}{2}f_1 a + a'\right)p_{k-1}, bp_l\right\}_{\mathcal{E}} + O(k+l-3)$$

$$= D_x^l(ap_k)b - D_x^k(bp_l)a + D_x^{l-1}(ap_k)\left(\frac{1}{2}\dot{b}x + \frac{l}{2}f_1 b + b'\right)$$

$$- D_x^k\left[\left(\frac{1}{2}\dot{b}x + \frac{l}{2}f_1 b + b'\right)p_{l-1}\right]a + D_x^l\left[\left(\frac{1}{2}\dot{a}x + \frac{l}{2}f_1 a + a'\right)p_{k-1}\right]b$$

$$- D_x^{k-1}(bp_l)\left(\frac{1}{2}\dot{a}x + \frac{k}{2}f_1 a + a'\right) + O(k+l-3)$$

$$= \frac{1}{2}(l\dot{a}b - k\dot{b}a)p_{k+l-2} + O(k+l-3).$$

Hence, since the algebra of higher symmetries is closed with respect to the Jacobi bracket and by the fact that the functions $\varphi_k[a]$ and $\varphi_l[b]$ are symmetries, we obtain that the function $\varphi_{k+l-2}[c]$, where

$$c = \frac{1}{2}\left(l\dot{a}b - k\dot{b}a\right), \tag{4.19}$$

is also a symmetry of the equation $\mathcal{E}$.

Let us recall that in the preceding chapter we computed classical symmetries of the Burgers equation an they were of the form

$$\varphi_1^0 = p_1,$$
$$\varphi_1^1 = tp_1 + 1,$$
$$\varphi_2^0 = p_2 + p_0 p_1,$$
$$\varphi_2^1 = tp_2 + (tp_0 + \frac{1}{2}x)p_1 + \frac{1}{2}p_0,$$
$$\varphi_2^2 = t^2 p_2 + (t^2 p_0 + tx)p_1 + tp_0 + x.$$

Similar computations show that for the heat equation the classical symmetries are

$$\varphi_{-\infty} = \varphi_{-\infty}(x, t),$$
$$\varphi_0^0 = p_0,$$
$$\varphi_1^0 = p_1,$$
$$\varphi_1^1 = tp_1 + \frac{1}{2}xp_0,$$
$$\varphi_2^0 = p_2,$$
$$\varphi_2^1 = tp_2 + \frac{1}{2}xp_1,$$
$$\varphi_2^2 = t^2 p_2 + txp_1 + \left(\frac{1}{4}x^2 + \frac{1}{2}t\right)p_0,$$

where $\varphi_{-\infty}$ is an arbitrary solution of the heat equation.

Let $\varphi_k[a]$, $k > 2$, be a symmetry; then $\{\varphi_k[a], \varphi_1^0\}_\mathcal{E}$ is a symmetry as well and by the above computations it is to be of the form

$$\{\varphi_k[a], \varphi_1^0\}_\mathcal{E} = \frac{1}{2}\dot{a}p_{k-1} + O(k-2).$$

Applying the operator $\{\cdot, \varphi_1^0\}_\mathcal{E}$ to the function $\varphi_k[a]$ $k-2$ times, we shall obtain a classical symmetry of the form

$$2^{-k+2}\frac{d^{k-2}a}{dt^{k-2}}p_2 + O(1).$$

But classical symmetries of the equations under consideration have the polynomial coefficient in t of order ≤ 2 at p_2. Therefore, a is also a polynomial in t and its order is not greater than k.

Let us show that any such a polynomial determines some symmetry. To do this, note that the equation at hand possesses a symmetry of the form $\varphi_3[t]$. Namely, by direct computations we can find that the Burgers equation has the symmetry

$$\varphi_3^1 = tp_3 + \frac{1}{2}(x + 3tp_0)p_2 + \frac{3}{2}tp_1^2 + \left(\frac{1}{2}x + \frac{3}{4}tp_0\right)p_0p_1 + \frac{1}{4}p_0^2,$$

while

$$\varphi_3^1 = tp_3 + \frac{1}{2}xp_2$$

is a symmetry of the heat equation. By (4.19), the symmetry φ_3^1 acts on the functions $\varphi_k[a]$ as follows:

$$\{\varphi_k[a], \varphi_3^1\}_\mathcal{E} = \frac{1}{2}(3\dot{a}t - ka)p_{k+1} + O(k).$$

In particular, applying the operator $\{\cdot, \varphi_3^1\}_\mathcal{E}$ to the function $\varphi_1^0 = p_1$ k times, we obtain a symmetry of the form

$$(-2)^k k! p_{k+1} + O(k),$$

which proves existence of symmetries

$$\varphi_k[1] \stackrel{\text{def}}{=} \varphi_k^0 = p_k + O(k-1), \quad k = 1, 2, \dots$$

Finally, consider the symmetry φ_2^2, which acts on functions $\varphi_k[a]$ as follows:

$$\{\varphi_k[a], \varphi_2^2\}_\mathcal{E} = t(t\dot{a} - ka)p_k + O(k-1).$$

Consequently, applying the operator $\{\cdot, \varphi_2^2\}_\mathcal{E}$ to the symmetry φ_k^0 i times, $i \leq k$, we obtain the symmetry, which up to a constant factor equals

$$\varphi_k[t^i] \stackrel{\text{def}}{=} \varphi_k^i = t^i p_k + O(k-1).$$

The above said is equally valid both for the Burgers equation and for the heat equation. Let us make some remarks specific for the latter. First note that any symmetry of the heat equation is linear with respect to all variables $p_0, p_1, \dots, p_k$, i.e., is of the form

$$\varphi = A(x,t) + \sum_{i=0}^{k} A_i(x,t)p_i,$$

where $\partial A/\partial t = \partial^2 A/\partial x^2$. This fact easily follows either from straightforward analysis of the equation $\ell_F(\varphi) = 0$, or from study of the system (4.7). Since $A(x,t)$

is also a symmetry, the functions φ_k^i can be considered to be linear homogeneous functions of the variables $p_0, \dots, p_k$. Hence

$$\{p_0, \varphi_k^i\}_{\mathcal{E}} = (\eth_{p_0} - \ell_{p_0})\varphi_k^i = \left(\sum_{\alpha \geq 0} \frac{\partial}{\partial p_\alpha} - 1\right)\varphi_k^i = 0,$$

and the bracket

$$\{\varphi_k^i, \varphi_{-\infty}\}_{\mathcal{E}} = \eth_{\varphi_k^i}(\varphi_{-\infty}) - \ell_{\varphi_k^i}(\varphi_{-\infty}) = -\ell_{\varphi_k^i}(\varphi_{-\infty})$$

$$= -\sum_{\alpha=0}^{k} \frac{\partial \varphi_k^i}{\partial p_\alpha} D_x^\alpha(\varphi_{-\infty}) = -\sum_{\alpha=0}^{k} \frac{\partial \varphi_k^i}{\partial p_\alpha} \frac{\partial^\alpha \varphi_{-\infty}}{\partial x^\alpha}$$

depends on x and t only and thus is a solution of the heat equation. Note also that

$$\{p_0, \varphi_{-\infty}\}_{\mathcal{E}} = \varphi_{-\infty}, \qquad \{\varphi'_{-\infty}, \varphi''_{-\infty}\}_{\mathcal{E}} = 0.$$

As a result of all preceding considerations, we have the following theorem:

THEOREM 4.4. 1. *Any equation of the form* $u_t = u_{xx} + f(u, u_x)$ *possessing symmetries of arbitrary high order is equivalent either to the Burgers equation* $u_t = u_{xx} + uu_x$, *or to the heat equation* $u_t = u_{xx}$.

2. *For any* $k > 0$, *these equations possess* $k + 1$ *symmetries of order* k *of the form*

$$\varphi_k^i = t^i p_k + O(k-1), \qquad i = 1, \dots, k.$$

3. *Symmetries* φ_k^i *form a Lie* $\mathbb{R}$-*algebra* $A_+(\mathcal{E})$ *and*

$$\{\varphi_k^i, \varphi_l^j\}_{\mathcal{E}} = \frac{1}{2}(li - kj)\varphi_{k+l-2}^{i+j-1} + \mathcal{S}_{<k+l-2},$$

where $\mathcal{S}_{<k+l-2}$ *are symmetries of order* $< k + l - 2$. *The algebra* $A_+(\mathcal{E})$ *has three generators* φ_1^0, φ_2^2, *and* φ_3^1, *where* $\varphi_1^0 = p_1$, *while*

$$\varphi_2^2 = t^2 p_2 + (t^2 p_0 + tx) + p_1 + tp_0 + x,$$

$$\varphi_3^1 = tp_3 + \frac{1}{2}(x + 3tp_0)p_2 + \frac{3}{2}tp_1^2 + \left(\frac{1}{2}x + \frac{3}{4}tp_0\right)p_0 p_1 + \frac{1}{4}p_0^2$$

for the Burgers equation and

$$\varphi_2^2 = t^2 p_2 + txp_1 + \left(\frac{1}{4}x^2 + \frac{1}{2}t\right)p_0,$$

$$\varphi_3^1 = tp_3 + \frac{1}{2}xp_2$$

for the heat equation.

4. *In the case of the Burgers equation, the algebra* $\mathrm{sym}\,\mathcal{E}$ *of higher symmetries coincides with the algebra* $A_+(\mathcal{E})$. *For the heat equation, the algebra* $\mathrm{sym}\,\mathcal{E}$ *is a semi-direct product of* $A_+(\mathcal{E})$ *with the ideal* $A_0(\mathcal{E})$ *consisting of functions of the form* $ap_0 + \varphi_{-\infty}(x, t)$, *where* $a = \mathrm{const}$ *and* $\varphi_{-\infty}$ *is an arbitrary solution of the heat equation. The functions* φ_k^i *are linear with respect to all variables* p_α, $\alpha = 0, 1, \dots, k$,

$$\{\varphi_k^i, ap_0 + \varphi_{-\infty}\}_{\mathcal{E}} = -\sum_{\alpha=1}^{k} \frac{\partial \varphi_k^i}{\partial p_\alpha} \frac{\partial^\alpha \varphi_{-\infty}}{\partial x^\alpha}$$

and

$$\{a'p_0 + \varphi'_{-\infty}, a''p_0 + \varphi''_{-\infty}\}_{\mathcal{E}} = a''\varphi'_{-\infty} - a'\varphi''_{-\infty}.$$

REMARK 4.2. All above constructed symmetries φ^i_k are determined up to symmetries of lower order. The corresponding ambiguity arises in commutation relations between φ^i_k as well (see Theorem 4.4 (3)). For the Burgers equation, this ambiguity can essentially diminished using the following trick.

Let us assign to the variables x, t, and u *weights* in the following way

$$\operatorname{gr} x = 1, \qquad \operatorname{gr} t = 2, \qquad \operatorname{gr} u = -1.$$

The Burgers equation becomes homogeneous with respect to this system of weights. Let us also set $\operatorname{gr} p_k = -k - 1$ and for any monomial $\mathcal{M} = x^{\alpha} t^{\beta} p_0^{\gamma_0} p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ define its weight as the sum of weights of all factors

$$\operatorname{gr} \mathcal{M} = \alpha + 2\beta - \sum_{i=0}^{k} \gamma_i(i+1).$$

In the ring $\mathcal{F}(\mathcal{E})$ of smooth functions on the infinitely prolonged Burgers equation, consider the subring $\mathcal{P}(\mathcal{E})$ consisting of functions polynomial with respect to all variables. Then, as it is easily seen, $\mathcal{P}(\mathcal{E})$ is closed with respect to the operators $\ell_F^{\mathcal{E}}$ and $\{\cdot,\cdot\}_{\mathcal{E}}$ and restrictions of these operators to $\mathcal{P}(\mathcal{E})$ are homogeneous with respect to the weighting. Moreover, if $\varphi, \psi \in \mathcal{P}(\mathcal{E})$ are homogeneous polynomials, then

$$\operatorname{gr} \ell_F^{\mathcal{E}}(\varphi) = \operatorname{gr} \varphi - 2, \quad \operatorname{gr}\{\varphi, \psi\}_{\mathcal{E}} = \operatorname{gr} \varphi + \operatorname{gr} \psi + 1.$$

Consequently, if $\varphi \in \mathcal{P}(\mathcal{E})$ is a solution of the equation $\ell_F^{\mathcal{E}}(\varphi) = 0$, then any homogeneous component of the polynomial φ is also a solution of this equation.

Further, the symmetries φ_1^0, φ_2^2, and φ_3^1 are polynomials and are generators of the Lie algebra $\operatorname{sym} \mathcal{E}$; therefore, $\operatorname{sym} \mathcal{E} \subset \mathcal{P}(\mathcal{E})$. Thus from the above said it follows that the functions φ_k^i can be considered to be homogeneous and $\operatorname{gr} \varphi_k^i = 2i - k - 1$. From here it is seen that the homogeneity condition *uniquely* determines classical symmetry of the Burgers equation as well as the symmetries of the form φ_k^0 and φ_k^k.

Let φ_k^0, φ_l^0 be two homogeneous symmetries. Then, since the order of the symmetry $\{\varphi_k^0, \varphi_l^0\}_{\mathcal{E}}$, if it is not equal to 0, is less than $k + l - 2$, its weight is not greater than $k + l - 3$ and not less than $1 - k - l$. But on the other hand,

$$\operatorname{gr}\{\varphi_k^0, \varphi_l^0\}_{\mathcal{E}} = \operatorname{gr} \varphi_k^0 + \operatorname{gr} \varphi_l^0 = -1 - k - l.$$

The contradiction obtained shows that $\{\varphi_k^0, \varphi_l^0\}_{\mathcal{E}} = 0$, i.e., the symmetries of the form φ_k^0, $k = 1, 2, \ldots$, commute with each other.

Let us consider the action of the operators $\{\varphi_1^0, \cdot\}_{\mathcal{E}}$ and $\{\varphi_2^0, \cdot\}_{\mathcal{E}}$ in more details. One has

$$\{\varphi_1^0, \cdot\}_{\mathcal{E}} = \eth_{\varphi_1^0} - \ell_{\varphi_1^0} = \sum_{\alpha \geq 0} p_{\alpha+1} \frac{\partial}{\partial p_\alpha} - D_x = -\frac{\partial}{\partial x}$$

and

$$\{\varphi_2^0, \cdot\}_{\mathcal{E}} = \eth_{\varphi_2^0} - \ell_{\varphi_2^0} = \sum_{\alpha \geq 0} \left[D^\alpha(p_2 + p_0 p_1) \frac{\partial}{\partial p_\alpha} \right] - p_1 - p_0 D_x - D_x^2 = \ell_F^{\mathcal{E}} - \frac{\partial}{\partial t}.$$

Since the action $\{\varphi_2^0, \cdot\}_\mathcal{E}$ is considered on symmetries of the Burgers equation, i.e., on solutions of the equation $\ell_F^\mathcal{E}(\varphi) = 0$, we eventually obtain

$$\{\varphi_2^0, \cdot\}_\mathcal{E} = \frac{\partial}{\partial t}.$$

Hence, the symmetries of the form φ_k^0 do not depend on x and t. Further, for homogenous components one has obvious equalities

$$\{\varphi_1^0, \varphi_k^1\}_\mathcal{E} = -\frac{1}{2}\varphi_{k-1}^0, \qquad \{\varphi_2^0, \varphi_k^1\}_\mathcal{E} = -\varphi_k^1.$$

Therefore, the symmetries φ_k^1 are linear in x and t. In the same way, by elementary induction it is proved that φ_k^i is a polynomial of i-th degree in t and x.

REMARK 4.3. Let us return back to the heat equation and show a simple way to construct its higher symmetries. In this case, the operator ℓ_F is of the form $D_t - D_x^2$ and consequently commutes with the operator D_x. Therefore, if φ is a symmetry, then

$$\ell_F(D_x\varphi) = D_x(\ell_F(\varphi)) = 0,$$

i.e., $D_x\varphi$ is also a symmetry of the heat equation. This fact is a particular case of a more general result.

Let $\mathcal{E} = \{F = 0\}$ be a linear equation and $\Delta = \Delta_F$ be the corresponding linear differential operator. Then (see formula (2.33)) $\ell_F = \widehat{\Delta}$. Let ∂ another linear operator for which

$$\Delta \circ \partial = \partial' \circ \Delta,$$

where ∂' is also a linear differential operator. Let us set $\mathcal{R} = \widehat{\partial}$. Then

$$\ell_F \circ \mathcal{R} = \widehat{\Delta} \circ \widehat{\partial} = \widehat{\Delta \circ \partial} = \widehat{\partial' \circ \Delta} = \widehat{\partial'} \circ \ell_F$$

and thus the operator $\mathcal{R}$ acts on the algebra $\operatorname{sym}\mathcal{E}$. Such operators are called *recursion operators* and are widely used for constructing higher symmetries [**59**]. In particular, applications of recursion operators will be shown in the next example. Nevertheless, for nonlinear equations the situation is more complicated and we shall discuss it in Ch. 6.

4.3. The plasticity equations. Consider the system of equations $\mathcal{E}$

$$\begin{cases} \sigma_x = 2k(\theta_x \cos 2\theta + \theta_y \sin 2\theta), \\ \sigma_y = 2k(\theta_x \sin 2\theta - \theta_y \cos 2\theta), \end{cases} \tag{4.20}$$

describing the plane strained state of the medium with von Mises conditions, where σ is hydrostatic pressure, θ is the angle between the x-axis and the first main direction of stress tensor, $k \neq 0$ is the plasticity constant. Describing symmetries of this equation[13] we follow paper the [**104**].

By the change of variables

$$\begin{cases} \sigma = k(\xi + \eta), \\ \theta = \frac{1}{2}(\eta - \xi), \end{cases} \quad \begin{cases} x = u\cos\frac{1}{2}(\eta - \xi) - v\sin\frac{1}{2}(\eta - \xi), \\ y = u\sin\frac{1}{2}(\eta - \xi) + v\cos\frac{1}{2}(\eta - \xi), \end{cases} \tag{4.21}$$

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[13]In what follows, we do not stress technical details, but pay main attention to the structure of proofs and to most interesting specific features. To reconstruct omitted details is a useful exercise in symmetry computations.

where u, v are new dependent and ξ, η are new independent variables, the system (4.20) is transformed to the system

$$\begin{cases} u_\xi + \dfrac{1}{2}v = 0, \\[3mm] v_\eta + \dfrac{1}{2}u = 0, \end{cases} \tag{4.22}$$

which will also denoted by $\mathcal{E}$. Its universal linearization operator is of the form

$$\ell_F = \begin{pmatrix} D_\xi & 1/2 \\ 1/2 & D_\eta \end{pmatrix}.$$

Choose on $\mathcal{E}^\infty$ internal coordinates ξ, η, u_k, v_k in such a way that u_k corresponds to the partial derivative $\partial^k u/\partial\eta^k$ and v_k corresponds to $\partial^k v/\partial\xi^k$. Then restrictions of total derivatives to $\mathcal{E}^\infty$ will be written in these coordinates in the form

$$D_\xi = \frac{\partial}{\partial\xi} - \frac{1}{2}v_0\frac{\partial}{\partial u_0} + \frac{1}{4}\sum_{k\geq 1} u_{k-1}\frac{\partial}{\partial u_k} + \sum_{k\geq 0} v_{k+1}\frac{\partial}{\partial v_k},$$

$$D_\eta = \frac{\partial}{\partial\eta} - \frac{1}{2}u_0\frac{\partial}{\partial v_0} + \sum_{k\geq 0} u_{k+1}\frac{\partial}{\partial u_k} + \frac{1}{4}\sum_{k\geq 1} v_{k-1}\frac{\partial}{\partial v_k},$$

and if $\Phi = \begin{pmatrix}\varphi\\\psi\end{pmatrix}$ is a symmetry of order k, then the corresponding defining equations will be

$$\begin{cases} \dfrac{\partial\varphi}{\partial\xi} - \dfrac{1}{2}v_0\dfrac{\partial\varphi}{\partial u_0} + v_1\dfrac{\partial\varphi}{\partial v_0} + \displaystyle\sum_{\alpha=1}^{k}\left(\dfrac{1}{4}u_{\alpha-1}\dfrac{\partial\varphi}{\partial u_\alpha} + v_{\alpha+1}\dfrac{\partial\varphi}{\partial v_\alpha}\right) + \dfrac{1}{2}\psi = 0, \\[5mm] \dfrac{\partial\psi}{\partial\eta} + u_1\dfrac{\partial\psi}{\partial u_0} - \dfrac{1}{2}u_0\dfrac{\partial\psi}{\partial v_0} + \displaystyle\sum_{\alpha=1}^{k}\left(u_{\alpha+1}\dfrac{\partial\psi}{\partial u_\alpha} + \dfrac{1}{4}v_{\alpha-1}\dfrac{\partial\psi}{\partial v_\alpha}\right) + \dfrac{1}{2}\varphi = 0. \end{cases} \tag{4.23}$$

Solving system (4.23) for $k \leq 1$, we shall obtain classical symmetries of equation (4.22); any such a symmetry is a linear combination of the following ones:

$$S_0 = \begin{pmatrix}u_0\\v_0\end{pmatrix}, \quad f_1^0 = \begin{pmatrix}u_1\\-\dfrac{u_0}{2}\end{pmatrix}, \quad g_1^0 = \begin{pmatrix}-\dfrac{v_0}{2}\\v_1\end{pmatrix}, \quad S_1 = \begin{pmatrix}\eta u_1 + \dfrac{u_0}{3} + \xi\dfrac{v_0}{2}\\-\xi v_1 - \dfrac{v_0}{2} - \eta\dfrac{u_0}{2}\end{pmatrix},$$

and of the symmetry of the form $H = \begin{pmatrix}f\\g\end{pmatrix}$, where $u = f$, $v = g$ is an arbitrary solution of equation (4.22).

For "large" k, the solutions of (4.23), if any, have the following "asymptotics"

$$\begin{pmatrix} Au_k - \dfrac{1}{2}Bv_{k-1} + au_{k-1} + \dfrac{1}{2}(B'-b)v_{k-2} + \left(\alpha - \dfrac{1}{4}\xi A'\right)u_{k-2} + O(k-3) \\[4mm] Bv_k - \dfrac{1}{2}Au_{k-1} + bv_{k-1} + \dfrac{1}{2}(A'-a)u_{k-2} + \left(\beta - \dfrac{1}{4}\eta B'\right)v_{k-2} + O(k-3) \end{pmatrix},$$

where A, a, α are functions in η, B, b, β are functions in ξ, while the "prime" denotes the derivatives with respect to ξ or η. Let us denote this solution by $\Phi_k(A, B)$.

Following Remark 4.3, let us construct first order recursion operators for equation (4.22). As it is easily seen, such operators exist and are of the form

$$\mathcal{R} = \begin{pmatrix} a_{11}D_\xi + b_{11}D_\eta + c_{11} & b_{12}D_\eta + c_{12} \\ a_{21}D_\xi + c_{21} & a_{22}D_\xi + b_{22}D_\eta + c_{22} \end{pmatrix}, \tag{4.24}$$

where a_{11}, c_{11}, b_{22}, c_{22} are arbitrary functions in ξ and η,

$$a_{21} = 2c_{11} + \alpha_1, \quad b_{12} = 2c_{22} + \alpha_2, \quad \alpha_1, \alpha_2 = \text{const},$$

$$a_{22} = \frac{1}{2}(\alpha_1 - \alpha_2)\xi + \beta_1, \quad b_{11} = \frac{1}{2}(\alpha_2 - \alpha_1)\eta + \beta_2, \quad \beta_1, \beta_2 = \text{const},$$

$$c_{12} = \frac{1}{2}(a_{11} - a_{22}), \quad c_{21} = \frac{1}{2}(b_{22} - b_{11}).$$

In particular, among operators (4.24) one has the operator

$$\mathcal{R}_1 = \begin{pmatrix} D_\xi & 0 \\ 0 & D_\xi \end{pmatrix},$$

application of which to the functions of the form $\Phi_k(A, B)$ gives the following expression:

$$\mathcal{R}_1\Phi_k(A, B) = \Phi_{k+1}(0, B) + \Phi_k(0, B') + \frac{1}{4}\Phi_{k-1}(A, 0) + O(k - 2).$$

On the other hand, if one acts on the function $\Phi_k(A, B)$ by the symmetry f_1^0, one obtains

$$\{\Phi_k(A, B), f_1^0\}_{\mathcal{E}} = \frac{\partial}{\partial \eta}\Phi_k(A, B) = \Phi_k(A', 0) - \frac{1}{4}\Phi_{k-2}(0, B') + O(k - 3).$$

Therefore,

$$\mathcal{R}_1\{\Phi_k(A, B), f_1^0\}_{\mathcal{E}} = \frac{1}{4}\Phi_{k-1}(A', -B') + O(k - 2). \tag{4.25}$$

Now, by induction on k (the base of induction is the above given representation of classical symmetries (4.22) while the induction step consists in using equation (4.25)) it is easily shown that any symmetry of order $k > 0$, if it exists, is a linear combination of the following symmetries:

$$S_k = \Phi_k(\eta^k, (-\xi)^k), \quad f_k^i = \Phi_k(\eta^i, 0), \quad g_k^i = \Phi_k(0, \xi^i), \qquad 0 \le i \le k.$$

To prove existence of these symmetries, let us note that among the recursion operators of the form (4.24) one has the operator

$$\mathcal{R}_2 = \begin{pmatrix} -\eta D_\xi + \eta D_\eta + \frac{1}{2} & \frac{1}{2}(\xi - \eta) \\ \frac{1}{2}(\xi - \eta) & \xi D_\xi - \xi D_\eta - \frac{1}{2} \end{pmatrix}.$$

Application of this operator to the symmetry S_0 k times gives, up to a constant factor, the symmetry S_k. Further, application of the operator $\{\cdot, f_1^0\}_{\mathcal{E}}$ to the symmetry S_k $k - i$ times leads to the symmetry f_k^i. Finally, applying $k - i$ times the operator $\{\cdot, g_1^0\}_{\mathcal{E}}$ to the symmetry S_k, we shall prove existence of g_k^i.

The above said can be combined in the following result.

THEOREM 4.5. *The algebra of higher symmetries of the system* (4.22) *is a semi-direct product of the commutative ideal* $\text{sym}_0\,\mathcal{E}$ *consisting of symmetries* $H = \binom{f}{g}$, *where* f, g *is an arbitrary solution of* (4.22)*, with the algebra* $\text{sym}_+\,\mathcal{E}$*. The latter, as a vector space is spanned by the elements* S_k, f_k^i, g_k^i, $0 \le i < k$, $k = 0, 1, \ldots,$

and as a Lie algebra is generated by the elements f_1^0, g_1^0, S_0, $S_1, \ldots, S_k, \ldots$ *Any symmetry* S_k *is of the form* $S_k = \mathcal{R}_2^k(S_0)$.

4.4. Transformation of symmetries under change of variables. Let us rewrite the results obtained in terms of the initial equation (4.20). To this end, it is necessary to find out how generating sections and recursion operators transform under change of variables. To do it, let us note the following.

Let $\mathcal{S} = (x_1, \ldots, x_n, u^1, \ldots, u^m, p_1^1, \ldots, p_n^m)$ be a local canonical coordinate system in $J^1(\pi)$, where $p_i^j \stackrel{\text{def}}{=} p_{1_i}^j$. Consider in the same neighborhood another canonical coordinate system $\widetilde{\mathcal{S}} = (\widetilde{x}_1, \ldots, \widetilde{x}_n, \widetilde{u}^1, \ldots, \widetilde{u}^m, \widetilde{p}_1^1, \ldots, \widetilde{p}_n^m)$ compatible with the contact structure in $J^1(\pi)^{14}$. The Cartan distribution on $J^1(\pi)$ is determined by the system of the Cartan forms

$$\omega^1 = du^1 - \sum_\alpha p_\alpha^1 \, dx_\alpha, \, \ldots, \, \omega^m = du^m - \sum_\alpha p_\alpha^m \, dx_\alpha,$$

which will be represented as a column $\Omega = (\omega^1, \ldots, \omega^m)^t$. On the other hand, in the coordinate system $\widetilde{\mathcal{S}}$ the set of forms $\widetilde{\Omega}$ determines, by the above said, the same distribution and consequently the equality

$$\widetilde{\Omega} = \Lambda\Omega, \tag{4.26}$$

holds, where $\Lambda = \|\lambda_\beta^\alpha\|$ is a nondegenerate transformation matrix: $\widetilde{\omega}^\alpha = \sum_\beta \lambda_\beta^\alpha \omega^\beta$.

Let X be a Lie field on $J^1(\pi)$. Then its generating section in the coordinate system $\mathcal{S}$, also represented as a column $\varphi = (\varphi^1, \ldots, \varphi^m)^t$, is determined by the equality $\varphi = X \lrcorner \Omega = (X \lrcorner \omega^1, \ldots, X \lrcorner \omega^m)^t$ (see Ch. 3). By the same reason, the generating section of the field X in the new coordinate system is represented in the form $\widetilde{\varphi} = X \lrcorner \widetilde{\Omega}$. Hence, using (4.26), we obtain

$$\widetilde{\varphi} = X \lrcorner \widetilde{\Omega} = X \lrcorner \Lambda\Omega = \Lambda\varphi, \tag{4.27}$$

and it gives us the desired transformation rule for generating sections.

Let now $\mathcal{R}$ be an operator acting in the space of generating sections and represented in the coordinate system $\mathcal{S}$. Then from (4.27) it follows that its representation in the system $\widetilde{\mathcal{S}}$ is of the form

$$\widetilde{\mathcal{R}} = \Lambda\mathcal{R}\Lambda^{-1}. \tag{4.28}$$

Since we are interested in recursion operators of the form $\mathcal{R} = \|\sum_\alpha a_{ij}^\alpha D_\alpha + b_{ij}\|$, we also need to find out how the operators of total derivatives transform under change of coordinates.

Let $D_i = D_{x_i}$ and $\widetilde{D}_i = D_{\widetilde{x}_i}$, $i = 1, \ldots, n$. Since the fields D_i, as well as the fields $\widetilde{D}_i$, form a basis of the Cartan distribution in $J^\infty(\pi)$, the equalities $\widetilde{D}_i = \sum_\alpha \mu_i^\alpha D_\alpha$ must hold, and since $D_i(x_\alpha) = \delta_{i\alpha}$, where $\delta_{i\alpha}$ are the Kronecker symbols, we have $\mu_i^\alpha = \widetilde{D}_i(x_\alpha)$ and consequently

$$\widetilde{D}_i = \sum_{\alpha=1}^n \widetilde{D}_i(x_\alpha)D_\alpha, \qquad i = 1, \ldots, n. \tag{4.29}$$

---

[14]This means that the change of coordinates $\widetilde{x} = \widetilde{x}(x, u)$, $\widetilde{u} = \widetilde{u}(x, u)$, $\widetilde{p} = \widetilde{p}(x, u, p)$ is a Lie transformation (see Ch. 3).

Let us come back to equation (4.20). For the transformation (4.21), the transformation matrix is of the form

$$\Lambda = \begin{pmatrix} -\sigma_x \cos\theta - \sigma_y \sin\theta, & \sigma_x \sin\theta - \sigma_y \cos\theta \\ -\theta_x \cos\theta - \theta_y \sin\theta, & \theta_x \sin\theta - \theta_y \cos\theta \end{pmatrix},$$

while the total derivative operators are represented as

$$D_\xi = \frac{k}{I} \left[(\frac{\sigma_y}{2k} + \theta_y)D_x - (\frac{\sigma_x}{2k} + \theta_x)D_y \right],$$

$$D_\eta = \frac{k}{I} \left[(\theta_y - \frac{\sigma_y}{2k})D_x - (\theta_x - \frac{\sigma_x}{2k})D_y \right],$$

where $I = \sigma_x\theta_y - \sigma_y\theta_x$. Therefore, to the symmetries H, S_0, f_1^0, and g_1^0 of the transformed equation there correspond the following symmetries of the initial equation:

$$\widetilde{H} = D \begin{pmatrix} (-\sigma_x \cos\theta - \sigma_y \sin\theta)f, & (\sigma_x \sin\theta - \sigma_y \cos\theta)g \\ (-\theta_x \cos\theta - \theta_y \sin\theta)f, & (\theta_x \sin\theta - \theta_y \cos\theta)g \end{pmatrix},$$

where $f = f(\sigma/2k - \theta, \sigma/2k + \theta)$, $g = g(\sigma/2k - \theta, \sigma/2k + \theta)$, and

$$\widetilde{S}_0 = \begin{pmatrix} -x\sigma_x - y\sigma_y \\ -x\theta_x - y\theta_y \end{pmatrix},$$

$$\widetilde{f}_1^0 = \begin{pmatrix} k + \frac{1}{2}(y\sigma_x - x\sigma_y) \\ -\frac{1}{2} + \frac{1}{2}(y\theta_x - x\theta_y) \end{pmatrix}, \quad \widetilde{g}_1^0 = \begin{pmatrix} k - \frac{1}{2}(y\sigma_x - x\sigma_y) \\ \frac{1}{2} - \frac{1}{2}(y\theta_x - x\theta_y) \end{pmatrix}$$

and to the recursion operator $\mathcal{R}_2$ there corresponds the operator

$$\widetilde{\mathcal{R}}_2 = \begin{pmatrix} r_{11}, & r_{12} \\ r_{21}, & r_{22,} \end{pmatrix}$$

where

$$r_{11} = \frac{\sigma}{k}\Delta + \frac{2k\theta}{I}(\theta_x^2 + \theta_y^2 + c\Delta(d) - d\Delta(c)) - \frac{\Delta(I)}{I},$$

$$r_{12} = 2k^2\theta\Delta + \frac{2k}{I}(c^2 - d^2 - \Delta(I)) + 4k(c\Delta(d) - d\Delta(c)),$$

$$r_{21} = \frac{\theta}{4k}\Delta + \frac{\theta}{I}(c\Delta(d) - d\Delta(c) + c^2 - d^2) + \frac{k}{4},$$

$$r_{22} = \frac{\sigma}{k}\Delta + \frac{2k\theta}{I}(c\Delta(d) - d\Delta(c) - \theta_x^2 - \theta_y^2) - \frac{\Delta(I)}{I}$$

and

$$c = -\theta_x \cos\theta - \theta_y \sin\theta, \quad d = \theta_x \sin\theta - \theta_y \cos\theta, \quad \Delta = \sigma_x D_y - \sigma_y D_x.$$

Applying the operator $\widetilde{\mathcal{R}}_2^k$ to the symmetry $\widetilde{S}_0$, we shall obtain the symmetries $\widetilde{S}_k$, and applying $(k - i)$ times the operators $\{\cdot, \widetilde{f}_1^0\}_\mathcal{E}$ and $\{\cdot, \widetilde{g}_1^0\}_\mathcal{E}$ to $\widetilde{S}_k$, we shall obtain (up to constant factors) the corresponding symmetries $\widetilde{f}_k^i$ and $\widetilde{g}_k^i$. Thus we can efficiently compute any symmetry of equation (4.20) (though the explicit expressions will certainly be quite cumbersome).

4.5. Ordinary differential equations. To finish this section, we shall study higher symmetries of ordinary differential equations. Besides the facts concerning higher symmetries, we also mention some results on classical symmetries of these equations.

Let $M = \mathbb{R}$ and $\pi\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a trivial bundle. Consider a determined system of ordinary differential equations $\mathcal{E} \subset J^k(\pi)$, i.e., a system such that $\operatorname{codim}\mathcal{E} = \dim\pi = m$. Since the base of the bundle π is one-dimensional, this means that dimension of the manifold $\mathcal{E}$ coincides with dimension of the space $J^{k-1}(\pi)$ which equals $m(k-1)+1$. Let us call a point $\theta \in \mathcal{E}$ *generic*, if the projection $\pi_{k,k-1}|_{\mathcal{E}}$ at this point is of the maximal rank $m(k-1)+1$. We shall confine ourselves with equations such that all their points are generic (otherwise all considerations below will be valid in a neighborhood of such a point). This means that $\mathcal{E}$ is projected to $J^{k-1}(\pi)$ diffeomorphically or, which is the same, that the equation under consideration is resolvable with respect to all derivatives $d^k u^1/dx^k, \ldots, d^k u^m/dx^k$ of highest order, where $x = x_1$ is the sole independent variable. Hence, $\mathcal{E}$ can be represented in the form

$$\mathcal{E} = s(J^{k-1}(\pi)), \tag{4.30}$$

where $s = s_0\colon J^{k-1}(\pi) \to J^k(\pi)$ is a section of the bundle $\pi_{k,k-1}$. Representation (4.30) allows one to obtain a convenient description of the manifold $\mathcal{E}^\infty$ and of the Cartan distribution on this manifold.

Namely, consider an arbitrary point $\theta \in \mathcal{E}$ and some R-plane L (in our case such a plane is one-dimensional) lying in the space $T_\theta(\mathcal{E})$. Let L' be another R-plane and $v \in L$, $v' \in L'$ be vectors such that $\pi_{k,k-1}(v) = \pi_{k,k-1}(v')$. Then $v - v'$ is a $\pi_{k,k-1}$-vertical vector lying in $T_\theta(\mathcal{E})$. But by (4.30), the intersection of $T_\theta(\mathcal{E})$ with the tangent space to the fiber of the bundle $\pi_{k,k-1}$ at θ is trivial. Hence, $v = v'$ and $L = L'$, i.e., the Cartan distribution on $\mathcal{E}$ at every point contains one R-plane at most. Let us show that such a plane always exists. In fact, let us represent the point $\theta \in \mathcal{E}$ as a pair (θ', L_θ), where $\theta' = \pi_{k,k-1}(\theta)$ and L_θ is the R-plane at the point θ' determined by the point θ. Then obviously the R-plane $L_\theta^0 = s_*(L_\theta)$ lies in $T_\theta(\mathcal{E})$. Thus the Cartan distribution on $\mathcal{E}$ coincides with the field of directions $\mathcal{L}^0 = \{ L_\theta^0 \mid \theta \in \mathcal{E} \}$. Note that the section s is an isomorphism between the manifold $J^{k-1}(\pi)$ endowed with the field of directions $\mathcal{L} = \{ L_\theta \mid \theta \in \mathcal{E} \}$ and the pair $(\mathcal{E}, \mathcal{L}^0)$.

Further, a pair (θ, L_θ^0), $\theta \in \mathcal{E}$, determines a point $\theta_1 \in J^{k+1}(\pi)$, and the set of such points coincides with the manifold $\mathcal{E}^{(1)}$. It is obvious that $\mathcal{E}^{(1)}$ represents as $s_1(J^{k-1}(\pi))$, where s_1 is a section of the bundle $\pi_{k+1,k-1}$, while the Cartan distribution on $\mathcal{E}^{(1)}$ coincides with the field of directions

$$\mathcal{L}^{(1)} = \{ L_{\theta_1}^1 \mid L_{\theta_1}^1 = (s_1)_*(L_\theta), \theta_1 = s_1(\theta), \theta \in \mathcal{E} \}.$$

Proceeding with this construction, we shall come to the following statement:

PROPOSITION 4.6. *Let $\mathcal{E} \subset J^k(\pi)$, $\dim(M) = 1$, $\operatorname{codim}(\mathcal{E}) = \dim(\pi)$, be an ordinary differential equation resolvable with respect to derivatives of highest degree. Then for any $l = 0, 1, \ldots, \infty$ the manifold $\mathcal{E}^{(l)}$ is representable in the form $\mathcal{E}^{(l)} = s_l(J^{k-1}(\pi))$, where $s_l \in \Gamma(\pi_{k+l,k-1})$ and $s_l = \pi_{k+l+1,k+l} \circ s_{l+1}$, while the Cartan distribution on $\mathcal{E}^{(l)}$ coincides with the field of directions*

$$\mathcal{L}^{(l)} = \{ L_{\theta_l}^l \mid L_{\theta_l}^l = (s_l)_*(L_\theta), \theta_l = s_l(\theta), \theta \in \mathcal{E} \}.$$

Thus, all $\mathcal{E}^{(l)}$ ($\mathcal{E}^\infty$ included) as manifolds with distributions are mutually isomorphic and are isomorphic to the manifold $J^{k-1}(\pi)$ endowed with the field of directions $\mathcal{L}$.

Suppose that the equation $\mathcal{E}$ satisfies the assumptions of Proposition 4.6 and consider a higher symmetry $X \in \operatorname{sym}\mathcal{E}$ of this equation. By Proposition 4.6, for any l the field X projects to the vector field X_l on $\mathcal{E}^{(l)}$ and the field preserves the corresponding Cartan distribution. In particular, $X_0 \in \mathrm{D}(\mathcal{E})$ is a classical interior symmetry of the equation $\mathcal{E}$. Taking the set $\operatorname{Sym}_i\mathcal{E}$ of such symmetries, we thus obtain the Lie algebra homomorphism

$$\mathrm{sL}\colon \operatorname{sym}\mathcal{E} \longrightarrow \operatorname{Sym}_i\mathcal{E}. \tag{4.31}$$

Let us study this homomorphism in more details and note the following. Since in the situation under consideration $\mathcal{E}^\infty$ is a finite-dimensional manifold, the symmetry X determines a one-parameter group of transformations $\{A_t\}$ on this manifold. Applying the transformation A_t to a solution $f \in \Gamma(\pi)$ of the equation $\mathcal{E}$, we again obtain a solution $f_t = A_t^*(f)$ of this equation. By the theorem on smooth dependence of solutions of ordinary differential equations on initial values, the set $\operatorname{Sol}(\mathcal{E})$ of solutions of the equation $\mathcal{E}$ is a smooth (possibly, with singularities) manifold, while A_t is a one-parameter group of its diffeomorphisms. Denoting by X^* the corresponding vector field, we obtain the mapping $X \mapsto X^*$ which determines the Lie algebra homomorphism

$$\mathrm{sD}\colon \operatorname{sym}\mathcal{E} \longrightarrow \mathrm{D}(\operatorname{Sol}(\mathcal{E})). \tag{4.32}$$

Exactly in the same way one can construct the homomorphism

$$\mathrm{LD}\colon \operatorname{Sym}_i(\mathcal{E}) \longrightarrow \mathrm{D}(\operatorname{Sol}(\mathcal{E})). \tag{4.33}$$

Conversely, let $Y \in \mathrm{D}(\operatorname{Sol}(\mathcal{E}))$ and $\{B_t\}$ be the corresponding one-parameter group of transformations. Consider a point $\theta \in \mathcal{E}^\infty$ and a solution f_θ passing through this point (such a solution exists and is unique). Let us put into correspondence to the point θ the point $B_t'(\theta) = [B_t(f_\theta)]_x^\infty$, $x = \pi_\infty(\theta)$. Then $\{B_t'\}$ is a one-parameter group of transformations of the manifold $\mathcal{E}^\infty$ and these transformations preserve the Cartan distribution. The corresponding vector field is vertical and determines a symmetry of the equation $\mathcal{E}$. Thus we constructed the homomorphism

$$\mathrm{Ds}\colon \mathrm{D}(\operatorname{Sol}(\mathcal{E})) \longrightarrow \operatorname{sym}\mathcal{E}, \tag{4.34}$$

inverse to the homomorphism (4.32) in an obvious way. It is easily seen that the homomorphisms constructed are mutually compatible in the following sense.

PROPOSITION 4.7. *Let $\mathcal{E}$ be an equation satisfying the assumptions of Proposition 4.6. Then the above constructed mappings* sD, Ds, sL, *and* LD *are Lie algebra isomorphisms with* $\mathrm{sD} \circ \mathrm{Ds} = \mathrm{id}$ *and* $\mathrm{LD} \circ \mathrm{sL} = \mathrm{sD}$. *In other words, in the situation under consideration the Lie algebras of higher symmetries, of internal symmetries and of vector fields on the manifold of solutions are isomorphic.*

PROOF. The equalities $\mathrm{sD} \circ \mathrm{Ds} = \mathrm{id}$ and $\mathrm{LD} \circ \mathrm{sL} = \mathrm{sD}$ follow by construction. Hence sL is a monomorphism while LD is an epimorphism. Consequently, to conclude the proof it suffices, for example, to show that the kernel of LD is trivial. From the definition of the homomorphism LD it follows that its kernel consists of symmetries such that all solutions are invariant with respect to them. On the other hand, from the above given description of the Cartan distribution on $\mathcal{E}$ it can be
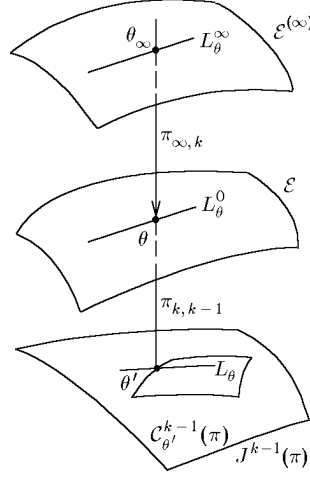
FIGURE 4.7. The field of directions L_θ on $J^{k-1}(\pi)$

seen that these symmetries must lie in the Cartan distribution. But the Cartan distribution on $\mathcal{E}^\infty$ does not contain vertical fields. $\square$

Thus, in the situation considered, the analogy between symmetries and vector field on the manifold $\mathrm{Sol}(\mathcal{E})$ acquires exact sense.

Note that Proposition 4.6 gives an obvious way to show that the algebras of exterior and interior symmetries do not coincide in the case of ordinary differential equations (see §7 of Ch. 3). In fact, interior symmetries are identified with vector fields on $J^{k-1}(\pi)$ preserving the field of directions $\mathcal{L} = \{L_\theta\}$, which contains in the Cartan distribution on $J^{k-1}(\pi)$ (see Figure 4.7). On the other hand, exterior symmetries must preserve both this field of directions and the Cartan distribution on $J^{k-1}(\pi)$ itself. Therefore, if the order of the equation is greater than 1, exterior symmetries form a proper subalgebra in the Lie algebra of interior ones.

Let us now pass to coordinate computations and consider an equation of the form

$$L\boldsymbol{f}(x) \equiv \left[\left(\frac{d}{dx} \right)^n + \sum_0^{n-1} A_i(x) \cdot \left(\frac{d}{dx} \right)^i \right] \boldsymbol{f}(x) = \boldsymbol{g}(x), \qquad (4.35)$$

where A_i are $m \times m$ matrices and $\boldsymbol{f}$, $\boldsymbol{g}$ are m-vectors. By a symmetry below we mean a higher infinitesimal symmetry.

Equation (4.35) is equivalent to the equation

$$L\boldsymbol{f}(x) = \left[\left(\frac{d}{dx} \right)^n + \sum_0^{n-1} A_i(x) \cdot \left(\frac{d}{dx} \right)^i \right] \boldsymbol{f}(x) = 0. \qquad (4.36)$$

This equivalence is given by the transformation

$$\boldsymbol{f}(x) \mapsto \boldsymbol{f}(x) - \boldsymbol{f}^*(x),$$

where $\boldsymbol{f}^*(x)$ is a solution of (4.35). Equation (4.36) determines a submanifold $\mathcal{E}$ in $J^n(\mathbb{R}, \mathbb{R}^m)$ of the form

$$\mathcal{E} = \left\{ y \in J^n(\mathbb{R}, \mathbb{R}^m) \mid \boldsymbol{p}_n + \sum_{i=0}^{n-1} A_i(x) \cdot \boldsymbol{p}_i = 0 \right\}$$

Prolongations of $\mathcal{E}$ denoted by $\mathcal{E}^{(N)} \subset J^{n+N}(\mathbb{R}, \mathbb{R}^m)$ are given by the equations

$$\boldsymbol{p}_{n+k} + D^k \left(\sum_{i=0}^{n-1} A_i(x) \cdot \boldsymbol{p}_i \right) = 0, \qquad 0 \le k \le N < \infty,$$

where

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} \sum_{k=1}^{m} p_{i+1}^k \frac{\partial}{\partial p_i^k}$$

is the total derivative along x. The space of solutions of (4.36) is isomorphic to $\mathbb{R}^d$, $d = m \cdot n$, since n initial values for m-vectors $\boldsymbol{f}^{(0)}, \dots, \boldsymbol{f}^{(n-1)}$ uniquely determine a solution. Let us fix a basis $\{\, \boldsymbol{R}_i \mid 1 \le i \le m \cdot n \,\}$ in the space $\ker L$ of solutions of (4.36). Then for any solution $\boldsymbol{f}$ of equation (4.36) we shall have

$$\boldsymbol{f} = \sum_{i=1}^{d} c_i \cdot \boldsymbol{R}_i, \tag{4.37}$$

where $c_i = c_i(\boldsymbol{f}) \in \mathbb{R}$ are constants. Dependence of c on $\boldsymbol{f}$ is explicitly given by the Wronskians:

$$c_i = \frac{W_i(\boldsymbol{f})}{W}. \tag{4.38}$$

Here $W = W(\boldsymbol{R}_1, \dots, \boldsymbol{R}_i, \dots, \boldsymbol{R}_d)$,

$$W = \det \begin{pmatrix} \boldsymbol{R}_1 & \dots & \boldsymbol{R}_i & \dots & \boldsymbol{R}_d \\ \boldsymbol{R}_1' & \dots & \boldsymbol{R}_i' & \dots & \boldsymbol{R}_d' \\ \dotfill \\ \boldsymbol{R}_1^{(n-1)} & \dots & \boldsymbol{R}_i^{(n-1)} & \dots & \boldsymbol{R}_d^{(n-1)} \end{pmatrix} \tag{4.39}$$

and $W_i(f) = W(\boldsymbol{R}_1, \dots, \boldsymbol{f}(x), \dots, \boldsymbol{R}_d)$,

$$W_i(\boldsymbol{f}) = \det \begin{pmatrix} \boldsymbol{R}_1 & \dots & \boldsymbol{f}(x) & \dots & \boldsymbol{R}_d \\ \boldsymbol{R}_1' & \dots & \boldsymbol{f}'(x) & \dots & \boldsymbol{R}_d' \\ \dotfill \\ \boldsymbol{R}_1^{(n-1)} & \dots & \boldsymbol{f}^{(n-1)}(x) & \dots & \boldsymbol{R}_d^{(n-1)} \end{pmatrix}. \tag{4.40}$$

The correspondences $\boldsymbol{f} \mapsto W_i(\boldsymbol{f})$ or $\boldsymbol{f} \mapsto c_i(\boldsymbol{f})$ are linear differential operators of order $n-1$. We shall understand them as functions on the jet space $J^{n-1}(\mathbb{R}, \mathbb{R}^m)$. Let us define

$$\widetilde{W}_i = \det \begin{pmatrix} \boldsymbol{R}_1 & \dots & \boldsymbol{p}_0 & \dots & \boldsymbol{R}_d \\ \boldsymbol{R}_1' & \dots & \boldsymbol{p}_1 & \dots & \boldsymbol{R}_d' \\ \dotfill \\ \boldsymbol{R}_1^{(n-1)} & \dots & \boldsymbol{p}_{n-1} & \dots & \boldsymbol{R}_d^{(n-1)} \end{pmatrix}. \tag{4.41}$$

Then $W_i(\boldsymbol{f}) = \widetilde{W}\Big|_{j_{n-1}(\boldsymbol{f})}$.

The defining equation for symmetries in this case is of the form

$$\left[D^n + \sum_{i=0}^{n-1} A_i(x) \cdot D^i \right] \mathcal{G}|_{\mathcal{E}^{(\infty)}} = 0, \tag{4.42}$$

where D is the restriction of the operator D_x to $\mathcal{E}^\infty$.

Any point symmetry is to be of the form

$$\beta(x, \boldsymbol{p}_0) + \xi(x, \boldsymbol{p}_0) \cdot \boldsymbol{p}_1 \tag{4.43}$$

and has to satisfy (4.42). Here x, ξ are scalars, $\boldsymbol{p}_0$, $\boldsymbol{p}_1$ and β are m-vectors.

A coordinate version of Proposition 3.4 is the following statement:

PROPOSITION 4.8. *The complete symmetry algebra* $\mathrm{sym}\,\mathcal{E}$ *is isomorphic to the Lie algebra of vector fields on the space* $\mathrm{Sol}(\mathcal{E})$ *of solutions of* (4.36). *This isomorphism is given by the formula*

$$\sum_{i=1}^d F_i(c_1, \ldots, c_d) \cdot \frac{\partial}{\partial c_i} \longmapsto \sum_{i=1}^d F_i\left(\frac{W_1}{W}, \ldots, \frac{W_d}{W}\right) \cdot \boldsymbol{R}_i.$$

The following fact specifying the structure of the Lie algebra $\mathrm{sym}\,\mathcal{E}$ also takes place.

THEOREM 4.9. *Let* $d = \dim \mathrm{Sol}(\mathcal{E})$. *Then the sets of functions*

$$\boldsymbol{p}_0, \quad \boldsymbol{R}_k, \quad \frac{\widetilde{W_l}}{W}\boldsymbol{R}_k, \quad \frac{\widetilde{W_l}}{W}\boldsymbol{p}_0, \qquad l, k = 1, \ldots, d$$

and

$$\frac{\widetilde{W_l}}{W}\boldsymbol{R}_k, \qquad i, k = 1, \ldots, d,$$

generate in $\mathrm{sym}\,\mathcal{E}$ *subalgebras isomorphic to* $\mathfrak{gl}(d+1, \mathbb{R})$ *and to* $\mathfrak{gl}(d, \mathbb{R})$ *respectively.*

The following result describes the subalgebra of point symmetries in $\mathrm{sym}\,\mathcal{E}^{15}$.

THEOREM 4.10. *Any point symmetry of equation* (4.36) *is of the form*

$$\mathcal{G} = \left(-\frac{n-1}{2} \cdot \xi'(x) + M\right)\boldsymbol{p}_0 + \xi(x) \cdot \boldsymbol{p}_1 + \boldsymbol{b}(x), \tag{4.44}$$

where ξ *is a scalar function,* $\boldsymbol{b}$ *is an arbitrary solution of* (4.36) *and* M *is a constant* $m \times m$ *matrix commuting with all matrices* $A_i(x)$ *(coefficients of* (4.36)).

COROLLARY 4.11. *Dimension* N *of the algebra of point symmetries of the equation* (4.35) *satisfies the inequality*

$$m \cdot n + 1 \le N \le (m+n) \cdot m + 3.$$

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[15]Detailed computation can be found in [**99**].

CHAPTER 5

Conservation Laws

This chapter is concerned with the theory of conservation laws of differential equations. We begin with the analysis of the concept itself. It turns out that the most natural and efficient way to study the space of conservation laws is to identify it with a term of the so-called $\mathcal{C}$-*spectral sequence* [**125, 127, 132**]. This spectral sequence arises from the filtration in the de Rham complex on $\mathcal{E}^\infty$ by powers of the ideal of Cartan forms. The theory of the $\mathcal{C}$-spectral sequence, set forth in §2, makes it possible to represent any conservation law by a *generating function*. The space of generating functions is essentially the kernel of a differential operator and so is accessible for computing. In §3, we illustrate the generating function method for computing conservation laws by several examples. In §4, we discuss the connection between symmetries and conservation laws (the Noether theorem) and Hamiltonian formalism on infinite jet spaces.

For the convenience of the reader with primary interests in results and examples we repeat in §§3 and 4 the relevant material from §2 without proofs, so that he could jump ahead to §3 immediately after §1.

1. Introduction: What are conservation laws?

Let us take the equation of continuity of fluid dynamics as a prototypical conservation law. If $\rho(x, t)$ is the density of the fluid at the point $x = (x_1, x_2, x_3)$ at the time t and $\boldsymbol{v}(x, t) = (v^1(x, t), v^2(x, t), v^3(x, t))$ is the vector of fluid velocity, then this equation reads

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^1)}{\partial x_1} + \frac{\partial(\rho v^2)}{\partial x_2} + \frac{\partial(\rho v^3)}{\partial x_3} = 0. \tag{1.1}$$

Using the Gauss theorem, we can rewrite the equation of continuity in the integral form

$$-\frac{d}{dt} \int_V \rho(x, t)\, dx = \int_{\partial V} (\rho \boldsymbol{v} \cdot \boldsymbol{n})\, d\sigma, \tag{1.2}$$

where $V \subset \mathbb{R}^3$ is a space domain, ∂V is its boundary, $\boldsymbol{n}$ is a unit external vector normal to ∂V, and $d\sigma$ is a surface area element. The surface integral equals the flow of the fluid from the region V, thus equation (1.2) states that the time rate of fluid mass decrease within the volume V is equal to the net outflow of the fluid through the boundary ∂V. In particular, when the normal velocity component $\boldsymbol{v} \cdot \boldsymbol{n}$ vanishes, the mass is conserved:

$$\int_V \rho(x, t)\, dx = \text{const.}$$

All usually considered conservation laws can be described in the same manner. Let S be the density of some conserved quantity, e.g., the energy density, a

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component of the linear or angular momentum density, etc. In the example just mentioned, $S = \rho$. Then there exists a flux $\bar{S} = (S_1, \ldots, S_{n-1})$ of S, where $(n-1)$ is the number of spatial variables. In our example, $\bar{S} = \rho\boldsymbol{v}$. The equation describing conservation of S and generalizing (1.1) has the form

$$\frac{\partial S}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial S_i}{\partial x_i} = 0, \tag{1.3}$$

or, in the integral form,

$$-\frac{d}{dt} \int_V S\, dx = \int_{\partial V} \bar{S} \cdot \boldsymbol{n}\, d\sigma.$$

A *conserved current* is an n-dimensional vector function $(S_1, \ldots, S_{n-1}, S)$ satisfying (1.3).

Consider now equation (1.3) in terms of the theory of differential equations. Suppose that we study a physical system described by an equation $\mathcal{E} = \{F = 0\} \subset J^k(\pi)$. Then S and S_i can be thought of as functions on $\mathcal{E}^\infty$, and equation (1.3) takes the form

$$\sum_{i=1}^{n} \bar{D}_i(S_i) = 0, \tag{1.4}$$

where $S_n = S$, $\bar{D}_i$ is the restriction of the total derivative D_i to $\mathcal{E}^\infty$, n is the number of independent variables. This observation makes it possible to define a *conserved current for the equation* $\mathcal{E}$ as a vector function $(S_1, \ldots, S_n)$, $S_i \in \mathcal{F}(\mathcal{E})$, satisfying equation (1.4) on $\mathcal{E}^\infty$.

In the case when one of independent variables is chosen to be the time $t = x_n$ and the other variables $(x_1, \ldots, x_{n-1})$ are considered as spatial variables, the component S_n is said to be a conserved *density* (or *charge density*), while the vector function $(S_1, \ldots, S_{n-1})$, as it was mentioned already, is called the *flux* of S.

There is one very simple method of constructing conserved currents. Take some set of functions $\mathcal{L}_{ij} \in \mathcal{F}(\pi)$ and put

$$S_i = \sum_{j<i} D_j(\mathcal{L}_{ji}) - \sum_{i<j} D_j(\mathcal{L}_{ij}).$$

It is clear that such conserved currents, called *trivial* (or *identically conserved*), are in no way related to the equation under consideration[1]. To get rid of them, let us say that two conserved currents are *equivalent*, if they differ by a trivial current, and define *conservation laws* for an equation $\mathcal{E}$ as equivalence classes of conserved currents of this equation. (The term "integral of motion" is also used in the same sense.)

EXERCISE 1.1. Let $\mathcal{E}$ be a system of ordinary differential equations. Check that the notion of a conservation law is identical to the notion of a first integral.

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[1]This does not necessarily imply that trivial currents are not interesting from the physical point of view. E.g., for gauge theories the conserved currents associated by the Noether theorem with gauge symmetries are trivial and, thus, do not relate to the field equations. Nevertheless, they carry some information on the gauge group and, therefore, are physically relevant (see, e.g., [**90, 56, 11, 10**]).

REMARK 1.1. Unlike ordinary equations, nonlinear partial differential equations do not have (even in a small neighborhood) a complete set of conservation laws[2]. The question whether there is a remedy, remains an interesting and intriguing problem. Here we just mention a conjecture proposed in [**134, 136**]: each regular equation has a complete set of *nonlocal* conservation laws, at least in a sufficiently small neighborhood (see §1.8 of Ch. 6 for the definition of nonlocal conservation laws).

The definition of conservation laws (as it was formulated) is unusable for finding conservation laws for a given differential equation. To this end, we need the means to describe entire classes of conserved currents. One solves this problem by reformulating the definition of conservation laws in terms of *horizontal de Rham cohomology* (i.e., the cohomology of the horizontal de Rham complex, see §1.6 of Ch. 4), which permits the use of homological algebra. In §2, we discuss this cohomological theory. (The reader with the prime interest in practical aspects of finding conservation laws may skip this section.)

2. The $\mathcal{C}$-spectral sequence

The $\mathcal{C}$-spectral sequence, introduced in [**125, 127**], is of the fundamental importance in the theory of conservation laws. For a more detailed discussion of this and related subjects see, e.g., [**132, 138, 139, 126, 8, 117, 118, 119, 32, 120, 82, 16, 4, 5, 148, 21, 24, 81, 64, 91**].

2.1. The definition of the $\mathcal{C}$-spectral sequence. Let $\mathcal{E}^\infty \subset J^\infty(\pi)$ be an infinitely prolonged equation and $\Lambda^*(\mathcal{E}) = \sum_{i \geq 0} \Lambda^i(\mathcal{E})$ be the exterior algebra of differential forms on $\mathcal{E}^\infty$. Consider the ideal $\mathcal{C}\Lambda^*(\mathcal{E}) = \sum_{i \geq 0} \mathcal{C}\Lambda^i(\mathcal{E})$ of the algebra $\Lambda^*(\mathcal{E})$ consisting of the Cartan forms (i.e., forms vanishing on the Cartan distribution: $\omega \in \mathcal{C}\Lambda^i(\mathcal{E})$ if and only if $\omega(X_1, \ldots, X_i) = 0$ for all $X_1, \ldots, X_i \in \mathcal{C}D(\mathcal{E})$; see §2.1 of Ch. 4). Denote by $\mathcal{C}^k\Lambda^*(\mathcal{E})$ the k-th power of the ideal $\mathcal{C}\Lambda^*(\mathcal{E})$, i.e., the submodule of $\Lambda^*(\mathcal{E})$ generated by forms $\omega_1 \wedge \cdots \wedge \omega_k$, where $\omega_i \in \mathcal{C}\Lambda^*(\mathcal{E})$. It is obvious that the ideal $\mathcal{C}\Lambda^*(\mathcal{E})$ is stable with respect to the operator d, and, hence, all ideals $\mathcal{C}^k\Lambda^*(\mathcal{E})$ have the same property:

$$d(\mathcal{C}^k\Lambda^*(\mathcal{E})) \subset \mathcal{C}^k\Lambda^*(\mathcal{E}).$$

Thus, we get the filtration

$$\Lambda^*(\mathcal{E}) \supset \mathcal{C}\Lambda^*(\mathcal{E}) \supset \mathcal{C}^2\Lambda^*(\mathcal{E}) \supset \cdots \supset \mathcal{C}^k\Lambda^*(\mathcal{E}) \supset \cdots, \tag{2.1}$$

in the de Rham complex on $\mathcal{E}^\infty$. The spectral sequence[3] $(E_r^{p,q}(\mathcal{E}), d_r^{p,q})$ determined by this filtration is called the $\mathcal{C}$-*spectral sequence* for the equation $\mathcal{E}$.

The filtration involved is finite in each degree, i.e.,

$$\Lambda^k(\mathcal{E}) \supset \mathcal{C}^1\Lambda^k(\mathcal{E}) \supset \mathcal{C}^2\Lambda^k(\mathcal{E}) \supset \cdots \supset \mathcal{C}^k\Lambda^k(\mathcal{E}) \supset \mathcal{C}^{k+1}\Lambda^k(\mathcal{E}) = 0.$$

Therefore the $\mathcal{C}$-spectral sequence converges to the de Rham cohomology $H^*(\mathcal{E}^\infty)$ of the infinite prolonged equation $\mathcal{E}^\infty$.

As usual, $p + q$ denotes the degree and p is the filtration degree.

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[2]A set of conservation laws is called *complete*, if the values of the corresponding conserved quantities at a solution of the equation in question completely determine this solution.

[3]For the theory of spectral sequences consult, e.g., [**79, 84, 31, 35**].
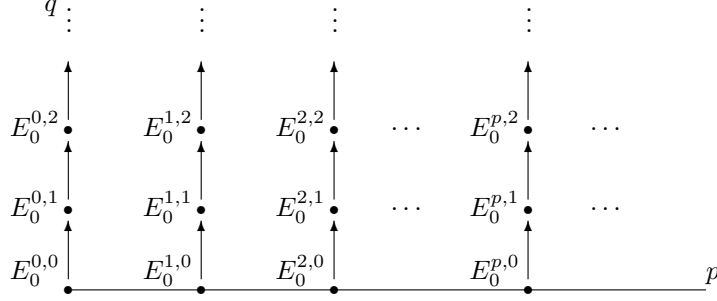
FIGURE 5.1

2.2. The term E_0. Consider the term $E_0(\mathcal{E}) = \sum_{p,q} E_0^{p,q}(\mathcal{E})$ of the $\mathcal{C}$-spectral sequence.

By definition we have

$$E_0^{p,q}(\mathcal{E}) = \mathcal{C}^p \Lambda^{p+q}(\mathcal{E}) \big/ \mathcal{C}^{p+1} \Lambda^{p+q}(\mathcal{E}).$$

The differential $d_0^{p,q} \colon E_0^{p,q}(\mathcal{E}) \to E_0^{p,q+1}(\mathcal{E})$ is induced by the exterior differential d. As usual when dealing with spectral sequences, we depict the spaces $E_r^{p,q}$ as integer points on the plane with coordinates (p, q). Then the differential d_0 is represented by arrows pointing upwards (see Figure 5.1).

Thus, $E_0(\mathcal{E})$ is the direct sum of complexes

$$0 \longrightarrow E_0^{p,0}(\mathcal{E}) \longrightarrow E_0^{p,1}(\mathcal{E}) \longrightarrow \cdots \longrightarrow E_0^{p,q}(\mathcal{E}) \longrightarrow E_0^{p,q+1}(\mathcal{E}) \longrightarrow \cdots,$$

which corresponds to columns. The zero column of the term E_0 is the *horizontal de Rham complex* (see §1.6 of Ch. 4):

$$E_0^{0,q}(\mathcal{E}) = \Lambda^q(\mathcal{E}) \big/ \mathcal{C}\Lambda^q(\mathcal{E}) = \Lambda_0^q(\mathcal{E}), \qquad d_0^{0,q} = \widehat{d}.$$

The cohomology $\bar{H}^q(\mathcal{E})$ of the horizontal de Rham complex is intimately related to the conservation laws of the equation $\mathcal{E}$. Indeed, let us assign to each conserved current $S = (S_1, \ldots, S_n)$ the horizontal $(n-1)$-form

$$\omega_S = \sum_{i=1}^{n} (-1)^{i-1} S_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n.$$

Then condition (1.4) means that $\widehat{d}\omega_S = 0$, with the current S being trivial if and only if the form ω_S is exact: $\omega_S = \widehat{d}\omega'$.

DEFINITION 2.1. *A conservation law for the equation $\mathcal{E}$ is an $(n-1)$-cohomology class of the horizontal de Rham complex on $\mathcal{E}^\infty$.*

Thus, $\bar{H}^{n-1}(\mathcal{E})$ is the group of conservation laws for the equation $\mathcal{E}$. Sometimes elements of the groups $\bar{H}^q(\mathcal{E})$ for $q < n-1$ are also said to be conservation laws, although for the majority of equations of mathematical physics they are of pure topological nature[4]. The group $\bar{H}^n(\mathcal{E})$ should be interpreted as the set of

---

[4]One encounters physically relevant "conservation laws" belonging to $\bar{H}^q(\mathcal{E})$, $q < n-1$, in gauge theories (see, e.g., [**11, 10, 42**]).

Lagrangians (see §§2.5 and 4.1) in variational problems constrained by the equation $\mathcal{E}$.[5]

To complete the description of the term $E_0(\mathcal{E})$, note that from the obvious decomposition $\Lambda^1(\mathcal{E}) = \mathcal{C}\Lambda^1(\mathcal{E}) \oplus \Lambda^1_0(\mathcal{E})$ it follows that $E_0^{p,q}(\mathcal{E}) = \mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes \Lambda^q_0(\mathcal{E})$, with the differential $d_0^{p,q}$ being the composition

$$\mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes \Lambda^q_0(\mathcal{E}) \xrightarrow{\ d\ } \mathcal{C}^{p+1}\Lambda^{p+1}(\mathcal{E}) \otimes \Lambda^q_0(\mathcal{E}) \oplus \mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes \Lambda^{q+1}_0(\mathcal{E})$$

$$\xrightarrow{\ \alpha\ } \mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes \Lambda^{q+1}_0(\mathcal{E}),$$

where α is the projection onto the second summand. In particular, all nontrivial terms $E_0^{p,q}(\mathcal{E})$ are situated in the domain $0 \le q \le n$.

EXERCISE 2.1. Using the direct decomposition $d = \widehat{d} + U_1$ and the properties of the operator U_1 (see §2.1 of Ch. 4), show that the following inclusion takes place:

$$d(\mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes \Lambda^q_0(\mathcal{E})) \subset \left(\mathcal{C}^{p+1}\Lambda^{p+1}(\mathcal{E}) \otimes \Lambda^q_0(\mathcal{E})\right) \oplus \left(\mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes \Lambda^{q+1}_0(\mathcal{E})\right).$$

2.3. The term E_1: preparatory results. Now we prove some facts needed to describe the term E_1 of the $\mathcal{C}$-spectral sequence.

Let us begin with the following definition. An $\mathcal{F}(\mathcal{E})$-module P is called *horizontal*, if it is isomorphic to the module of sections $\mathcal{F}(\mathcal{E}, \xi)$ for a finite-dimensional vector bundle $\xi \colon N_\xi \to M$. For instance, as it was shown in §1.6 of Ch. 4, $\Lambda^q_0(\mathcal{E})$ is a horizontal module.

EXERCISE 2.2. Show that $P = \Gamma(\xi) \otimes_{C^\infty(M)} \mathcal{F}(\mathcal{E})$.

Note also that a horizontal module P is filtered by its submodules $P_k = \mathcal{F}_k(\mathcal{E}, \xi) = \Gamma(\xi) \otimes_{C^\infty(M)} \mathcal{F}_k(\mathcal{E})$.

Consider now two vector bundles $\xi \colon N_\xi \to M$ and $\eta \colon N_\eta \to M$ and introduce the notation $P = \mathcal{F}(\mathcal{E}, \xi)$ and $Q = \mathcal{F}(\mathcal{E}, \eta)$. A differential operator $\Delta \colon P \to Q$ is called $\mathcal{C}$-*differential* (or *horizontal*), if it is of the form $\Delta = f_1\widehat{\square}_1 + f_2\widehat{\square}_2 + \cdots + f_r\widehat{\square}_r$, where $\widehat{\square}_i$ are the liftings (see Remark 1.1 of Ch. 4) of some differential operators $\square_i \colon \Gamma(\xi) \to \Gamma(\eta)$, $f_i \in \mathcal{F}(\mathcal{E})$, $i = 1, \ldots, r$. In particular, operators of the form ℓ_φ (universal linearizations) are $\mathcal{C}$-differential. We shall denote the $\mathcal{F}(\mathcal{E})$-module of all $\mathcal{C}$-differential operators acting from P to Q by $\mathcal{C}\mathrm{Diff}(P, Q)$. For its submodules consisting of operators of order $\le k$, we shall use the notation $\mathcal{C}\mathrm{Diff}_k(P, Q)$.

EXERCISE 2.3. Check that in local coordinates scalar $\mathcal{C}$-differential operators of order $\le k$ are written as follows

$$\Delta = \sum_{|\sigma|=0}^{k} a_\sigma \bar{D}_\sigma,$$

where $\bar{D}_\sigma = \bar{D}_1^{\sigma_1} \circ \cdots \circ \bar{D}_n^{\sigma_n}$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, $a_\sigma \in \mathcal{F}(\mathcal{E})$.

EXERCISE 2.4. Show that $\mathcal{C}\mathrm{Diff}(P, Q)$ is a horizontal module.

Let $P_1, P_2, \ldots, P_l, Q$ be horizontal modules. Set

$$\mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_l; Q) = \mathcal{C}\mathrm{Diff}(P_1, \mathcal{C}\mathrm{Diff}(P_2, \ldots, \mathcal{C}\mathrm{Diff}(P_l, Q) \ldots))$$

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[5]There is a number of nonequivalent formulations of the Lagrangian formalism with constrains. We consider here the case when the sections subject to variation satisfy the "constraint equation" $\mathcal{E}$. This implies, in particular, that the Euler–Lagrange equations are determined by the restriction of the Lagrangian to $\mathcal{E}^\infty$.

and consider the complex

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_l; \mathcal{F}(\mathcal{E})) \xrightarrow{w} \mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_l; \Lambda_0^1(\mathcal{E})) \xrightarrow{w} \cdots$$
$$\xrightarrow{w} \mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_l; \Lambda_0^n(\mathcal{E})) \longrightarrow 0, \quad (2.2)$$

where $w(\nabla) = \widehat{d} \circ \nabla$, $\nabla \in \mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_l; \Lambda_0^q(\mathcal{E}))$.

THEOREM 2.1. *The cohomology of complex (2.2) is equal to zero at the terms* $\mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_l; \Lambda_0^q(\mathcal{E}))$ *for $q < n$ while at the term* $\mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_l; \Lambda_0^n(\mathcal{E}))$ *its cohomology is* $\mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_{l-1}; \widehat{P}_l)$, *where* $\widehat{P}_l = \mathrm{Hom}_{\mathcal{F}(\mathcal{E})}(P_l, \Lambda_0^n(\mathcal{E}))$.

PROOF. Let $\mathcal{C}\mathrm{Diff}_k(P_1, P_2, \ldots, P_l; Q) \subset \mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_l; Q)$ be the module consisting of operators ∇ such that the operator $\nabla(p_1, p_2, \ldots, p_{l-1}) \colon P_l \to Q$ is of order $\leq k$ for all $p_1 \in P_1$, $p_2 \in P_2$, $\ldots$, $p_{l-1} \in P_{l-1}$ (here and subsequently, $\nabla(p_1, p_2, \ldots, p_{l-1})$ stands for $\nabla(p_1)(p_2)\ldots(p_{l-1})$). Thus,

$$\mathcal{C}\mathrm{Diff}_k(P_1, \ldots, P_l; Q) = \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_{l-1}; \mathcal{C}\mathrm{Diff}_k(P_l, Q))$$

and the embeddings $\mathcal{C}\mathrm{Diff}_k(P_l, Q) \subset \mathcal{C}\mathrm{Diff}_{k+1}(P_l, Q)$ induce the homomorphisms of complexes ($\bullet$ denotes "$P_1, \ldots, P_l$")

$$\cdots \longrightarrow \mathcal{C}\mathrm{Diff}_{k-n+i}(\bullet; \Lambda_0^i(\mathcal{E})) \xrightarrow{w} \mathcal{C}\mathrm{Diff}_{k-n+i+1}(\bullet; \Lambda_0^{i+1}(\mathcal{E})) \longrightarrow \cdots$$
$$\Big\uparrow \qquad\qquad\qquad\qquad\qquad \Big\uparrow \qquad\qquad\qquad (2.3)$$
$$\cdots \longrightarrow \mathcal{C}\mathrm{Diff}_{k-n+i-1}(\bullet; \Lambda_0^i(\mathcal{E})) \xrightarrow{w} \mathcal{C}\mathrm{Diff}_{k-n+i}(\bullet; \Lambda_0^{i+1}(\mathcal{E})) \longrightarrow \cdots$$

Hence, there exists the quotient complex

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(\bullet; S_{k-n}(P_l)) \xrightarrow{\delta} \mathcal{C}\mathrm{Diff}(\bullet; S_{k-n+1}(P_l) \otimes \Lambda_0^1(\mathcal{E})) \xrightarrow{\delta} \cdots$$
$$\xrightarrow{\delta} \mathcal{C}\mathrm{Diff}(\bullet; S_k(P_l) \otimes \Lambda_0^n(\mathcal{E})) \longrightarrow 0, \quad (2.4)$$

where $\bullet$ stands for "$P_1, \ldots, P_{l-1}$" and

$$S_r(P) = \mathcal{C}\mathrm{Diff}_r(P, \mathcal{F}(\mathcal{E}))\big/ \mathcal{C}\mathrm{Diff}_{r-1}(P, \mathcal{F}(\mathcal{E})).$$

The proof will be accomplished if we prove that complex (2.4) is exact for $k > 0$. Indeed, the cohomologies of complexes (2.3) for $k > 0$ coincide in this case and, hence, the cohomology of (2.2) equals to the cohomology of the complex

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}_0(P_1, \ldots, P_l; \Lambda_0^n(\mathcal{E})) \longrightarrow 0.$$

The differentials δ of complex (2.4) are $\mathcal{F}(\mathcal{E})$-linear. Therefore, it suffices to check its exactness at every point $\theta \in \mathcal{E}$. Since the functor $\mathcal{C}\mathrm{Diff}(P, \cdot)$ is exact, we can take $l = 1$. Thus, we are to prove that the complexes

$$0 \longrightarrow S_{k-n}(P)_\theta \longrightarrow S_{k-n+1}(P)_\theta \otimes_{\mathbb{R}} \Lambda_0^1(\mathcal{E})_\theta \longrightarrow \cdots \longrightarrow S_k(P)_\theta \otimes_{\mathbb{R}} \Lambda_0^n(\mathcal{E})_\theta \longrightarrow 0$$

are acyclic for $k > 0$.

Let $\{e_\xi\}$ be a basis of $(P_\theta)^*$. Pick the elements $e_\xi \otimes D_\sigma|_\theta$, $|\sigma| = i$, as a basis of the space $S_i(P)_\theta$ and the elements $dx_{i_1} \wedge \cdots \wedge dx_{i_q}\big|_\theta$, $1 \leq i_1 < \cdots < i_q \leq n$, as a basis of $\Lambda_0^q(\mathcal{E})_\theta$. Then the differential δ takes the form

$$\delta(e_\xi \otimes D_\sigma \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_q}\big|_\theta) = \sum_{i=1}^n e_\xi \otimes D_{\sigma+1_i} \otimes dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}\big|_\theta.$$

Consequently, complex (2.4) is the Koszul complex of the polynomial algebra (see, e.g., [**15**]) and, consequently, is exact for $k > 0$. $\square$

EXERCISE 2.5. Prove that the embedding

$$i_l \colon \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_{l-1}; \widehat{P}_l) \longrightarrow \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_l; \Lambda_0^n(\mathcal{E}))$$

induced by the embedding $\widehat{P}_l = \mathrm{Hom}_{\mathcal{F}(\mathcal{E})}(P_l, \Lambda_0^n(\mathcal{E})) \to \mathcal{C}\mathrm{Diff}(P_l, \Lambda_0^n(\mathcal{E}))$ splits the natural projection

$$\mu_l \colon \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_l; \Lambda_0^n(\mathcal{E})) \longrightarrow \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_l; \Lambda_0^n(\mathcal{E}))\big/ \mathrm{im}\, w$$
$$= \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_{l-1}; \widehat{P}_l).$$

Thus the module $\mathcal{C}\mathrm{Diff}(P_1, \ldots, P_{l-1}; \widehat{P}_l) = \mathrm{im}\, i_l$ is a direct summand in the module $\mathcal{C}\mathrm{Diff}(P_1, \ldots, P_l; \Lambda_0^n(\mathcal{E}))$.

Take a $\mathcal{C}$-differential operator $\Delta \colon P \to Q$. It induces the cochain map of complexes (2.2)

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(P, \mathcal{F}(\mathcal{E})) \xrightarrow{w} \mathcal{C}\mathrm{Diff}(P, \Lambda_0^1(\mathcal{E})) \xrightarrow{w} \cdots \xrightarrow{w} \mathcal{C}\mathrm{Diff}(P, \Lambda_0^n(\mathcal{E})) \longrightarrow 0$$
$$\Delta' \Big\uparrow \qquad\qquad \Delta' \Big\uparrow \qquad\qquad\qquad \Delta' \Big\uparrow$$
$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(Q, \mathcal{F}(\mathcal{E})) \xrightarrow{w} \mathcal{C}\mathrm{Diff}(Q, \Lambda_0^1(\mathcal{E})) \xrightarrow{w} \cdots \xrightarrow{w} \mathcal{C}\mathrm{Diff}(Q, \Lambda_0^n(\mathcal{E})) \longrightarrow 0$$

where $\Delta'(\nabla) = \nabla \circ \Delta$ for any $\nabla \in \mathcal{C}\mathrm{Diff}(Q, \Lambda_0^q(\mathcal{E}))$. Theorem 2.1 implies that the map Δ' gives rise to the cohomology map $\Delta^* \colon \widehat{Q} \to \widehat{P}$. The operator Δ^* is called *adjoint* to the operator Δ.

EXERCISE 2.6. Show that:

1. If $\Delta = \sum_\sigma a_\sigma D_\sigma$ is a scalar $\mathcal{C}$-differential operator, then

$$\Delta^* = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_\sigma.$$

2. If $\Delta = \|\Delta_{ij}\|$ is a matrix $\mathcal{C}$-differential operator, then $\Delta^* = \|\Delta_{ji}^*\|$.

In particular, this exercise implies that Δ^* is a $\mathcal{C}$-differential operator of the same order as Δ.

EXERCISE 2.7. Prove that for all $\Delta \in \mathcal{C}\mathrm{Diff}(P, Q)$, $p \in P$, and $\widehat{q} \in \widehat{Q}$ there exists a form $\omega_{p,\widehat{q}}(\Delta) \in \Lambda_0^{n-1}(\mathcal{E})$ such that

$$\widehat{q}(\Delta(p)) - (\Delta^*(\widehat{q}))(p) = \widehat{d}\omega_{p,\widehat{q}}(\Delta). \tag{2.5}$$

EXERCISE 2.8. Check that for any two $\mathcal{C}$-differential operators $\Delta_1 \colon P \to Q$ and $\Delta_2 \colon Q \to R$ one has $(\Delta_2 \circ \Delta_1)^* = \Delta_1^* \circ \Delta_2^*$.

EXERCISE 2.9. Consider $X \in \mathcal{C}\mathrm{Diff}_1(\mathcal{F}(\mathcal{E}), \Lambda_0^n(\mathcal{E}))$. Show that $X + X^* \in \Lambda_0^n(\mathcal{E})$, i.e., $X + X^*$ is an operator of zero order.

EXERCISE 2.10. Show that the natural projection

$$\mu_l \colon \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_l; \Lambda_0^n(\mathcal{E})) \longrightarrow \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_l; \Lambda_0^n(\mathcal{E}))\big/ \mathrm{im}\, w$$
$$= \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_{l-1}; \widehat{P}_l).$$

is given by the formula

$$\mu_l(\nabla)(p_1, \ldots, p_{l-1}) = (\nabla(p_1, \ldots, p_{l-1}))^*(1).$$

Let $\mathcal{C}\mathrm{Diff}_{(l)}(P;Q)$ denote $\mathcal{C}\mathrm{Diff}(\underbrace{P,\ldots,P}_{l\ \mathrm{times}};Q)$ and $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(l)}(P;Q)$ denote the submodule of $\mathcal{C}\mathrm{Diff}_{(l)}(P;Q)$ consisting of skew-symmetric operators, i.e., of operators $\nabla \in \mathcal{C}\mathrm{Diff}_{(l)}(P;Q)$ such that

$$\nabla(p_1,\ldots,p_i,p_{i+1},\ldots,p_l) = -\nabla(p_1,\ldots,p_{i+1},p_i,\ldots,p_l)$$

for all $p_1,\ldots,p_l \in P$, $i = 1,\ldots,l-1$.

Consider complex (2.2) for $P_1 = P_2 = \cdots = P_l = P$:

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}_{(l)}(P;\mathcal{F}(\mathcal{E})) \overset{w}{\longrightarrow} \mathcal{C}\mathrm{Diff}_{(l)}(P;\Lambda_0^1(\mathcal{E})) \overset{w}{\longrightarrow} \cdots$$

$$\overset{w}{\longrightarrow} \mathcal{C}\mathrm{Diff}_{(l)}(P;\Lambda_0^n(\mathcal{E})) \longrightarrow 0. \quad (2.6)$$

The permutation group S_l acts in this complex as follows:

$$\tau(\nabla)(p_1,\ldots,p_l) = \nabla(p_{\tau(1)},\ldots,p_{\tau(l)}), \qquad \tau \in S_l.$$

It is obvious that this action commutes with the differential w. Hence from Theorem 2.1 it follows that the skew-symmetric part of complex (2.6), i.e., the subcomplex

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(l)}(P;\mathcal{F}(\mathcal{E}))) \overset{w}{\longrightarrow} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(l)}(P;\Lambda_0^1(\mathcal{E})) \overset{w}{\longrightarrow} \cdots$$

$$\overset{w}{\longrightarrow} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(l)}(P;\Lambda_0^n(\mathcal{E})) \longrightarrow 0, \quad (2.7)$$

is exact in all degrees different from n while its n-th cohomology group is isomorphic to a submodule of $\mathcal{C}\mathrm{Diff}_{(l-1)}(P;\widehat{P})$, which shall be denoted by $K_l(P)$.

An explicit description of the module $K_l(P)$ can be obtained in the following manner. First, observe that the embedding (see Exercise 2.5)

$$i_l \colon \mathcal{C}\mathrm{Diff}(P_1,\ldots,P_{l-1};\widehat{P}_l) \longrightarrow \mathcal{C}\mathrm{Diff}(P_1,\ldots,P_l;\Lambda_0^n(\mathcal{E}))$$

commutes with the action of the subgroup $S_{l-1} \subset S_l$ which preserves the l-th index. For this reason

$$K_l(P) \subset \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(l-1)}(P;\widehat{P}).$$

Consider the transposition $\tau \in S_l$ interchanging the j-th and the l-th indices, $j < l$, and let us describe its action on an operator $\Delta \in \mathcal{C}\mathrm{Diff}_{(l-1)}(P,\widehat{P})$. Let us fix the elements $p_1,\ldots,p_{j-1},p_{j+1},\ldots,p_{l-1} \in P$ and consider the operator

$$\square(p) = \Delta(p_1,\ldots,p_{j-1},p,p_{j+1},\ldots,p_{l-1}).$$

From (2.5) it follows that

$$\square(p)(p') - \square^*(p')(p) \in \mathrm{im}\,\widehat{d}$$

for all $p,p' \in P$. Thus the operator $\tau(\Delta)$ has the form $\tau(\Delta)(p_1,\ldots,p_{l-1}) = \square^*(p_j)$. This proves the following

THEOREM 2.2. *Complex (2.7) is acyclic in the terms* $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(l)}(P;\Lambda_0^q(\mathcal{E}))$ *for* $q < n$. *The cohomology group in the term* $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(l)}(P;\Lambda_0^n(\mathcal{E}))$ *is isomorphic to the module* $K_l(P) \subset \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(l-1)}(P;\widehat{P})$ *consisting of all operators* ∇ *satisfying the condition*

$$(\nabla(p_1,\ldots,p_{l-2}))^* = -\nabla(p_1,\ldots,p_{l-2})$$

for all $p_1,\ldots,p_{l-2} \in P$.

2.4. Generalizations. We shall need to generalize Theorems 2.1 and 2.2. To do this, we use the following result:

EXERCISE 2.11. To each $\mathcal{C}$-differential operator $\Delta\colon P_1 \to P_2$ assign the family of operators $\Delta(p_1, p_2^*) \in \mathcal{C}\mathrm{Diff}(\mathcal{F}(\mathcal{E}), \mathcal{F}(\mathcal{E}))$

$$\Delta(p_1, p_2^*)(f) = p_2^*(\Delta(fp_1)), \qquad p_1 \in P_1, \quad p_2^* \in \mathrm{Hom}_{\mathcal{F}(\mathcal{E})}(P_2, \mathcal{F}(\mathcal{E})).$$

Prove that the family $\Delta(p_1, p_2^*)$ uniquely determines the operator Δ and that for any family $\Delta[p_1, p_2^*] \in \mathcal{C}\mathrm{Diff}(\mathcal{F}(\mathcal{E}), \mathcal{F}(\mathcal{E}))$, $p_1 \in P_1$, $p_2^* \in \mathrm{Hom}_{\mathcal{F}(\mathcal{E})}(P_2, \mathcal{F}(\mathcal{E}))$ satisfying

$$\Delta\big[p_1, \textstyle\sum_i f_i p_2^{*i}\big] = \textstyle\sum_i f_i \Delta\big[p_1, p_2^{*i}\big], \qquad \Delta\big[\textstyle\sum_i f_i p_1^i, p_2^*\big] = \textstyle\sum_i \Delta\big[p_1^i, p_2^*\big] \circ f_i,$$

there exists an operator $\Delta \in \mathcal{C}\mathrm{Diff}(P_1, P_2)$ such that $\Delta\big[p_1, p_2^*\big] = \Delta(p_1, p_2^*)$.

Let Q be a left module over the ring $\mathcal{C}\mathrm{Diff}(\mathcal{F}(\mathcal{E}), \mathcal{F}(\mathcal{E}))$. Since

$$\mathcal{F}(\mathcal{E}) = \mathcal{C}\mathrm{Diff}_0(\mathcal{F}(\mathcal{E}), \mathcal{F}(\mathcal{E})) \subset \mathcal{C}\mathrm{Diff}(\mathcal{F}(\mathcal{E}), \mathcal{F}(\mathcal{E})),$$

the module Q is an $\mathcal{F}(\mathcal{E})$-module as well. Suppose that it is a horizontal $\mathcal{F}(\mathcal{E})$-module. By Exercise 2.11, for any operator $\Delta \in \mathcal{C}\mathrm{Diff}(P_1, P_2)$ there exists a unique operator $\Delta_Q \in \mathcal{C}\mathrm{Diff}(P_1 \otimes Q, P_2 \otimes Q)$ such that

$$\Delta_Q(p_1 \otimes q, p_2^* \otimes q^*) = q^*(\Delta(p_1, p_2^*)q).$$

Note that operators Δ and Δ_Q are of the same order.

It is easy to see that $(\Delta_1 \circ \Delta_2)_Q = (\Delta_1)_Q \circ (\Delta_2)_Q$, thus any complex of $\mathcal{C}$-differential operators

$$\cdots \longrightarrow P_k \xrightarrow{\Delta_k} P_{k+1} \longrightarrow \cdots$$

can be multiplied by Q:

$$\cdots \longrightarrow P_k \otimes Q \xrightarrow{(\Delta_k)_Q} P_{k+1} \otimes Q \longrightarrow \cdots .$$

In particular, multiplying complex (2.2) by Q, we get

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_l; \mathcal{F}(\mathcal{E})) \otimes Q \longrightarrow \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_l; \Lambda_0^1(\mathcal{E})) \otimes Q \longrightarrow \cdots$$
$$\longrightarrow \mathcal{C}\mathrm{Diff}(P_1, \ldots, P_l; \Lambda_0^n(\mathcal{E})) \otimes Q \longrightarrow 0. \quad (2.8)$$

Literally repeating the proof of Theorem 2.1, we obtain the following

THEOREM 2.3. *The cohomology of complex* (2.8) *in the term*

$$\mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, P_l; \Lambda_0^i(\mathcal{E})) \otimes Q$$

has the form $H^i = \begin{cases} 0 & \text{for } i < n, \\ \mathcal{C}\mathrm{Diff}(P_1, P_2, \ldots, \widehat{P_l}) \otimes Q & \text{for } i = n. \end{cases}$

Theorem 2.2 can be generalized in the same way.

EXERCISE 2.12. Show that the set $\mathcal{C}^p\Lambda^p(\mathcal{E})$ can be supplied with a unique left $\mathcal{C}\mathrm{Diff}(\mathcal{F}(\mathcal{E}), \mathcal{F}(\mathcal{E}))$-module structure such that

$$f \cdot \omega = f\omega, \qquad f \in \mathcal{F}(\mathcal{E}),$$
$$X \cdot \omega = L_X(\omega), \qquad X \in \mathcal{C}\mathrm{D}(\mathcal{E}).$$

2.5. The term E_1 for $J^\infty(\pi)$. Consider now the term E_1 of the $\mathcal{C}$-spectral sequence for the "empty equation", i.e., for the case $\mathcal{E}^\infty = J^\infty(\pi)$. We shall follow the notation of §2.1, but abbreviate "$J^0(\pi)$" to "π", e.g., $\mathcal{C}^k\Lambda^q(\pi)$ stands for $\mathcal{C}^k\Lambda^q(J^0(\pi))$, etc.

By definition, the first term E_1 of a spectral sequence is the cohomology of its zero term E_0. So the zero column of E_1 consists of the horizontal cohomology groups of the space $J^\infty(\pi)$: $E_1^{0,q}(\pi) = \bar{H}^q(\pi)$. To describe the terms $E_1^{p,q}(\pi)$ for $p > 0$, we need to compute the cohomology of complexes (see §2.3)

$$0 \longrightarrow \mathcal{C}^p\Lambda^p(\pi) \overset{\widehat{d}}{\longrightarrow} \mathcal{C}^p\Lambda^p(\pi) \otimes \Lambda_0^1(\pi) \overset{\widehat{d}}{\longrightarrow} \cdots \overset{\widehat{d}}{\longrightarrow} \mathcal{C}^p\Lambda^p(\pi) \otimes \Lambda_0^1(\pi) \longrightarrow 0. \quad (2.9)$$

Let $\varkappa(\pi) = \mathcal{F}(\pi, \pi)$ be the $\mathcal{F}(\pi)$-module of evolutionary derivations. To each form $\omega \in \mathcal{C}^p\Lambda^p(\pi)$, we assign the operator $\nabla_\omega \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa(\pi); \mathcal{F}(\pi))$ by

$$\nabla_\omega(\chi_1, \dots, \chi_p) = \eth_{\chi_p} \lrcorner (\dots (\eth_{\chi_1} \lrcorner \omega) \dots), \quad (2.10)$$

where $\chi_i \in \varkappa(\pi)$.

LEMMA 2.4. *Operation* (2.10) *establishes an isomorphism of the $\mathcal{F}(\mathcal{E})$-modules $\mathcal{C}^p\Lambda^p(\pi)$ and $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa(\pi); \mathcal{F}(\pi))$.*

PROOF. Let us construct the map inverse to the given map $\omega \mapsto \nabla_\omega$. Take $\nabla \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa(\pi); \mathcal{F}(\mathcal{E}))$. Any vertical tangent vector ξ at the point $\theta \in J^\infty(\pi)$ may be realized in the form $\eth_\chi|_\theta$ for some χ (this can be seen, e.g., from the coordinate expression for $\eth_\chi$ (see equality (2.15) of Ch. 4)). Define the form $\omega_\nabla \in \mathcal{C}^p\Lambda^p(\pi)$ by putting

$$\omega_\nabla|_\theta (\xi_1, \dots, \xi_p) = \nabla(\chi_1, \dots, \chi_p)(\theta),$$

where $\xi_i = \eth_{\chi_i}|_\theta$. It is clear that the map $\nabla \mapsto \omega_\nabla$ is the desired inverse map. Other details of the proof are left to the reader as an exercise. $\quad\square$

Thus, we can rewrite complex (2.9) in the form

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa(\pi); \mathcal{F}(\pi)) \overset{w}{\longrightarrow} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa(\pi); \Lambda_0^1(\pi)) \overset{w}{\longrightarrow} \cdots$$

$$\overset{w}{\longrightarrow} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa(\pi); \Lambda_0^n(\pi)) \longrightarrow 0. \quad (2.11)$$

EXERCISE 2.13. Prove that the differential w corresponds to the differential $\widehat{d}$ in complex (2.9).

Computing the cohomology of complex (2.11) by Theorem 2.2, we get the following result:

THEOREM 2.5. *Let π be a smooth vector bundle over a manifold M, $\dim M = n$. Then:*

$$\begin{aligned}
&E_1^{0,q}(\pi) = \bar{H}^q(\pi) && \text{for all } q \geq 0; \\
&E_1^{p,q}(\pi) = 0 && \text{for } p > 0,\ q \neq n; \\
&E_1^{p,n}(\pi) = K_p(\varkappa(\pi)) && \text{for } p > 0.
\end{aligned}$$

Since the $\mathcal{C}$-spectral sequence converges to the de Rham cohomology of the manifold $J^\infty(\pi)$ in the case under consideration, Theorem 2.5 has the following obvious corollary:

COROLLARY 2.6. *Under the assumptions of Theorem 2.5, we have*

1. $E_r^{p,q}(\pi) = 0$, *for all* $1 \leq r \leq \infty$ *if* $p > 0$, $q \neq n$, *or* $p = 0$, $q > n$.
2. $E_1^{0,q}(\pi) = E_\infty^{0,q}(\pi) = H^q(J^\infty(\pi)) = H^q(J^0(\pi))$ *for* $q < n$.
3. $E_2^{p,n}(\pi) = E_\infty^{p,n}(\pi) = H^{p+n}(J^\infty(\pi)) = H^{p+n}(J^0(\pi))$ *for* $p \geq 0$.

EXERCISE 2.14. Prove the equality $H^q(J^\infty(\pi)) = H^q(J^0(\pi))$.

Let us now consider the differentials $d_1^{p,n}$.

EXERCISE 2.15. a. Show that the operator

$$E_1^{0,n}(\pi) = \bar{H}^n(\pi) \xrightarrow{d_1^{0,n}} E_1^{1,n}(\pi) = \widehat{\varkappa}(\pi)$$

is given by the formula $d_1^{0,n}([\omega]) = \ell_\omega^*(1)$, where $\omega \in \Lambda_0^n(\pi)$ and $[\omega]$ is the horizontal cohomology class of ω. (Note that the expression ℓ_ω makes sense, because ω is a horizontal n-form, i.e., a nonlinear differential operator acting from $\Gamma(\pi)$ to $\Lambda^n(M)$.)

b. Write down the coordinate expression for the operator $d_1^{0,n}$ and check that this operator is the standard Euler operator (i.e., the operator that takes each Lagrangian to the corresponding Euler–Lagrange equation (see §4.1 below)).

Now let us describe the differentials $d_1^{p,n}$, $p > 0$.

Consider an operator $\nabla \in K_p(\varkappa(\pi))$. Define $\square \in \mathcal{C}\mathrm{Diff}_{(p+1)}(\varkappa(\pi), \Lambda_0^n(\pi))$ by putting

$$\square(\chi_1, \ldots, \chi_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \partial_{\chi_i}(\nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_{p+1}))$$

$$+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \nabla(\{\chi_i, \chi_j\}, \chi_1, \ldots, \widehat{\chi}_i, \ldots, \widehat{\chi}_j, \ldots, \chi_{p+1}). \quad (2.12)$$

EXERCISE 2.16. Prove that $d_1^{p,n}(\nabla) = \mu_{p+1}(\square)$.

REMARK 2.1. It is needless to say that this fact follows from the standard formula for the exterior differential. However, it is necessary to prove that one may use this formula even though ∇, as an element of $\mathcal{C}\mathrm{Diff}_{(p)}(\varkappa(\pi); \Lambda_0^n(\pi))$, is not skew-symmetric.

From (2.12) we get

$$\Box(\chi_1, \ldots, \chi_{p+1}) = \sum_{i=1}^{p} (-1)^{i+1} \eth_{\chi_i}(\nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p))(\chi_{p+1})$$

$$+ \sum_{i=1}^{p} (-1)^{i+1} \nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p, \eth_{\chi_i}(\chi_{p+1})) + (-1)^p \eth_{\chi_{p+1}}(\nabla(\chi_1, \ldots, \chi_p))$$

$$+ \sum_{1 \le i < j \le p} (-1)^{i+j} \nabla(\{\chi_i, \chi_j\}, \chi_1, \ldots, \widehat{\chi}_i, \ldots, \widehat{\chi}_j, \ldots, \chi_{p+1})$$

$$+ \sum_{i=1}^{p} (-1)^{i+p+1} \nabla(\{\chi_i, \chi_{p+1}\}, \chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p)$$

$$= \sum_{i=1}^{p} (-1)^{i+1} \eth_{\chi_i}(\nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p))(\chi_{p+1})$$

$$+ \sum_{1 \le i < j \le p} (-1)^{i+j} \nabla(\{\chi_i, \chi_j\}, \chi_1, \ldots, \widehat{\chi}_i, \ldots, \widehat{\chi}_j, \ldots, \chi_{p+1})$$

$$+ \sum_{i=1}^{p} (-1)^{i+1} \nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p, \ell_{\chi_i}(\chi_{p+1})) + (-1)^p \ell_{\nabla(\chi_1, \ldots, \chi_p)}(\chi_{p+1}).$$

Therefore

$$d_1^{p,n}(\nabla)(\chi_1, \ldots, \chi_p) = \mu_{p+1}(\Box)(\chi_1, \ldots, \chi_p)$$

$$= \sum_{i=1}^{p} (-1)^{i+1} \eth_{\chi_i}(\nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p))$$

$$+ \sum_{i<j} (-1)^{i+j} \nabla(\{\chi_i, \chi_j\}, \chi_1, \ldots, \widehat{\chi}_i, \ldots, \widehat{\chi}_j, \ldots, \chi_p)$$

$$+ \sum_{i=1}^{p} (-1)^{i+1} \ell_{\chi_i}^*(\nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p)) + (-1)^p \ell_{\nabla(\chi_1, \ldots, \chi_p)}^*(1). \quad (2.13)$$

Using the obvious equality

$$\ell_{\psi(\varphi)}^*(1) = \ell_\psi^*(\varphi) + \ell_\varphi^*(\psi), \qquad \varphi \in \varkappa(\pi), \quad \psi \in \widehat{\varkappa}(\pi),$$

(cf. Exercise 2.17), let us rewrite the last term of (2.13) in the following way:

$$(-1)^p \ell_{(\nabla(\chi_1, \ldots, \chi_p))}^*(1) = \frac{1}{p} \sum_{i=1}^{p} (-1)^i \ell_{(\nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p, \chi_i))}^*(1)$$

$$= \frac{1}{p} \sum_{i=1}^{p} (-1)^i (\ell_{\nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p)}^*(\chi_i) + \ell_{\chi_i}^*(\nabla(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_p))).$$

Finally, we obtain

$$(d_1^{p,n}(\nabla))(\chi_1,\ldots,\chi_p) = \sum_{i=1}^{p}(-1)^{i+1}\partial_{\chi_i}(\nabla(\chi_1,\ldots,\widehat{\chi_i},\ldots,\chi_p))$$

$$+ \sum_{i<j}(-1)^{i+j}\nabla(\{\chi_i,\chi_j\},\chi_1,\ldots,\widehat{\chi_i},\ldots,\widehat{\chi_j},\ldots,\chi_p)$$

$$+ \frac{1}{p}\sum_{i=1}^{p}(-1)^{i+1}((p-1)\ell_{\chi_i}^{*}(\nabla(\chi_1,\ldots,\widehat{\chi_i},\ldots,\chi_p)) - \ell_{\nabla(\chi_1,\ldots,\widehat{\chi_i},\ldots,\chi_p)}^{*}(\chi_i)).$$

In particular, for $p = 1$ we have $d_1^{1,n}(\psi)(\varphi) = \partial_{\varphi}(\psi) - \ell_{\psi}^{*}(\varphi) = \ell_{\psi}(\varphi) - \ell_{\psi}^{*}(\varphi)$, $\psi \in \widehat{\varkappa}(\pi)$, $\varphi \in \varkappa(\pi)$, that is,

$$d_1^{1,n}(\psi) = \ell_{\psi} - \ell_{\psi}^{*}. \tag{2.14}$$

Note that the horizontal de Rham complex on $J^{\infty}(\pi)$ can be combined with the complex $(E_1^{p,n}(\pi), d_1^{p,n})$ to give the complex

$$0 \longrightarrow \mathcal{F}(\mathcal{E}) \xrightarrow{\widehat{d}} \Lambda_0^1(\pi) \xrightarrow{\widehat{d}} \cdots \xrightarrow{\widehat{d}} \Lambda_0^n(\pi) \xrightarrow{\mathsf{E}} E_1^{1,n}(\pi) \xrightarrow{d_1^{1,n}} E_1^{2,n}(\pi) \xrightarrow{d_1^{2,n}} \cdots, \tag{2.15}$$

where E is the composition of the natural projection $\Lambda_0^n(\pi) \to \bar{H}^n(\pi)$ and the differential $d_1^{0,n}\colon \bar{H}^n(\pi) \to E_1^{1,n}(\pi)$[6].

In view of Corollary 2.6, the i-th cohomology group of this complex coincides with $H^i(J^0(\pi))$. The operator E is the Euler operator (see Exercise 2.15b). It takes each Lagrangian density $\omega \in \Lambda_0^n(\pi)$ to the left-hand side of the corresponding Euler–Lagrange equation $\mathsf{E}(\omega) = 0$. Thus the action functional

$$s \mapsto \int_M j_{\infty}(s)^{*}(\omega), \qquad s \in \Gamma(\pi),$$

is stationary on a section s if and only if $j_{\infty}(s)^{*}(\mathsf{E}(\omega)) = 0$.

Complex (2.15) is often called the (global) *variational complex* of the bundle π. If the cohomology of the space $J^0(\pi)$ is trivial then this complex is exact. This immediately implies a number of consequences. The three most important ones are:

1. $\ker\mathsf{E} = \operatorname{im}\widehat{d}$ (variationally trivial Lagrangians are total divergences).
2. $\widehat{d}\omega = 0$ if and only if ω is of the form $\omega = \widehat{d}\eta$, $\omega \in \Lambda_0^{n-1}(\pi)$ (null total divergences are total curls).
3. $\ell_{\psi} = \ell_{\psi}^{*}$ if and only if ψ is of the form $\psi = \mathsf{E}(\omega)$, $\psi \in \widehat{\varkappa}(\pi)$ (this solves the inverse problem of the calculus of variations).

EXERCISE 2.17. Let P be a horizontal module, $\Delta \in \mathcal{C}\mathrm{Diff}(P, \Lambda_0^n(\pi))$, and $p \in P$. Show that

$$\mathsf{E}(\Delta(p)) = \ell_p^{*}(\Delta^{*}(1)) + \ell_{\Delta^{*}(1)}^{*}(p).$$

Use this fact to prove the formula

$$\mathsf{E}(\partial_{\varphi}(\omega)) = \partial_{\varphi}(\mathsf{E}(\omega)) + \ell_{\varphi}^{*}(\mathsf{E}(\omega))$$

for all $\varphi \in \varkappa(\pi)$ and $\omega \in \Lambda_0^n(\pi)$.

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[6]Below we use the same notation E for the operator $d_1^{0,n}\colon \bar{H}^n(\pi) \to E_1^{1,n}(\pi)$ as well.

2.6. The term E_1 in the general case. Let us compute the term $E_1(\mathcal{E})$ of the $\mathcal{C}$-spectral sequence of a differential equation $\mathcal{E}$.

Let $I(\mathcal{E}) \subset \mathcal{F}(\mathcal{E})$ be the ideal of the equation $\mathcal{E}$. Recall that if the equation $\mathcal{E} \subset J^k(\pi)$ has the form $\mathcal{E} = \{F = 0\}$, where $F \in P_k = \mathcal{F}_k(\pi, \xi)$, $\xi \colon N_\xi \to M$ is a vector bundle over M, and the projection $\mathcal{E} \to M$ is surjective, then

$$I(\mathcal{E}) = \mathcal{C}\mathrm{Diff}(P, \mathcal{F}(\pi))(F),$$

where $P = \mathcal{F}(\pi, \xi)$ (i.e., the components of the vector function F are differential generators of the ideal $I(\mathcal{E})$). We call such equations *regular*. As it was mentioned in Ch. 4, the regularity condition is not restrictive, and equations encountered in mathematical physics, as a rule, satisfy it.

For regular equations, the module $\mathcal{C}\Lambda^1(\mathcal{E})$ can be described as follows. Let $\varkappa = \varkappa(\pi)/(I(\mathcal{E}) \cdot \varkappa(\pi))$ be the restriction of the module $\varkappa(\pi)$ to $\mathcal{E}^\infty$. Consider the submodule L of $\mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E}))$ consisting of operators of the form $\square \circ \ell_F^{\mathcal{E}}$, where $\square \in \mathcal{C}\mathrm{Diff}(P, \mathcal{F}(\mathcal{E}))$, $\ell_F^{\mathcal{E}} = \ell_F|_{\mathcal{E}^\infty}$. Here $P = \mathcal{F}(\mathcal{E}, \xi)$ is the restriction of the module $\mathcal{F}(\pi, \xi)$ to $\mathcal{E}^\infty$. (In the sequel, we shall not distinguish between notation for a module and its restriction to $\mathcal{E}^\infty$.) Next two results are generalizations of Lemma 2.4.

LEMMA 2.7. *To each form $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$, we assign the $\mathcal{C}$-differential operator $\nabla_\omega \in \mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E}))$ by setting*

$$\nabla_\omega(\chi) = \omega(\eth_\chi), \qquad \chi \in \varkappa. \tag{2.16}$$

If the equation $\mathcal{E}$ is regular, then correspondence (2.16) establishes an isomorphism of the $\mathcal{F}(\mathcal{E})$-modules $\mathcal{C}\Lambda^1(\mathcal{E})$ and $\mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E}))/L$.

PROOF. It is obvious that $\mathcal{C}\Lambda^1 = \mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E}))$, where $\mathcal{C}\Lambda^1$ is the restriction of $\mathcal{C}\Lambda^1(\pi)$ to $\mathcal{E}^\infty$. The kernel L' of the natural projection $\mathcal{C}\Lambda^1 \to \mathcal{C}\Lambda^1(\mathcal{E})$ is generated by the forms $d^v f$, where $f \in I(\mathcal{E})$ and $d^v \colon \mathcal{F}(\pi) \to \mathcal{C}\Lambda^1$ is the composition of the differential $d \colon \mathcal{F}(\pi) \to \Lambda^1(\pi)$ and the projection $\Lambda^1(\pi) \to \mathcal{C}\Lambda^1$. In terms of $\mathcal{C}$-differential operators this means that L' is generated by the operators of the form $\ell_f^{\mathcal{E}}$, where $f \in I(\mathcal{E})$. Since $\mathcal{E}$ is regular, $f \in I(\mathcal{E})$ can be represented in the form $f = \nabla(F)$, $\nabla \in \mathcal{C}\mathrm{Diff}(P, \mathcal{F}(\pi))$. Thus $\ell_f^{\mathcal{E}} = \nabla \circ \ell_F^{\mathcal{E}}$ and, consequently, $L' = L$. $\qquad\square$

EXERCISE 2.18. Show that

$$\mathcal{C}^p\Lambda^p(\mathcal{E}) = \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(p)}(\varkappa; \mathcal{F}(\mathcal{E}))/L_p,$$

where $L_p = \mathrm{alt}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(p-1)}(\varkappa; \mathcal{F}(\mathcal{E})) \otimes L)$ and

$$\mathrm{alt} \colon \mathcal{C}\mathrm{Diff}_{(p)}(\varkappa; \mathcal{F}(\mathcal{E})) \to \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}{}_{(p)}(\varkappa; \mathcal{F}(\mathcal{E}))$$

is the alternation operation.

Thus, for a regular equation $\mathcal{E}$ there exists the exact sequence of $\mathcal{F}(\mathcal{E})$-modules

$$\mathcal{C}\mathrm{Diff}(P, \mathcal{F}(\mathcal{E})) \longrightarrow \mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E})) \longrightarrow \mathcal{C}\Lambda^1(\mathcal{E}) \longrightarrow 0.$$

Generally speaking, the left-hand arrow is not injective: in other words, an operator of the form $\square \circ \ell_F^{\mathcal{E}}$, $\square \in \mathcal{C}\mathrm{Diff}(P, \mathcal{F}(\mathcal{E}))$, may be trivial for a nontrivial operator $\square$.

A regular equation is called ℓ-*normal*, if the sequence

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(P, \mathcal{F}(\mathcal{E})) \longrightarrow \mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{F}(\mathcal{E})) \longrightarrow \mathcal{C}\Lambda^1(\mathcal{E}) \longrightarrow 0 \tag{2.17}$$

is exact.

The next proposition gives a way to find out whether a given equation is ℓ-normal.

PROPOSITION 2.8. *Let $\mathcal{E}$ be a regular equation in n independent variables. Assume that in each coordinate neighborhood on $\mathcal{E}^\infty$ one can choose internal coordinates θ such that the functions $D_i(\theta)$, $i = 1, \ldots, n-1$, can be expressed in terms of these coordinates. Then the equation $\mathcal{E}$ is ℓ-normal.*

PROOF. Locally, the equation at hand can be written in the form

$$p^{i_r}_{(0,\ldots,0,k_r)} = f_r, \qquad i_r \in A \subset \{1, \ldots, m\}, \quad r = 1, \ldots, l, \quad k_r > 0,$$

where f_r are functions of the coordinates $x_1, \ldots, x_n$ and p^i_σ, where $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_n < k_r$, if $i \in A$. We take these coordinates as internal coordinates on $\mathcal{E}^\infty$. Consequently the operator $\ell^{\mathcal{E}}_F$ has the form $\ell^{\mathcal{E}}_F = \Delta - \ell_f$, where $f = (f_1, \ldots, f_l)$, $\Delta = \|\Delta_{ij}\|$ is the $l \times m$ matrix such that $\Delta_{r i_r} = \bar{D}^{k_r}_n$, $r = 1, \ldots, l$, and the other elements of Δ are trivial. It follows in the standard way now that if $\square \circ \ell^{\mathcal{E}}_F = 0$, $\square \in \mathcal{C}\mathrm{Diff}(P, \mathcal{F}(\mathcal{E}))$, then $\square = 0$. $\qquad\square$

Thus, if the equation under consideration is not overdetermined, i.e., the number of equations is not greater than the number of unknowns ($l \leq m$), it is almost always ℓ-normal. Nevertheless, let us give an example of a determined but not ℓ-normal equation.

EXAMPLE 2.1. Consider the system

$$\begin{cases} p^3_2 - p^2_3 + u^2 p^3_4 - u^3 p^2_4 = 0, \\ p^1_3 - p^3_1 + u^3 p^1_4 - u^1 p^3_4 = 0, \\ p^2_1 - p^1_2 + u^1 p^2_4 - u^2 p^1_4 = 0, \end{cases} \qquad (2.18)$$

which is the condition for integrability of a three-dimensional distribution in $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4) given by the form $\omega = dx_4 - \sum_{i=1}^{3} u^i \, dx_i$.

EXERCISE 2.19. Find an operator $\square$ such that $\square \circ \ell^{\mathcal{E}}_F = 0$, where F is the left-hand part of (2.18).

(*The answer*: $\square = (\bar{D}_1 + u^1 \bar{D}_4 - p^1_4, \bar{D}_2 + u^2 \bar{D}_4 - p^2_4, \bar{D}_3 + u^3 \bar{D}_4 - p^3_4)$.)

REMARK 2.2. The Maxwell, Yang–Mills, and Einstein equations provide examples of equations which are not ℓ-normal. Roughly speaking, this can be explained as follows. All these equations are invariant under the action of a pseudogroup, i.e., there exists a nontrivial $\mathcal{C}$-differential operator $R\colon Q \to \varkappa$, where Q is a horizontal module, such that $\ell^{\mathcal{E}}_F \circ R = 0$. Hence $R^* \circ \left(\ell^{\mathcal{E}}_F\right)^* = 0$ and, since the equations at hand are Euler–Lagrange equations (that is, $\left(\ell^{\mathcal{E}}_F\right)^* = \ell^{\mathcal{E}}_F$), we get $R^* \circ \ell^{\mathcal{E}}_F = 0$. Thus the operator R^* disobeys the condition of ℓ-normality.

In spite of these examples, it is important to stress that the majority of equations of mathematical physics are ℓ-normal.

From now on we consider only ℓ-normal equations[7].

For such an equation $\mathcal{E}$, sequence (2.17) is exact, and hence there exists the exact sequence of complexes

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(P, \Lambda^q_0(\mathcal{E})) \longrightarrow \mathcal{C}\mathrm{Diff}(\varkappa, \Lambda^q_0(\mathcal{E})) \longrightarrow \Lambda^q_0(\mathcal{E}) \otimes \mathcal{C}\Lambda^1(\mathcal{E}) \longrightarrow 0.$$

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[7]Closer examination of the $\mathcal{C}$-spectral sequence makes it possible to compute of conservation laws for equations which are not ℓ-normal (see [**132, 120**]).

Let us multiply each of these complexes by $\mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E})$ (see Exercise 2.12):

$$0 \longrightarrow \mathcal{C}\mathrm{Diff}(P, \Lambda_0^q(\mathcal{E})) \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}) \longrightarrow \mathcal{C}\mathrm{Diff}(\varkappa, \Lambda_0^q(\mathcal{E})) \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E})$$
$$\longrightarrow \Lambda_0^q(\mathcal{E}) \otimes \mathcal{C}\Lambda^1(\mathcal{E}) \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}) \longrightarrow 0. \quad (2.19)$$

Using the long exact sequence corresponding to (2.19) and Theorem 2.3, we get the following

PROPOSITION 2.9. *Let $\mathcal{E}$ be an ℓ-normal equation. Then the cohomology of the complex*

$$0 \longrightarrow \mathcal{C}\Lambda^1(\mathcal{E}) \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}) \longrightarrow \mathcal{C}\Lambda^1(\mathcal{E}) \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}) \otimes \Lambda_0^1(\mathcal{E}) \longrightarrow \cdots$$
$$\longrightarrow \mathcal{C}\Lambda^1(\mathcal{E}) \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}) \otimes \Lambda_0^n(\mathcal{E}) \longrightarrow 0 \quad (2.20)$$

is trivial in all degrees different from $n-1$ and n. The cohomology groups in degrees $n-1$ and n are isomorphic to the kernel and cokernel of the operator

$$\left(\ell_F^{\mathcal{E}}\right)^*_{(p-1)} \stackrel{\mathrm{def}}{=} \left(\ell_F^{\mathcal{E}}\right)^*_{\mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E})} : \widehat{P} \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}) \longrightarrow \widehat{\varkappa} \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E})$$

respectively.

EXERCISE 2.20. Let $\nabla_\omega \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; P)$ be the operator that corresponds to a P-valued form $\omega \in \mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes P$ (see Lemma 2.7). Show that if $\Delta \in \mathcal{C}\mathrm{Diff}(P, P_1)$, then the operator

$$\Delta_{\mathcal{C}^p\Lambda^p(\mathcal{E})} : \mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes P \longrightarrow \mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes P_1$$

takes ∇_ω to $\Delta \circ \nabla_\omega$.

The complex

$$0 \longrightarrow \mathcal{C}^p\Lambda^p(\mathcal{E}) \longrightarrow \Lambda_0^1(\mathcal{E}) \otimes \mathcal{C}^p\Lambda^p(\mathcal{E}) \longrightarrow \cdots \longrightarrow \Lambda_0^n(\mathcal{E}) \otimes \mathcal{C}^p\Lambda^p(\mathcal{E}) \longrightarrow 0$$

is obviously a direct summand in the complex (2.20). Hence its cohomology is trivial in all degrees different from $n-1$ and n. The cohomology groups in degrees $n-1$ and n are isomorphic to the skew-symmetric parts of the kernel and cokernel of the operator $\left(\ell_F^{\mathcal{E}}\right)^*_{(p-1)}$ respectively.

EXERCISE 2.21. If $\omega \in \ker\left(\ell_F^{\mathcal{E}}\right)^*_{(p-1)} \subset \widehat{P} \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E})$, then, as it follows from the previous exercise, the corresponding operator $\nabla_\omega \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(\varkappa; \widehat{P})$ satisfies the equality

$$\left(\ell_F^{\mathcal{E}}\right)^* \left(\nabla_\omega(\chi_1, \ldots, \chi_{p-1})\right) = \sum_{i=1}^{p-1} \Delta_i(\chi_1, \ldots, \widehat{\chi}_i, \ldots, \chi_{p-1})(\ell_F^{\mathcal{E}}(\chi_i)),$$

where $\Delta_i \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-2)}(\varkappa; \mathcal{C}\mathrm{Diff}(P, \widehat{\varkappa}))$. Prove that ω lies in the skew-symmetric part of $\ker\left(\ell_F^{\mathcal{E}}\right)^*_{(p-1)}$ if and only if

$$\nabla_\omega = (-1)^{p-i}\Delta_i^* \mod L_{p-1} \otimes \widehat{P}, \qquad 1 \leq i \leq p-1,$$

where Δ_i^* is the adjoint operator to Δ_i with respect to the last argument (i.e., to that belonging to P) and L_{p-1} is the module defined in Exercise 2.18.

EXERCISE 2.22. Show that $\omega \in \operatorname{coker}\left(\ell_F^{\mathcal{E}}\right)^*_{(p-1)} \subset \widehat{\varkappa} \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E})$ lies in the skew-symmetric part of $\operatorname{coker}\left(\ell_F^{\mathcal{E}}\right)^*_{(p-1)}$ if and only if

$$\nabla_\omega = -\nabla_\omega^{*i} \mod L_{p-1} \otimes \widehat{\varkappa}, \qquad 1 \leq i \leq p-1,$$

where $\nabla_\omega \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(\varkappa; \widehat{\varkappa})$ is the operator corresponding to ω and ∇^{*i} denotes the operator adjoint to ∇ with respect to the i-th argument.

EXERCISE 2.23. a. Show that the operator

$$d_1^{0,n-1}\colon E_1^{0,n-1}(\mathcal{E}) = \bar{H}^{n-1}(\mathcal{E}) \longrightarrow E_1^{1,n-1}(\mathcal{E}) = \ker\left(\ell_F^{\mathcal{E}}\right)^* \subset \widehat{P}$$

has the form

$$d_1^{0,n-1}(h) = \square^*(1),$$

where $h = [\omega] \in \bar{H}^{n-1}(\mathcal{E})$, $\omega \in \Lambda_0^{n-1}(\mathcal{E})$, and $\square \in \mathcal{C}\mathrm{Diff}(P, \Lambda_0^n(\mathcal{E}))$ is an operator satisfying $\widehat{d}\omega = \square(F)$.
    b. Check that the term $E_1^{2,n-1}(\mathcal{E})$ can be described as the quotient set

$$\left\{ \nabla \in \mathcal{C}\mathrm{Diff}(\varkappa, \widehat{P}) \mid \left(\ell_F^{\mathcal{E}}\right)^* \circ \nabla = \nabla^* \circ \ell_F^{\mathcal{E}} \right\}/\theta,$$

where $\theta = \{ \square \circ \ell_F^{\mathcal{E}} \mid \square \in \mathcal{C}\mathrm{Diff}(P, \widehat{P}), \; \square^* = \square \}$.
    c. Show that the operator $d_1^{1,n-1}\colon E_1^{1,n-1}(\mathcal{E}) = \ker\left(\ell_F^{\mathcal{E}}\right)^* \to E_1^{2,n-1}(\mathcal{E})$ has the form

$$d_1^{1,n-1}(\psi) = (\ell_\psi^{\mathcal{E}} + \Delta^*) \mod \theta,$$

where $\Delta \in \mathcal{C}\mathrm{Diff}(P, \widehat{\varkappa})$ is an operator satisfying $\ell_F^*(\psi) = \Delta(F)$.
    d. Describe the operators $d_1^{p,n}$ and $d_1^{p,n-1}$ for all $p \geq 0$.

Bringing together the above results, we get the following description of the term $E_1(\mathcal{E})$ of the $\mathcal{C}$-spectral sequence:

THE TWO-LINE THEOREM. *Let $\mathcal{E}$ be an ℓ-normal equation. Then*:

1. $E_1^{p,q}(\mathcal{E}) = 0$, *if $p \geq 1$ and $q \neq n-1, n$.*
2. $E_1^{p,n-1}(\mathcal{E})$ *(resp., $E_1^{p,n}(\mathcal{E})$) coincides with the skew-symmetric part (see Exercises 2.21 and 2.22) of the kernel (resp., cokernel) of the operator*

$$\left(\ell_F^{\mathcal{E}}\right)^*_{(p-1)} : \widehat{P} \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}) \longrightarrow \widehat{\varkappa} \otimes \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}).$$

The following result is an obvious consequence of this theorem:

COROLLARY 2.10. *Under the assumptions of the two-line theorem we have:*

1. $E_r^{p,q}(\mathcal{E}) = 0$ *if $p \geq 1$, $q \neq n-1, n$, $1 \leq r \leq \infty$.*
2. $E_3^{p,q}(\mathcal{E}) = E_\infty^{p,q}(\mathcal{E})$.
3. $E_1^{0,q}(\mathcal{E}) = E_\infty^{0,q}(\mathcal{E}) = H^q(\mathcal{E}^\infty)$, $q \leq n-2$.
4. $E_2^{0,n-1}(\mathcal{E}) = E_\infty^{0,n-1}(\mathcal{E}) = H^{n-1}(\mathcal{E}^\infty)$.
5. $E_2^{1,n-1}(\mathcal{E}) = E_\infty^{1,n-1}(\mathcal{E})$.

2.7. Conservation laws and generating functions. Now we apply the results of the previous subsection to the problem of computing conservation laws of an ℓ-normal equation $\mathcal{E}$.

First of all, note that for a formally integrable equation $\mathcal{E}$ the projections $\mathcal{E}^{(k+1)} \to \mathcal{E}^{(k)}$ are affine bundles, therefore $\mathcal{E}^{(k+1)}$ and $\mathcal{E}^{(k)}$ are of the same homotopy type. Hence, $H^*(\mathcal{E}^\infty) = H^*(\mathcal{E})$.

Further, from the two-line theorem it follows that there exists the following exact sequence:

$$0 \longrightarrow H^{n-1}(\mathcal{E}) \longrightarrow \bar{H}^{n-1}(\mathcal{E}) \xrightarrow{d_1^{0,n-1}} \ker\left(\ell_F^{\mathcal{E}}\right)^*.$$

Recall that the group $\bar{H}^{n-1}(\mathcal{E})$ was interpreted as the group of conservation laws of the equation $\mathcal{E}$ (see Definition 2.1). Conservation laws $\omega \in H^{n-1}(\mathcal{E}) \subset \bar{H}^{n-1}(\mathcal{E})$ are called *topological* (or *rigid*), since they are determined only by the topology of the equation $\mathcal{E}$. In particular, the corresponding conserved quantities do not change under deformations of solutions of the equation $\mathcal{E}$. Thus topological conservation laws[8] are not very interesting for us and we consider the quotient group $\mathrm{cl}(\mathcal{E}) = \bar{H}^{n-1}(\mathcal{E})/H^{n-1}(\mathcal{E})$ called the group of *proper* conservation laws of the equation $\mathcal{E}$. The two-line theorem implies immediately the following

THEOREM 2.11. *If $\mathcal{E}$ is an ℓ-normal equation, then*

$$\mathrm{cl}(\mathcal{E}) \subset \ker\left(\ell_F^{\mathcal{E}}\right)^*.$$

If, moreover, $H^n(\mathcal{E}) \subset \bar{H}^n(\mathcal{E})$ (in particular, if $H^n(\mathcal{E}) = 0$), then

$$\mathrm{cl}(\mathcal{E}) = \ker d_1^{1,n-1}.$$

The element $\psi \in \ker\left(\ell_F^{\mathcal{E}}\right)^*$ corresponding to a conservation law $[\omega] \in \mathrm{cl}(\mathcal{E})$ is called its *generating function*.

Theorem 2.11 gives an effective method for computing the group of (proper) conservation laws $\mathrm{cl}(\mathcal{E})$. In §3 we demonstrate this method in action.

EXERCISE 2.24. Show that the generating function of any conservation law can be extended to the entire space $J^\infty(\pi)$ such that the following equality holds:

$$l_F^*(\psi) + l_\psi^*(F) = 0. \tag{2.21}$$

Prove that:

a. Equality (2.21) holds identically, if $\psi = \square(F)$ and $\square = -\square^*$.

b. The solutions to (2.21) of the form $\psi = \square(F)$ for a self-adjoint operator $\square$, $\square = \square^*$, correspond to topological conservation laws of $\mathcal{E}$.

c. Two different topological conservation laws correspond to the same solution of the above described form if and only if they differ by an element belonging to the image of the natural mapping $H^{n-1}(J^\infty(\pi)) = \bar{H}^{n-1}(\pi) \to \bar{H}^{n-1}(\mathcal{E})$.

To conclude this subsection, note yet another important result. Let $\varphi \in \ker \ell_F^{\mathcal{E}}$ be a symmetry and $[\omega] \in \bar{H}^{n-1}(\mathcal{E})$ be a conservation law of $\mathcal{E}$. Then $[\partial_\varphi(\omega)]$ is obviously a conservation law of $\mathcal{E}$ as well.

EXERCISE 2.25. Prove that if $\psi \in \ker\left(\ell_F^{\mathcal{E}}\right)^*$ is the generating function of a conservation law $[\omega]$ of an ℓ-normal equation $\mathcal{E}$, then the generating function of the conservation law $[\partial_\varphi(\omega)]$ has the form $\partial_\varphi(\psi) + \Delta^*(\psi)$, where the operator $\Delta \in \mathcal{C}\mathrm{Diff}(P, P)$ is defined by $\partial_\varphi(F) = \Delta(F)$.

---

[8]They are discussed, e.g., in the book [**103**].

2.8. Euler–Lagrange equations. Consider a Lagrangian $\mathcal{L} = [\omega] \in \bar{H}^n(\pi)$ and the corresponding Euler–Lagrange equation $\mathcal{E} = \{\mathsf{E}(\mathcal{L}) = 0\}$. Let $\varphi \in \varkappa(\pi)$ be a *Noether symmetry* of $\mathcal{L}$, i.e., $\eth_\varphi(\mathcal{L}) = 0$ on $J^\infty(\pi)$.

EXERCISE 2.26. Check that a Noether symmetry of $\mathcal{L}$ is a symmetry of the corresponding equation $\mathcal{E}$ as well, i.e., $\mathrm{sym}\,\mathcal{L} \subset \mathrm{sym}\,\mathcal{E}$.

EXERCISE 2.27. Show that if $E_2^{0,n}(\mathcal{E}) = 0$, then finding of Noether symmetries of the Lagrangian $\mathcal{L} = [\omega]$ amounts to solution of the equation $\mathsf{E}(\ell_\omega(\varphi)) = \ell_{\mathsf{E}(\mathcal{L})}(\varphi) + \ell_\varphi^*(\mathsf{E}(\mathcal{L})) = 0$. (Thus, to compute Noether symmetries of an Euler–Lagrange equation one has no need to know the Lagrangian.)

Let $\eth_\varphi(\omega) = \widehat{d}\nu$, where $\nu \in \Lambda_0^{n-1}(\pi)$. By (2.5), we have

$$\eth_\varphi(\omega) - \widehat{d}\nu = \ell_\omega(\varphi) - \widehat{d}\nu = \ell_\omega^*(1)(\varphi) + \widehat{d}\omega_{\varphi,1}(\ell_\omega) - \widehat{d}\nu$$
$$= \mathsf{E}(\mathcal{L})(\varphi) + \widehat{d}(\omega_{\varphi,1}(\ell_\omega) - \nu) = 0.$$

Set

$$\eta = (\nu - \omega_{\varphi,1}(\ell_\omega))|_{\mathcal{E}^\infty} \in \Lambda_0^{n-1}(\mathcal{E}).$$

Thus, $\widehat{d}\eta\big|_{\mathcal{E}^\infty} = 0$, i.e., $[\eta] \in \bar{H}^{n-1}(\mathcal{E})$ is a conservation law for $\mathcal{E}$. The map

$$\mathrm{sym}\,\mathcal{L} \longrightarrow \bar{H}^{n-1}(\mathcal{E}), \qquad \varphi \longmapsto [\eta],$$

is said to be the *Noether map*.

EXERCISE 2.28. Check that the Noether map is well defined up to the image of the natural homomorphism $H^{n-1}(J^\infty(\pi)) = \bar{H}^{n-1}(\pi) \to \bar{H}^{n-1}(\mathcal{E})$.

REMARK 2.3. The definition of the Noether map implies that it is defined for all Euler–Lagrange equations and not just for ℓ-normal ones.

EXERCISE 2.29. Prove that if the Euler–Lagrange equation $\mathcal{E}$ corresponding to a Lagrangian $\mathcal{L}$ is ℓ-normal, then the Noether map restricted to the set of Noether symmetries of $\mathcal{L}$ is inverse to the differential $d_1^{0,n-1}$:

$$d_1^{0,n-1}([\eta]) = \varphi. \tag{2.22}$$

REMARK 2.4. The Noether map can be understood as a procedure for finding a conserved current to a given generating function.

EXERCISE 2.30. Given an ℓ-normal Euler–Lagrange equation $\mathcal{E}^\infty$, describe the map inverse to the embedding $d_1^{0,n-1}\colon \mathrm{cl}(\mathcal{E}) \to \ker\ell\left(\ell_F^{\mathcal{E}}\right)^* = \ker\ell_F^{\mathcal{E}} = \mathrm{sym}\,\mathcal{L}$. (In view of the previous exercise, the required inverse map is an extension of the Noether map.)

2.9. Hamiltonian formalism on $J^\infty(\pi)$. Let $A \in \mathcal{C}\mathrm{Diff}(\widehat{\varkappa}(\pi), \varkappa(\pi))$ be a $\mathcal{C}$-differential operator. Define the *Poisson bracket* on $\bar{H}^n(\pi)$ corresponding to the operator A by the formula

$$\{\omega_1, \omega_2\}_A = \langle A(\mathsf{E}(\omega_1)), \mathsf{E}(\omega_2)\rangle,$$

where $\langle\,,\rangle$ denotes the natural paring $\varkappa(\pi) \times \widehat{\varkappa}(\pi) \to \bar{H}^n(\pi)$.

EXERCISE 2.31. a. Consider an operator $A \in \mathcal{C}\mathrm{Diff}(\widehat{\varkappa}(\pi), P)$, where P is a module over $\mathcal{F}(\pi)$. Prove that $A \circ \mathsf{E} = 0$ implies $A = 0$.

b. Consider an operator $A \in \mathcal{C}\mathrm{Diff}_{(l)}(\widehat{\varkappa}(\pi), \Lambda_0^n(\pi))$. Prove that if for all $\omega_1, \ldots, \omega_l \in \bar{H}^n(\pi)$ the element $A(\mathsf{E}(\omega_1), \ldots, \mathsf{E}(\omega_l))$ belongs to the image of the operator $\widehat{d}$, then $\mathrm{im}\, A \subset \mathrm{im}\, \widehat{d}$, i.e., $\mu_l(A) = 0$ (see Exercises 2.5 and 2.10). In particular, if for any $\omega \in \bar{H}^n(\pi)$ one has $\mathfrak{I}_\varphi(\omega) = \langle \varphi, \mathsf{E}(\omega) \rangle = 0$, $\varphi \in \varkappa(\pi)$, then $\varphi = 0$. Note that the equality $\mathfrak{I}_\varphi(\omega) = \langle \varphi, \mathsf{E}(\omega) \rangle$ follows from

$$\langle \varphi, \mathsf{E}(\omega) \rangle = \langle \varphi, \ell_\omega^*(1) \rangle = \langle \ell_\omega(\varphi), 1 \rangle = \langle \mathfrak{I}_\varphi(\omega), 1 \rangle.$$

c. Using the above said, check that if the Poisson bracket $\{\omega_1, \omega_2\}_A$ is trivial for any ω_1 and ω_2, then $A = 0$.

An operator A is called *Hamiltonian*, if the corresponding Poisson bracket defines a Lie algebra structure (over $\mathbb{R}$) on $\bar{H}^n(\pi)$, i.e., if

$$\{\omega_1, \omega_2\}_A = -\{\omega_2, \omega_1\}_A, \tag{2.23}$$

$$\{\{\omega_1, \omega_2\}_A, \omega_3\}_A + \{\{\omega_2, \omega_3\}_A, \omega_1\}_A + \{\{\omega_3, \omega_1\}_A, \omega_2\}_A = 0. \tag{2.24}$$

The bracket $\{\,,\,\}_A$ is said to be a *Hamiltonian structure*.

EXERCISE 2.32. Prove that the Poisson bracket corresponding to an operator A is skew-symmetric, i.e., condition (2.23) holds, if and only if the operator A is skew-adjoint, i.e., $A^* = -A$.

Now we derive criteria for a skew-adjoint operator $A \in \mathcal{C}\mathrm{Diff}(\widehat{\varkappa}(\pi), \varkappa(\pi))$ to be Hamiltonian. For this, we use the following observation. Let $P = \mathcal{F}(\pi, \xi)$ and $P' = \mathcal{F}(\pi, \xi')$ be horizontal modules and $\Delta \colon P \to P'$ be a $\mathcal{C}$-differential operator. Set $D_\sigma = D_1^{\sigma_1} \circ \cdots \circ D_n^{\sigma_n}$. One has $j_\infty(s)^*(D_\sigma f) = \partial^{|\sigma|}/\partial x_\sigma (j_\infty(s)^* f)$ for any section $s \in \Gamma(\pi)$ (see Ch. 4). Therefore Δ can be considered as a nonlinear operator acting from the sections of π to the set of linear operators $\Gamma(\xi) \to \Gamma(\xi')$. Thus, the universal linearization of this operator, which we denote by ℓ_Δ, belongs to $\mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{C}\mathrm{Diff}(P, P')) = \mathcal{C}\mathrm{Diff}(\varkappa, P; P')$.

For a $\mathcal{C}$-differential operator $A \colon \widehat{\varkappa}(\pi) \to \varkappa(\pi)$ and an element $\psi \in \widehat{\varkappa}(\pi)$, we can define the $\mathcal{C}$-differential operator $\ell_{A,\psi} \colon \varkappa(\pi) \to \varkappa(\pi)$ by

$$\ell_{A,\psi}(\varphi) = (\ell_A(\varphi))(\psi).$$

EXERCISE 2.33. Prove that

$$\ell_{A,\psi_1}^*(\psi_2) = \ell_{A^*,\psi_2}^*(\psi_1).$$

THEOREM 2.12. *Let $A \in \mathcal{C}\mathrm{Diff}(\widehat{\varkappa}(\pi), \varkappa(\pi))$ be a skew-adjoint operator; then the following conditions are equivalent:*

1. *A is a Hamiltonian operator.*
2. *$\langle \ell_A(A(\psi_1))(\psi_2), \psi_3 \rangle + \langle \ell_A(A(\psi_2))(\psi_3), \psi_1 \rangle + \langle \ell_A(A(\psi_3))(\psi_1), \psi_2 \rangle = 0$ for all $\psi_1, \psi_2, \psi_3 \in \widehat{\varkappa}(\pi)$.*
3. *$\ell_{A,\psi_1}(A(\psi_2)) - \ell_{A,\psi_2}(A(\psi_1)) = A(\ell_{A,\psi_2}^*(\psi_1))$ for all $\psi_1, \psi_2 \in \widehat{\varkappa}(\pi)$.*
4. *The expression $\ell_{A,\psi_1}(A(\psi_2)) + \frac{1}{2}A(\ell_{A,\psi_1}^*(\psi_2))$ is symmetric with respect to $\psi_1, \psi_2 \in \widehat{\varkappa}(\pi)$.*
5. *$[\mathfrak{I}_{A(\psi)}, A] = \ell_{A(\psi)} \circ A + A \circ \ell_{A(\psi)}^*$ for all $\psi \in \mathrm{im}\, \mathsf{E} \subset \widehat{\varkappa}(\pi)$[9].*

_____

[9]Let $P = \mathcal{F}(\pi, \xi)$ and $P' = \mathcal{F}(\pi, \xi')$ be horizontal modules and $F \colon P \to P'$ be a map. Then for any $\varphi \in \mathcal{F}(\pi, \pi)$ one can define the commutator $[\mathfrak{I}_\varphi, F] \stackrel{\text{def}}{=} \mathfrak{I}_\varphi^{\xi'} \circ F - F \circ \mathfrak{I}_\varphi^\xi$, since each field $\mathfrak{I}_\varphi$ defines the family of derivations $\mathfrak{I}_\varphi^\xi \colon \mathcal{F}(\pi, \xi) \to \mathcal{F}(\pi, \xi)$ (see Ch. 4).

Moreover, it is sufficient to verify conditions 2–4 for elements $\psi_i \in \operatorname{im} \mathsf{E}$ only.

PROOF. Let $\omega_1, \omega_2, \omega_3 \in \bar{H}^n(\pi)$ and $\psi_i = \mathsf{E}(\omega_i)$. The Jacobi identity (2.24) has the form

$$
\begin{aligned}
\{\{\omega_1, \omega_2\}_A, \omega_3\}_A + \text{(cyclic)} &= -\eth_{A(\psi_3)}\langle A(\psi_1), \psi_2 \rangle + \text{(cyclic)} \\
&= -\langle \eth_{A(\psi_3)}(A)(\psi_1), \psi_2 \rangle - \langle A(\ell_{\psi_1}(A(\psi_3))), \psi_2 \rangle - \langle A(\psi_1), \ell_{\psi_2}(A(\psi_3)) \rangle + \text{(cyclic)} \\
&= -\langle \ell_A(A(\psi_3))(\psi_1), \psi_2 \rangle + \langle A(\psi_2), \ell_{\psi_1}(A(\psi_3)) \rangle - \langle A(\psi_1), \ell_{\psi_2}(A(\psi_3)) \rangle + \text{(cyclic)} \\
&= -\langle \ell_A(A(\psi_3))(\psi_1), \psi_2 \rangle + \text{(cyclic)} = 0, \quad (2.25)
\end{aligned}
$$

where "(cyclic)" denotes terms obtained by the cyclic permutation of indices. It follows from Exercise 2.31 that equality (2.25) holds for all $\psi_i \in \widehat{\varkappa}(\pi)$. Criterion 2 is proved.

Rewrite now the Jacobi identity in the form

$$
\langle \ell_{A,\psi_1}(A(\psi_2)), \psi_3 \rangle + \langle A(\psi_1), \ell^*_{A,\psi_3}(\psi_2) \rangle - \langle A(\ell^*_{A,\psi_2}(\psi_1)), \psi_3 \rangle = 0.
$$

Using Exercise 2.33, we obtain

$$
\langle \ell_{A,\psi_1}(A(\psi_2)), \psi_3 \rangle - \langle \ell_{A,\psi_2}(A(\psi_1)), \psi_3 \rangle - \langle A\left(\ell^*_{A,\psi_2}(\psi_1)\right), \psi_3 \rangle = 0.
$$

In view of Exercise 2.31, this implies criterion 3.

Equivalence of criteria 3 and 4 follows from Exercise 2.33.

Finally, criterion 5 is equivalent to criterion 3 by virtue of the following obvious equalities:

$$
\begin{aligned}
[\eth_{A(\psi_2)}, A](\psi_1) &= \ell_{A,\psi_1}(A(\psi_2)), \\
\ell_{A,\psi} \circ A &= \ell_{A(\psi)} \circ A - A \circ \ell_\psi \circ A.
\end{aligned}
$$

This concludes the proof. $\qquad\square$

Let $A \colon \widehat{\varkappa}(\pi) \to \varkappa(\pi)$ be a Hamiltonian operator. For any $\omega \in \bar{H}^n(\pi)$, evolutionary derivation $X_\omega = \eth_{A(\mathsf{E}(\omega))}$ is called the *Hamiltonian vector field* corresponding to the Hamiltonian ω. Obviously,

$$
X_{\omega_1}(\omega_2) = \langle A\mathsf{E}(\omega_1), \mathsf{E}(\omega_2) \rangle = \{\omega_1, \omega_2\}_A.
$$

This yields

$$
\begin{aligned}
X_{\{\omega_1, \omega_2\}_A}(\omega) = \{\{\omega_1, \omega_2\}_A, \omega\}_A &= \{\omega_1, \{\omega_2, \omega\}_A\}_A - \{\omega_2, \{\omega_1, \omega\}_A\}_A \\
&= (X_{\omega_1} \circ X_{\omega_2} - X_{\omega_2} \circ X_{\omega_1})(\omega) = [X_{\omega_1}, X_{\omega_2}](\omega)
\end{aligned}
$$

for all $\omega \in \bar{H}^n(\pi)$. Thus,

$$
X_{\{\omega_1, \omega_2\}_A} = [X_{\omega_1}, X_{\omega_2}]. \tag{2.26}
$$

Similar to the finite-dimensional case, (2.26) implies a result similar to the Noether theorem.

For each $\mathcal{H} \in \bar{H}^n(\pi)$, evolution equation

$$
u_t = A(\mathsf{E}(\mathcal{H})), \tag{2.27}
$$

corresponding to the Hamiltonian $\mathcal{H}$ is called *Hamiltonian evolution equation*.

THEOREM 2.13. *Hamiltonian operators take generating functions of conservation laws of equation* (2.27) *to symmetries of this equation.*

PROOF. Let A be a Hamiltonian operator and

$$\widetilde{\omega}_0(t) + \widetilde{\omega}_1(t) \wedge dt \in \Lambda_0^n(\pi) \oplus \Lambda_0^{n-1}(\pi) \wedge dt$$

be a conserved current of equation (2.27). This means that $\bar{D}_t(\omega_0(t)) = 0$, where $\omega_0(t) \in \bar{H}^n(\pi)$ is the horizontal cohomology class corresponding to the form $\widetilde{\omega}_0(t)$, and $\bar{D}_t$ is the restriction of the total derivative in t to the infinite prolongation of the equation at hand. Further,

$$\bar{D}_t(\omega_0) = \frac{\partial \omega_0}{\partial t} + \eth_{A(\mathsf{E}(\mathcal{H}))}(\omega_0) = \frac{\partial \omega_0}{\partial t} + \{\mathcal{H}, \omega_0\}.$$

This yields

$$\frac{\partial}{\partial t} X_{\omega_0} + [X_{\mathcal{H}}, X_{\omega_0}] = 0.$$

Hence the field $X_{\omega_0} = \eth_{A(\mathsf{E}(\omega_0))}$ is a symmetry of (2.27). It remains to observe that $\mathsf{E}(\omega_0)$ is the generating function of the conservation law under consideration (cf. Exercise 3.2). $\square$

EXERCISE 2.34. a. A skew-symmetric operator

$$B \in E_1^{2,n}(\pi) \subset \mathcal{C}\mathrm{Diff}(\varkappa(\pi), \widehat{\varkappa}(\pi))$$

is called *symplectic* if $d_1^{2,n}(B) = 0$. Prove that the following conditions are equivalent:

1. B is a symplectic operator.
2. $\langle \ell_B(\varphi_1)(\varphi_2), \varphi_3 \rangle + \langle \ell_B(\varphi_2)(\varphi_3), \varphi_1 \rangle + \langle \ell_B(\varphi_3)(\varphi_1), \varphi_2 \rangle = 0$ for all elements $\varphi_1, \varphi_2, \varphi_3 \in \varkappa(\pi)$.
3. $\ell_{B,\varphi_1}(\varphi_2) - \ell_{B,\varphi_2}(\varphi_1) = \ell_{B,\varphi_1}^*(\varphi_2)$ for all $\varphi_1, \varphi_2, \in \varkappa(\pi)$.
4. The expression $\ell_{B,\varphi_1}(\varphi) - \frac{1}{2}\ell_{B,\varphi_1}^*(\varphi_2)$ is symmetric with respect to $\varphi_1, \varphi_2, \in \varkappa(\pi)$.
5. $\ell_B(\varphi) = \ell_{B,\varphi} - \ell_{B,\varphi}^*$, where, as above, the operator $\ell_{B,\varphi} \in \mathcal{C}\mathrm{Diff}(\varkappa(\pi), \widehat{\varkappa}(\pi))$, $\varphi \in \varkappa(\pi)$, is defined by $\ell_{B,\varphi_1}(\varphi_2) = \ell_B(\varphi_2)(\varphi_1)$.

b. Evolution equation $u_t = \varphi$, $\varphi \in \varkappa(\pi)$, is called *Hamiltonian* (with respect to a symplectic operator B), if $B(\varphi) = \mathsf{E}(\mathcal{H})$ for a Hamiltonian $\mathcal{H} \in \Lambda_0^n(\pi)$. Study the relationship between symmetries and conservation laws of such an equation.

3. Computation of conservation laws

In this section, we discuss applications of the $\mathcal{C}$-spectral sequence. To begin with, we show how one computes the space of conservation laws by the method of generating functions based on Theorem 2.11. Generating functions establishes an efficient means for describing conservation laws, i.e., conserved currents modulo trivial ones, and allow one to find *all* conservation laws of equation at hand. Thereupon we present concrete results, to illustrate the technique of computations for conservation laws.

3.1. Basic results. In this subsection, we present the basic results from the theory of conservation laws in coordinate form, suitable to the computations. This material can be read independently of §2.

Let $S = (S_1, \ldots, S_n)$ be a conserved current for the equation $\mathcal{E} = \{F = 0\}$ with n independent and m dependent variables. Equality (1.4), which is satisfied on $\mathcal{E}^\infty$, is equivalent to

$$\sum_{i=1}^{n} D_i(S_i) = \sum_{j=1}^{l} \Box_j(F_j), \tag{3.1}$$

where l is the number of equations, $\Box_j = \sum_\sigma a_\sigma^j D_\sigma$ are scalar differential operators, $F = (F_1, \ldots, F_l)$.

Recall that operators of the form $\sum_\sigma a_\sigma D_\sigma$ are called $\mathcal{C}$-*differential* or *horizontal*. If $\Delta = \sum_\sigma a_\sigma D_\sigma$ is a scalar $\mathcal{C}$-differential operator, then $\Delta^* = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_\sigma$ is the operator (*formally*) *adjoint* to Δ. If $\Delta = \|\Delta_{ij}\|$ is a matrix $\mathcal{C}$-differential operator, then $\Delta^* = \|\Delta_{ji}^*\|$.

The following fundamental result holds (see Theorem 2.11):

THEOREM 3.1. *Suppose that operators $\Box_1, \ldots, \Box_l$ satisfy (3.1). Then the restriction $\psi = (\Box_1^*(1), \ldots, \Box_l^*(1))|_{\mathcal{E}^\infty}$ of the vector function $(\Box_1^*(1), \ldots, \Box_l^*(1))$ to the infinite prolonged equation $\mathcal{E}^\infty$ satisfies the equation*

$$\left(\ell_F^\mathcal{E}\right)^* (\psi) = 0.$$

Note that, generally speaking, the vector function ψ is not uniquely defined by the conserved current S.

EXERCISE 3.1. Check that such an ambiguity takes place for equation (2.18).

If any conserved current S defines a unique vector function ψ for the equation at hand, then ψ is the same for all conserved currents equivalent to S and thus characterizes the corresponding conservation law. Such a vector function is called the *generating function* of this conservation law.

EXERCISE 3.2. Prove that for evolution equations

$$u_t = f(x, u, u_x, u_{xx}, \ldots)$$

the generating function corresponding to a conserved current $(S_0, S_1, \ldots, S_n)$, where S_0 is the t-component, has the form $\psi = \ell_{S_0}^*(1)$. In other words, ψ is the left-hand side of the Euler–Lagrange equations for the Lagrangian S_0 which is understood as a function of $x, u, u_x, u_{xx}, \ldots$.

One can prove that for regular and determined ($l = m$) equations generating functions are uniquely defined by conservation laws. Such equations are called ℓ-*normal* (see §2.6). The majority of equations of mathematical physics are ℓ-normal.

Now the following natural question arises. Given an ℓ-normal equation, is a conservation law uniquely defined by its generating function? To give the answer, note that every de Rham cohomology class $\xi \in H^{n-1}(\mathcal{E})$ can be understood as a conservation law for the equation $\mathcal{E}$. Such *topological* conservation laws are, of course, not very interesting (see, however, [**103**]), since they do not change under continuous deformations of solutions of $\mathcal{E}$. The results of §2.7 imply the following

THEOREM 3.2. *Let $\mathcal{E}$ be an ℓ-normal equation. Two conservation laws of the equation $\mathcal{E}$ have the same generating function if and only if they differ by a topological conservation law.*

The quotient group of all conservation laws by topological ones is called the group of *proper* conservation laws.

Thus, the problem of finding all the conservation laws for a given ℓ-normal equation reduces to the problem of finding the corresponding generating functions, i.e., the solutions of the following equation:

$$\left(\ell_F^{\mathcal{E}}\right)^*(\psi) = 0. \tag{3.2}$$

Not every solution of (3.2) corresponds to a conservation law. The redundant solutions can be removed by using the following fact.

Let ψ be a solution of (3.2). This means that $\ell_F^*(\psi) = \Delta(F)$, where Δ is an $m \times l$ matrix $\mathcal{C}$-differential operator. The solution ψ corresponds to a conservation law if and only if there exists an $l \times l$ matrix $\mathcal{C}$-differential operator ∇ such that

$$\ell_\psi^{\mathcal{E}} + (\Delta|_{\mathcal{E}^\infty})^* = \nabla|_{\mathcal{E}^\infty} \circ \ell_F^{\mathcal{E}},$$
$$\nabla^* = \nabla.$$

This was proved in §2.6 (see Exercise 2.23c and Corollary 2.10).

To conclude, let us describe the action of symmetries of the equation $\mathcal{E} = \{F = 0\}$ on its conservation laws in terms of generating functions. Let $\varphi \in \operatorname{sym}\mathcal{E}$ be a symmetry of $\mathcal{E}$ and $\psi \in \ker\left(\ell_F^{\mathcal{E}}\right)^*$ be the generating function of a conservation law of $\mathcal{E}$. Then $\partial_\varphi(F) = \Delta(F)$, where Δ is a $\mathcal{C}$-differential operator. From Exercise 2.25 it follows that ∂_φ acts on the space of generating functions by the formula $\psi \mapsto \partial_\varphi(\psi) + \Delta^*(\psi)$.

EXERCISE 3.3. Work out an analog of the commutator relation (4.1) of Ch. 4 for conservation laws.

3.2. Examples. Now we give several examples to illustrate the above-described algorithm of computing conservation laws[10].

EXAMPLE 3.1. Let us first consider the Burgers equation:

$$F = u_{xx} + uu_x - u_t = 0.$$

The operator $\ell_F^{\mathcal{E}}$, $\mathcal{E} = \{F = 0\}$, has the form $\ell_F^{\mathcal{E}} = D_x^2 + uD_x + p_1 - D_t$ (we use the notation of §4.1 of Ch. 4), therefore,

$$\left(\ell_F^{\mathcal{E}}\right)^* = D_x^2 - D_x \circ u + p_1 + D_t = D_x^2 - uD_x + D_t.$$

Let us look for a generating function ψ in the form $\psi(x, t, u, \ldots, p_k)$, where $k \geq 0$ and $\partial\psi/\partial p_k \neq 0$. Then, using the formulas for D_x and D_t from §4.1 of Ch. 4, we find that

$$D_x^2(\psi) = p_{k+2}\frac{\partial\psi}{\partial p_k} + O(k+1),$$
$$D_t(\psi) = p_{k+2}\frac{\partial\psi}{\partial p_k} + O(k+1).$$

Thus,

$$\left(\ell_F^{\mathcal{E}}\right)^*(\psi) = 2p_{k+2}\frac{\partial\psi}{\partial p_k} + O(k+1),$$

---

[10]The technique of computation of conservation laws is much the same as that of symmetries. Since the latter is fully considered in Ch. 4, we shall restrict ourselves to brief descriptions of the computations. For details on Examples 3.2, 3.7, 3.9–3.11 see [**137**]

and the equality $\left(\ell_F^{\mathcal{E}}\right)^* (\psi) = 0$ implies that $\partial \psi / \partial p_k = 0$. The contradiction obtained proves that $\psi = \psi(x,t)$ and, hence,

$$\left(\ell_F^{\mathcal{E}}\right)^* (\psi) = \psi_{xx} + \psi_t - u\psi_x = 0.$$

Since ψ does not depend on u, then $\psi_x = 0$. Consequently, $\psi_t = 0$, i.e., $\psi = \text{const}$. It is readily seen that the conserved current $(-u, u_x + u^2/2)$ corresponds to the generating function $\psi = 1$. Thus, we have proved that the group of conservation laws of the Burgers equation is one-dimensional and is generated by the conserved current $(-u, u_x + u^2/2)$.

EXAMPLE 3.2. Consider the n-dimensional quasi-linear isotropic heat equation:

$$u_t = \Delta(u^\alpha) + f(u), \qquad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad \alpha \neq 0.$$

In this case, equation (3.2) immediately implies that ψ depends on the variables t, x only and satisfies the equation

$$\left(\frac{\partial}{\partial t} + \alpha u^{\alpha-1}\Delta\right) \psi + f'(u)\psi = 0. \tag{3.3}$$

It is easily shown that equation (3.3) has a nontrivial solution only if

$$f(u) = au^\alpha + bu + c.$$

Now, if $\alpha \neq 1$, then $\psi = v(x)e^{-bt}$, with $v = v(x)$ satisfying the equation $\Delta v + av = 0$. If $\alpha = 1$, then $f(u) = bu + c$ and the function $\psi = \psi(x,t)$ satisfies the equation $\psi_t + \Delta\psi + b\psi = 0$.

It can easily be checked that the vector function

$$\left(\psi u, \psi_{x_1}u^\alpha - \alpha\psi u^{\alpha-1}u_{x_1} - c\int \psi\,dx_1, \psi_{x_2}u^\alpha - \alpha\psi u^{\alpha-1}u_{x_2}, \ldots,\right.$$
$$\left. \psi_{x_n}u^\alpha - \alpha\psi u^{\alpha-1}u_{x_n}\right).$$

is a conserved current corresponding to the generating function ψ.

EXAMPLE 3.3. If $\mathcal{E} = \{\Delta = 0\}$ is an arbitrary linear system of equations defined by an operator Δ, then each solution of the adjoint system $\mathcal{E}^* = \{\Delta^* = 0\}$ is the generating function of a conservation law of $\mathcal{E}$ (cf. Example 3.2 for $\alpha = 1$).

EXERCISE 3.4. Do the *linear conservation laws* from Example 3.3 form a complete set of conservation laws (see §1) for an ℓ-normal linear equation?

EXAMPLE 3.4 ([**110**]). The equation

$$u_t - \Delta u_t = \Delta u, \qquad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \tag{3.4}$$

which describes the filtration of fluid in crack-porous media, has a complete set of conservation laws. Let us demonstrate this with the example of the following boundary value problem:

$$u(x,0) = \varphi(x), \qquad x \in \bar{V},$$
$$u(x,t) = h(x,t), \qquad x \in \partial V, \quad t \geq 0,$$

where $V \subset \mathbb{R}^n$ is a bounded domain with piecewise smooth boundary ∂V, $\varphi(x) = h(x,0)$ if $x \in \partial V$. Let us look for generating functions $v(x,t)$ of linear conservation laws of equation (3.4), i.e., solutions of the equation

$$v_t - \Delta v_t + \Delta v = 0 \tag{3.5}$$

satisfying the homogeneous boundary conditions

$$v(x,t)|_{\partial V} = 0, \qquad t \geq 0. \tag{3.6}$$

Using the standard Fourier method of separation of variables, we obtain the following complete and orthogonal in $L^2(V)$ system of functions satisfying (3.5) and (3.6):

$$v_k(x,t) = e^{\lambda_k t} X_k(x), \qquad \lambda_k = \frac{\bar{\lambda}_k}{1 + \bar{\lambda}_k}, \qquad k = 1, 2, 3, \ldots,$$

where $\bar{\lambda}_k$ and $X_k(x)$ are eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$\begin{cases} \Delta X + \bar{\lambda} X = 0, \\ X|_{\partial V} = 0 \end{cases}$$

respectively. The conservation law corresponding the generating function $v(x,t)$ in the integral form reads:

$$-\frac{d}{dt} \int_V (v - \Delta v) u \, dx = \int_{\partial V} \left((u_t + u) \frac{\partial v}{\partial \boldsymbol{n}} - \left(\frac{\partial u_t}{\partial \boldsymbol{n}} + \frac{\partial u}{\partial \boldsymbol{n}} \right) v \right) d\sigma, \tag{3.7}$$

where $\boldsymbol{n}$ is a unit external vector normal to ∂V and $d\sigma$ is a surface area element. Given $v(x,t)$, one can compute the right-hand side of (3.7), so that the value of the integral $\int_V (v - \Delta v) u \, dx$ at the initial moment $t = 0$ uniquely determines its value at each instant of time. This makes it possible to find the Fourier coefficients $c_k(t)$ of $u(x,t)$ with respect to the orthogonal system $\{v_k\}$, since

$$c_k(t) = \frac{1}{\|v_k\|^2} \int_V v_k u \, dx = \frac{1}{(1 + \bar{\lambda}_k)\|v_k\|^2} \int_V (v_k - \Delta v_k) u \, dx.$$

Thus, completeness of the system of functions in the space $L^2(V)$ implies completeness of the set of conservation laws with generating functions $v_k(x,t)$.

EXAMPLE 3.5. It is well known that the Korteweg–de Vries equation

$$u_t = 6uu_x - u_{xxx}$$

possesses an infinite series of conservation laws (see e.g., [**81, 1, 89, 91, 21, 24**]), which starts with the conservation laws of mass, momentum, and energy:

$$u_t + (-3u^2 + u_{xx})_x = 0,$$
$$(u^2)_t + (-4u^3 + 2uu_{xx} - u_x^2)_x = 0,$$
$$(u^3 + u_x^2/2)_t + (-9u^4/2 + 3u^2 u_{xx} - 6uu_x^2 + u_x u_{xxx} - u_{xx}^2/2)_x = 0.$$

This infinite series of conservation laws has been constructed in various ways by many authors. The solution of this problem gave birth to the famous inverse scattering method (which today is dealt with in an extensive literature, see, e.g., the books [**1, 89, 22, 21, 24, 28**] and the references given there). The computation technique described above allows us not only to obtain easily the infinite series of conservation laws, but also to prove that the Korteweg–de Vries equation has no other conservation laws.

EXERCISE 3.5. Prove that an evolution equation $u_t = f(x, u_x, u_{xx}, \dots)$ of *even* order cannot have an infinite number of conservation laws that depend on derivatives of arbitrarily high order.

EXAMPLE 3.6. The nonlinear Schrödinger equation

$$i\psi_t = \Delta\psi + |\psi|^2\psi, \qquad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

has two physically obvious conservation laws:

$$(|\psi|^2)_t + i\nabla(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}) = 0, \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right);$$

$$(|\nabla\psi|^2 - \frac{1}{2}|\psi|^4)_t + i\nabla((\Delta\psi + |\psi|^2\psi)\nabla\bar{\psi} - (\Delta\bar{\psi} + |\psi|^2\bar{\psi})\nabla\psi) = 0.$$

If the number of spatial variables equals 1, then, like the Korteweg–de Vries equation, this equation possesses a well-known infinite series of conservation laws. For $n > 1$, there are no other conservation laws of the nonlinear Schrödinger equation.

REMARK 3.1. The explosion of interest to the Korteweg–de Vries equation and the nonlinear Schrödinger equation in the late 60s–early 70s has led to the discovery of a great deal of "integrable" equations. Such an equation possesses a complete set of conservation laws and is solvable by the inverse scattering method. The Boussinesq, Kadomtsev–Petviashvili, Harry Dym, sine-Gordon, etc., equations are the examples.

EXAMPLE 3.7. Consider the Zakharov equations describing the nonlinear interaction of high-frequency and low-frequency waves:

$$\begin{cases} i\psi_t + \psi_{xx} - n\psi = 0, \\ n_t + u_x = 0, \\ u_t + n_x + (|\psi|^2)_x = 0. \end{cases} \tag{3.8}$$

Though this system has soliton-like solutions, it is not integrable by the inverse scattering method. The computation of conservation laws by direct solving of (3.2) is hardly possible in this case. In fact, one needs the Hamiltonian technique (see §4.2) to prove that (3.8) is not integrable.

Equation (3.8) has eight conservation laws:

$(1/2i(\psi\bar{\psi}_x - \bar{\psi}\psi_x) - nu, 1/2(|\psi|^2)_{xx} - 2|\psi_x|^2 - n|\psi|^2 - n^2/2 - u^2/2)$
(conservation law of quasi-momentum).

$(|\psi_x|^2 + n|\psi|^2 + n^2/2 + u^2/2,$

$\qquad 1/i(\psi_{xx}\bar{\psi}_x - \bar{\psi}_{xx}\psi_x) + n/i(\bar{\psi}\psi_x - \bar{\psi}_x\psi)_x + nu - u|\psi|^2)$
(conservation law of quasi-energy).

$(|\psi|^2, i(\bar{\psi}_x\psi - \bar{\psi}\psi_x))$ (conservation law of quasi-particle number).

$(t|\psi|^2 + tn - xu, t/i(\bar{\psi}\psi_x - \psi\bar{\psi}_x) - x|\psi|^2 - xn + tu).$

$(t^2|\psi|^2 + (x^2 + t^2)n - 2txu, t^2/i(\bar{\psi}\psi_x - \psi\bar{\psi}_x) - 2tx|\psi|^2 - 2txn + (x^2 + t^2)u).$

$(tu - xn, tn - xu + t|\psi|^2).$

$(n, u).$

$(u, n + |\psi|^2).$

EXAMPLE 3.8 ([**104**]). The plasticity equations (see §4.3 of Ch. 4)

$$\begin{cases} u_\xi + \dfrac{1}{2}v = 0, \\ v_\eta + \dfrac{1}{2}u = 0, \end{cases}$$

possess an infinite series of conservation laws obtained from the conservation law
with the generating function

$$T_0 = \begin{pmatrix} u \\ -v \end{pmatrix}$$

by using the action of the symmetries f_{2k}^i, g_{2k}^i, $0 \le i \le 2k$, and the operators
$\left(\mathcal{R}_2^{2k}\right)^*$, $k = 0, 1, 2, \dots$ (see Theorem 4.5 of Ch. 4). This series, together with the
conservation laws corresponding to generating functions $(B_1(\xi, \eta), B_2(\xi, \eta))$ which
are the solutions of the system

$$\begin{cases} \dfrac{\partial B_1}{\partial \xi} - \dfrac{1}{2}B_2 = 0, \\ \dfrac{\partial B_2}{\partial \eta} - \dfrac{1}{2}B_1 = 0, \end{cases}$$

generate the whole space of conservation laws of the plasticity equations.

EXERCISE 3.6. Check that $(u\psi_1, v\psi_2)$ is a conserved current corresponding to
the generating function (ψ_1, ψ_2).

EXAMPLE 3.9. The Khokhlov–Zabolotskaya equation encountered in the non-
linear acoustics of bounded beams (see §5.3 of Ch. 3) has the form

$$-\frac{\partial^2 u}{\partial q_1 \partial q_2} + \frac{1}{2}\frac{\partial^2 (u^2)}{\partial q_1^2} + \frac{\partial^2 u}{\partial q_3^2} + \frac{\partial^2 u}{\partial q_4^2} = 0.$$

It has a large group of conservation laws, with the corresponding generating func-
tions being of the form:

$$\psi = q_1\mathcal{A}(q_2, q_3, q_4) + \mathcal{B}(q_2, q_3, q_4),$$

where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary solutions of the system

$$\begin{cases} \mathcal{A}_{q_3 q_3} + \mathcal{A}_{q_4 q_4} = 0, \\ \mathcal{B}_{q_3 q_3} + \mathcal{B}_{q_4 q_4} = \mathcal{A}_{q_2}. \end{cases}$$

The conserved current

$$((q_1\mathcal{A} - \mathcal{B})(uu_{q_1} - u_{q_2}) - \mathcal{A}u^2/2, \mathcal{A}u, (q_1\mathcal{A} + \mathcal{B})u_{q_3} - (q_1\mathcal{A}_{q_3} + \mathcal{B}_{q_3})u,$$
$$(q_1\mathcal{A} + \mathcal{B})u_{q_4} - (q_1\mathcal{A}_{q_4} + \mathcal{B}_{q_4})u).$$

corresponds to the generating function ψ.

The two-dimensional analog of the Khokhlov–Zabolotskaya equation is of the form:

$$-\frac{\partial^2 u}{\partial q_1 \partial q_2} + \frac{1}{2}\frac{\partial^2(u^2)}{\partial q_1^2} + \frac{\partial^2 u}{\partial q_3^2} = 0.$$

The generating functions of conservation laws for this equation are

$$\psi = q_1 a(q_2)q_3 + a'(q_2)q_3^3/6 + q_1 b(q_2) + b'(q_2)q_3^2/2 + c(q_2)q_3 + d(q_2), \qquad (3.9)$$

where a, b, c, and d are arbitrary functions of q_2.

For the axially symmetric Khokhlov–Zabolotskaya equation

$$-\frac{\partial^2 u}{\partial q_1 \partial q_2} + \frac{1}{2}\frac{\partial^2(u^2)}{\partial q_1^2} + \frac{\partial^2 u}{\partial q_3^2} + \frac{1}{q_3}\frac{\partial u}{\partial q_3} = 0$$

the generating functions of conservation laws have the form:

$$\psi = q_1 a(q_2)q_3 \ln q_3 + \frac{1}{4}a'(q_2)q_3^3(\ln q_3 - 1)$$

$$+ q_1 b(q_2)q_3 + \frac{1}{4}b'(q_2)q_3^3 + c(q_2)q_3 \ln q_3 + d(q_2)q_3, \quad (3.10)$$

where a, b, c, and d are again arbitrary functions of q_2.

EXERCISE 3.7. Find conserved currents corresponding to generating functions (3.9) and (3.10).

EXAMPLE 3.10. The Navier–Stokes equations of viscous incompressible fluid motion in a three-dimensional domain

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} = 0, \end{cases}$$

where $\boldsymbol{u} = (u^1, u^2, u^3)$ is the vector of fluid velocity and p is the pressure, has a 7-dimensional space of conservation laws. The basis generating functions are of the form:

$$\psi_1 = (x_2, -x_1, 0, u^1 x_2 - u^2 x_1),$$
$$\psi_2 = (x_3, 0, -x_1, u^1 x_3 - u^3 x_1),$$
$$\psi_3 = (0, x_3, -x_2, u^2 x_3 - u^3 x_2),$$
$$\psi_4 = (a_1(t), 0, 0, u^1 a_1(t) - a_1'(t)x_1),$$
$$\psi_5 = (0, a_2(t), 0, u^2 a_2(t) - a_2'(t)x_2),$$
$$\psi_6 = (0, 0, a_3(t), u^3 a_3(t) - a_3'(t)x_3),$$
$$\psi_7 = (0, 0, 0, f(t)),$$

where a_1, a_2, a_3, and f are arbitrary functions of t.

EXERCISE 3.8. Find conserved currents for these generating functions and explain their physical meaning.

EXAMPLE 3.11. The Kadomtsev–Pogutse equations (see §5.3.3 of Ch. 3), describing nonlinear processes in high-temperature plasma, are written as follows:

$$\begin{cases} \dfrac{\partial \psi}{\partial t} + [\nabla_\perp \varphi, \nabla_\perp \psi]_z = \dfrac{\partial \varphi}{\partial z}, \\ \dfrac{\partial}{\partial t} \Delta_\perp \varphi + [\nabla_\perp \varphi, \nabla_\perp \Delta_\perp \varphi]_z = \dfrac{\partial}{\partial t} \Delta_\perp \psi + [\nabla_\perp \psi, \nabla_\perp \Delta_\perp \psi]_z, \end{cases}$$

where $\nabla_\perp = (\partial/\partial x, \partial/\partial y)$, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $[u, v]_z = u_x v_y - u_y v_x$, (x, y, z, t) are the standard coordinates in the space-time, φ and ψ are the potentials of the velocity and the cross-component of the magnetic field respectively.

Here are the generating functions of conservation laws for this equation depending on the derivatives of order no greater than three[11]:

$$\theta_1 = ((ax + by + c)G(z, t), 0),$$
$$\theta_2 = (\alpha(x^2 + y^2), -4\alpha),$$
$$\theta_3 = (\beta(x^2 + y^2), 4\beta),$$
$$\theta_4 = (\gamma(\varphi - \psi), \gamma(\Delta_\perp \psi - \Delta_\perp \varphi)),$$
$$\theta_5 = (\delta(\varphi + \psi), \delta(\Delta_\perp \varphi + \Delta_\perp \psi)),$$

where $G = G(z, t)$, $\alpha = \alpha(z - t)$, $\beta = \beta(z + t)$, $\gamma = \gamma(z - t)$, $\delta = \delta(z + t)$, a, b, c are arbitrary constant.

EXERCISE 3.9. Find conserved currents for the functions $\theta_1, \ldots, \theta_5$. What is the physical meaning of the conservation law corresponding to θ_4?

EXERCISE 3.10. Show that the transformation $t \mapsto -z$, $\varphi \mapsto -\psi$, which is a discrete symmetry of the Kadomtsev–Pogutse equations, takes the conservation laws with generating functions θ_2 and θ_4 to the conservation laws with generating functions θ_3 and θ_5 respectively.

4. Symmetries and conservation laws

4.1. The Noether theorem. It is well known that for equations derived from a variational principle the conservation laws are produced by the symmetries of the action. This is the Noether theorem [**90**]. Since prior to the works [**125, 127, 132**] on the $\mathcal{C}$-spectral sequence this theorem had remained the only general method of finding conservation laws, until the present time it is widely believed that the existence of conserved quantities are always caused by symmetry properties of the equation at hand. However, the examples of §3 demonstrate that, in general, this is not true. For instance, the Burgers equation has only one conservation law, whereas its symmetry algebra is infinite-dimensional (see Theorem 4.4 of Ch. 4).

On the other hand, symmetries and conservation laws cannot be treated as totally independent concepts, since the generating functions of symmetries and of conservation laws satisfy the mutually adjoint equations (3.19) of Ch. 4 and (3.2). This fact clarifies the nature of the Noether theorem. Recall (see §2.5) that an equation $F = 0$ is derived (locally) from a variational principle if and only if

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[11]The Kadomtsev–Pogutse equations do not supposedly have other conservation law.

$\ell_F = \ell_F^*$. Therefore, in this case equations (3.19) of Ch. 4 and (3.2) coincide and, consequently, to every conservation law there corresponds a symmetry. It follows from equality (2.22) that this correspondence is inverse to the Noether map that takes each Noether symmetry of the action to a conservation law of the equation at hand.

Recall that a *Lagrangian* (or an *action*, or a *variational functional*) $\mathcal{L} = \int L(x, u, p_\sigma^i) \, dx$, where $dx = dx_1 \wedge \cdots \wedge dx_n$, is a functional on the set of sections of a bundle π and is of the form

$$s \mapsto \int_M L\left(x, s(x), \ldots, \frac{\partial^{|\sigma|} s^i}{\partial x_\sigma}, \ldots\right) \, dx_1 \wedge \cdots \wedge dx_n, \qquad s \in \Gamma(\pi).$$

The function $L \in \mathcal{F}(\pi)$ (or the form $L \, dx \in \Lambda_0^n(\pi)$) is called the *density* of the Lagrangian. The *Lagrangian* (or *variational*) *derivative* of the Lagrangian $\mathcal{L} = \int L \, dx$ is the function

$$\frac{\delta L}{\delta u^i} = \sum_\sigma (-1)^{|\sigma|} D_\sigma \left(\frac{\partial L}{\partial p_\sigma^i}\right) \in \mathcal{F}(\pi).$$

EXERCISE 4.1. Check that the Lagrangian derivative $\delta L / \delta u^i$ is uniquely determined by the functional $\mathcal{L} = \int L \, dx$ and does not depend on the choice of the density L.

The operator $\mathsf{E}(\mathcal{L}) = (\delta L / \delta u^1, \ldots, \delta L / \delta u^m)$ is called the *Euler operator*, while the equation $\mathsf{E}(\mathcal{L}) = 0$ is called the Euler–Lagrange equation corresponding to the Lagrangian $\mathcal{L}$. In §2.5 we have proved that an Euler–Lagrange equation is trivial if and only if the corresponding density L is a total divergence, i.e., if for some functions $P_1, P_2, \ldots, P_n \in \mathcal{F}(\pi)$ one has

$$L = D_1(P_1) + D_2(P_2) + \cdots + D_n(P_n).$$

Furthermore, it was shown in §2.5 that an equation $F = 0$ is an Euler–Lagrange equation if and only if

$$\ell_F = \ell_F^*.$$

An evolutionary derivation ∂_φ is called a *Noether symmetry* of a Lagrangian $\mathcal{L}$, if $\partial_\varphi(\mathcal{L}) = 0$, i.e., if there exist functions $P_1, P_2, \ldots, P_n \in \mathcal{F}(\pi)$ such that

$$\partial_\varphi(L) = D_1(P_1) + D_2(P_2) + \cdots + D_n(P_n).$$

If $F = 0$ is an Euler–Lagrange equation corresponding to a Lagrangian $\mathcal{L}$, i.e., $F = \mathsf{E}(\mathcal{L})$, then ∂_φ is a Noether symmetry of $\mathcal{L}$ if and only if

$$\partial_\varphi(F) + \ell_\varphi^*(F) = 0 \tag{4.1}$$

(see Exercise 2.30). In particular, φ is a symmetry of the equation $F = 0$ (the converse statement is *not* true!).

The results of §2.8 imply the following fundamental theorem:

THE NOETHER THEOREM. *Suppose that $\mathcal{E} = \{\mathsf{E}(\mathcal{L}) = 0\}$ is an ℓ-normal Euler–Lagrange equation corresponding to a Lagrangian $\mathcal{L}$. Then an evolutionary derivation ∂_φ is a Noether symmetry of $\mathcal{L}$ if and only if φ is a generating function of a conservation law for $\mathcal{E}$*[12].

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[12]As noted above, equations encountered in gauge theories, like the Maxwell, Yang–Mills, and Einstein equations, make up an important class of Euler–Lagrange equations which are not

EXAMPLE 4.1. Consider the sine-Gordon equation

$$u_{xy} = \sin u.$$

It is an Euler–Lagrange equation (check the condition $\ell_F = \ell_F^*$!), the corresponding Lagrangian density being equal to

$$L = \frac{1}{2} u_x u_y - \cos u.$$

EXERCISE 4.2. Using (4.1), show that

$$\varphi_1 = u_x,$$

$$\varphi_2 = u_{xxx} + \frac{1}{2} u_x^3,$$

$$\varphi_3 = u_{xxxxx} + \frac{5}{2} u_x^2 u_{xxx} + \frac{5}{2} u_x u_{xx}^2 + \frac{3}{8} u_x^5$$

are Noether symmetries of the Lagrangian under consideration.

REMARK 4.1. The symmetries φ_2 and φ_3 are produced from φ_1 by means of the recursion operator $R = D_x^2 + u_x^2 - u_x D_x^{-1} \cdot u_{xx}$: $\varphi_2 = R(\varphi_1)$, $\varphi_3 = R(\varphi_2) = R^2(\varphi_1)$.

By the Noether theorem, the function φ_1, φ_2, and φ_3 are the generating functions of conservation laws for the sine-Gordon equation.

EXAMPLE 4.2. Consider the problem on the motion of r particles of masses $m_1, \ldots, m_r$ in a potential field. Let $x^j = (x_1^j, x_2^j, x_3^j)$ be the coordinates of j-th particle. It is well-known that the Newton equations of motion

$$m_j x_{tt}^j = -\operatorname{grad}_j U = \left(-\frac{\partial U}{\partial x_1^j}, -\frac{\partial U}{\partial x_2^j}, -\frac{\partial U}{\partial x_3^j} \right), \qquad j = 1, \ldots, r,$$

where $U(t, x_i^j)$ is the potential energy, are Euler–Lagrange equations with the Lagrangian density

$$L = \sum_{j=1}^{r} \frac{1}{2} m_j ((\dot{x}_1^j)^2 + (\dot{x}_2^j)^2 + (\dot{x}_3^j)^2) - U.$$

In this example, we consider the classical Noether symmetries with generating functions

$$\varphi_l^k = b_l^k(t, x_i^j) - a(t, x_i^j) \dot{x}_l^k, \qquad k = 1, \ldots, r, \quad l = 1, 2, 3.$$

Moreover, we restrict ourselves to symmetries satisfying the condition

$$X^{(1)}(L \, dt) = 0,$$

where $X = a(t, x_i^j) \partial/\partial t + \sum_{k,l} b_l^k(t, x_i^j) \partial/\partial x_l^k$ and $X^{(1)}$ is the lifting of X to the space of 1-jets (see Ch. 3).

By the Noether theorem, this symmetry gives rise to the first integral

$$\sum_{i,j} m_j b_i^j \dot{x}_i^j - aE = \text{const},$$

where $E = \sum_{j=1}^{r} m/2((\dot{x}_1^j)^2 + (\dot{x}_2^j)^2 + (\dot{x}_3^j)^2) + U$ is the total energy (check this).

ℓ-normal. For such an equation the kernel of the Noether map, i.e., the set of Noether symmetries that produce trivial conservation laws, coincides with the set of gauge symmetries.

If the potential energy U does not explicitly depend on the time, i.e., $\partial U/\partial t = 0$, then, as is easy to see, $\varphi_i^j = \dot{x}_i^j$ is a Noether symmetry. The resulting first integral is the energy $E = \text{const}$. Here we have a commonly encountered situation: the conservation of energy manifests of symmetry under time translations.

If the function U is invariant under spatial shifts in a fixed direction $l = (l_1, l_2, l_3) \in \mathbb{R}^3$, then $\varphi_i^j = l_i$ is a Noether symmetry. The corresponding first integral is the momentum

$$\sum_{i,j} m_j l_i \dot{x}_i^j = \text{const}.$$

We see that conservation of momentum follows from invariance under spatial translations.

Yet another example of this kind: invariance of U under rotations implies conservation of angular momentum. Take, e.g., the z-axis. The corresponding Noether symmetry has the form $\varphi_1^j = -x_2^j$, $\varphi_2^j = x_1^j$, $\varphi_3^j = 0$. It implies conservation of the z-component of angular momentum:

$$\sum_j m_j (x_1^j \dot{x}_2^j - \dot{x}_1^j x_2^j) = \text{const}.$$

REMARK 4.2. From our discussion of the Noether theorem it is readily seen that the connection between symmetries and conservation laws, given by the inverse Noether theorem, can be generalized to equations $\mathcal{E} = \{F = 0\}$ satisfying the condition

$$\left(\ell_F^{\mathcal{E}}\right)^* = \lambda \ell_F^{\mathcal{E}}$$

for a function $\lambda \in \mathcal{F}(\mathcal{E})$. The simplest example of this kind is the equation $u_x = u_y$. It is skew-adjoint: $\left(\ell_F^{\mathcal{E}}\right)^* = -\ell_F^{\mathcal{E}}$.

4.2. Hamiltonian equations. In this subsection we discuss Hamiltonian differential equations. Further details are available, for example, in [**126, 8, 21, 24, 64, 91, 81**].

An $m \times m$ matrix $\mathcal{C}$-differential operator $A = \|A^{ij}\|$ is called *Hamiltonian* if the corresponding Poisson bracket

$$\{\mathcal{L}_1, \mathcal{L}_2\}_A = \int \sum_{i,j} A^{ij} \left(\frac{\delta \mathcal{L}_1}{\delta u^j}\right) \frac{\delta \mathcal{L}_2}{\delta u^i} \, dx,$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are Lagrangians, defines a Lie algebra structure on the space of Lagrangians. The skew-symmetry of the Poisson bracket is equivalent to the skew-adjointness of the operator A, i.e., to the equality $A^* = -A$. To check the Jacobi identity one uses the following equivalent criteria (see §2):

1. $\langle \ell_A(A(\psi_1))(\psi_2), \psi_3 \rangle + \langle \ell_A(A(\psi_2))(\psi_3), \psi_1 \rangle + \langle \ell_A(A(\psi_3))(\psi_1), \psi_2 \rangle = 0$ for all $\psi_1, \psi_2, \psi_3 \in \widehat{\varkappa}(\pi)$.
2. $\ell_{A,\psi_1}(A(\psi_2)) - \ell_{A,\psi_2}(A(\psi_1)) = A(\ell_{A,\psi_2}^*(\psi_1))$ for all $\psi_1, \psi_2 \in \widehat{\varkappa}(\pi)$.
3. The expression $\ell_{A,\psi_1}(A(\psi_2)) + \frac{1}{2} A(\ell_{A,\psi_1}^*(\psi_2))$ is symmetric with respect to $\psi_1, \psi_2 \in \widehat{\varkappa}(\pi)$.
4. $[\partial_{A(\psi)}, A] = \ell_{A(\psi)} \circ A + A \circ \ell_{A(\psi)}^*$ for all $\psi \in \operatorname{im} \mathsf{E}$.

Here $\psi_i = (\psi_i^1, \ldots, \psi_i^n)$ are vector functions, $\ell_{A,\psi}$ is $n \times n$ matrix $\mathcal{C}$-differential operator defined by the formula

$$(\ell_{A,\psi})^{ij}_\tau = \sum_{k,\sigma} \frac{\partial A_\sigma^{ik}}{\partial p_\tau^j} D_\sigma(\psi^k) D_\tau.$$

Moreover, it is sufficient to verify conditions 1–3 for elements $\psi_i \in \text{im}\, \mathsf{E}$ only.

EXERCISE 4.3. a. Write down the Hamiltonian criteria in coordinate form.
b. Show that $\ell^*_{A,\psi_1}(\psi_2) = \ell^*_{A^*,\psi_2}(\psi_1)$ (see Exercise 2.33).

EXAMPLE 4.3. Consider the case of one dependent and one independent variable $n = m = 1$. The simplest Hamiltonian operator of order one is the operator D_x.

EXERCISE 4.4. Describe all Hamiltonian operators of order one in this case.

EXAMPLE 4.4. It is clear that any skew-adjoint operator with coefficients dependent on x only is a Hamiltonian operator.

EXERCISE 4.5. Prove that in the case $n = 1$ the operator of the form $A = 2LD_x + D_x(L)$, where L is a symmetric matrix with elements dependent on x and u^i only, is Hamiltonian.

EXERCISE 4.6. Prove that in the case $n = m = 1$ there are two-parametric family of Hamiltonian operators of order three

$$\Gamma_{\alpha,\beta} = D_x^3 + (\alpha + \beta u)D_x + \frac{1}{2}\beta u_x. \tag{4.2}$$

An evolution differential equation is said to be *Hamiltonian* (with respect to a Hamiltonian operator A), if it has the form

$$u_t^i = A^{ij}\frac{\delta \mathcal{H}}{\delta u^j}$$

for some action functional $\mathcal{H}$, called the *Hamiltonian*. One can prove (see Theorem 2.13) that the operator A takes the generating functions of conservation laws of a Hamiltonian equation to its symmetries.

EXAMPLE 4.5. The Zakharov equations (3.8)

$$\begin{cases} i\psi_t + \psi_{xx} - n\psi = 0, \\ n_t + u_x = 0, \\ u_t + n_x + \left(|\psi|^2\right)_x = 0 \end{cases}$$

are Hamiltonian with respect to the operator

$$A = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -D_x \\ 0 & 0 & -D_x & 0 \end{pmatrix}$$

with the energy functional

$$\mathcal{H} = \int \left(|\psi_x|^2 + n|\psi|^2 + \frac{n^2}{2} + \frac{u^2}{2}\right) dx$$

being the Hamiltonian.

EXERCISE 4.7. Using Example 3.7, find all Hamiltonian symmetries of the Zakharov equations[13].

EXAMPLE 4.6. The Korteweg–de Vries equation

$$u_t = 6uu_x - u_{xxx}$$

can be written in the Hamiltonian form in two different ways. The first Hamiltonian structure is quite obvious:

$$u_t = D_x(3u^2 - u_{xx}) = D_x \left(\frac{\delta}{\delta u} \left(u^3 + \frac{1}{2}u_x^2 \right) \right) = A_1 \frac{\delta}{\delta u} \mathcal{H}_1,$$

where $A_1 = D_x$ is the Hamiltonian operator, $\mathcal{H}_1 = \int (u^3 + 1/2 u_x^2) \, dx$ is the Hamiltonian.

The second Hamiltonian structure is written as follows:

$$u_t = (D_x^3 - 4uD_x - 2u_x) \left(\frac{\delta}{\delta u} \left(-\frac{u^2}{2} \right) \right) = A_2 \frac{\delta}{\delta u} \mathcal{H}_0.$$

Here $A_2 = D_x^3 - 4uD_x - 2u_x$, $\mathcal{H}_0 = \int (-u^2/2) \, dx$. The Hamiltonian operator A_2 belongs to the family (4.2) with $A_2 = \Gamma_{0,4}$.

We know (see §5.2 of Ch. 3) that the basis of the space of classical symmetries for the Korteweg–de Vries equation has the form:

$$\varphi_1 = u_x \qquad \text{(translation along } x\text{)},$$
$$\varphi_2 = u_t \qquad \text{(translation along } t\text{)},$$
$$\varphi_3 = 6tu_x + 1 \qquad \text{(Galilean boost)},$$
$$\varphi_4 = xu_x + 3tu_t + 2u \qquad \text{(scale symmetry)}.$$

The first three symmetries are Hamiltonian with respect to the operator $A_1 = D_x$, i.e., they are of the form

$$\varphi_i = A_1 \frac{\delta}{\delta u} \mathcal{Q}_i, \qquad i = 1, 2, 3, \tag{4.3}$$

where

$$\mathcal{Q}_1 = \int \frac{u^2}{2} \, dx,$$
$$\mathcal{Q}_2 = \int \left(u^3 + \frac{u_x^2}{2} \right) \, dx,$$
$$\mathcal{Q}_3 = \int (xu + 3tu^2) \, dx.$$

Thus, $\mathcal{Q}_i$ are conserved quantities ($\psi_i = \delta \mathcal{Q}_i / \delta u$ are the corresponding generating functions). The fourth symmetry φ_4 is not of the form (4.3) and does not correspond to a conservation law. Note that the Hamiltonian operator $A_1 = D_x$ has a one-dimensional kernel, so that there exists one more conservation law with the generating function $\psi = 1$ (which is the conservation law of mass $\mathcal{Q}_0 = \int u \, dx$).

Let us turn to the second Hamiltonian operator $A_2 = D_x^3 - 4uD_x - 2u_x$. In this case, the symmetries φ_1, φ_2, and φ_4 are Hamiltonian:

$$\varphi_i = A_2 \frac{\delta}{\delta u} \mathcal{P}_i, \qquad i = 1, 2, 4,$$

---

[13]In fact, the Zakharov equations have no other symmetries.

where

$$\mathcal{P}_1 = \int -\frac{u}{2}\, dx = -\frac{\mathcal{Q}_0}{2},$$

$$\mathcal{P}_2 = \int -\frac{u^2}{2}\, dx = -\mathcal{Q}_1,$$

$$\mathcal{P}_4 = \int \left(-\frac{xu}{2} - \frac{3tu^2}{2} \right) dx = -\frac{\mathcal{Q}_3}{2}.$$

The symmetry φ_3 is not Hamiltonian.

None of the classical symmetries corresponds to the conservation law of energy $\mathcal{Q}_2$. Therefore this conservation law is produced by a higher symmetry, the corresponding generating function being of the form

$$\varphi_5 = A_2 \frac{\delta}{\delta u}\mathcal{Q}_2 = -u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} - 30u^2 u_x.$$

It turns out that the symmetry φ_5 satisfies the Hamiltonian condition (4.3) for the operator A_1, and the corresponding functional

$$\mathcal{Q}_5 = \int \left(-\frac{u_{xx}^2}{2} - 5uu_x^2 - \frac{5u^4}{2} \right) dx$$

is one more conservation law for the Korteweg–de Vries equation. The procedure whereby the conserved quantity $\mathcal{Q}_5$ is obtained from the conserved quantity $\mathcal{Q}_2$ can be applied to $\mathcal{Q}_5$ to give a new conservation law, and so on. In other words, we have the following

PROPOSITION 4.1. *There exists an infinite series of conserved quantities of the Korteweg–de Vries equation* $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$, ... , *such that*

1. $\mathcal{H}_0 = \int -u^2/2\, dx$.
2. $A_1\delta/\delta u(\mathcal{H}_i) = A_2\delta/\delta u(\mathcal{H}_{i-1})$.
3. *The functionals* $\mathcal{H}_i$ *are mutually in involution with respect to both the Poisson brackets*:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_{A_1} = \{\mathcal{H}_k, \mathcal{H}_l\}_{A_2} = 0 \qquad \textit{for all } k, l \geq 0.$$

4. *The symmetries* $\varphi_i = A_1\delta/\delta u(\mathcal{H}_i) = A_2\delta/\delta u(\mathcal{H}_{i-1})$, $i > 0$, *corresponding to the Hamiltonians* $\mathcal{H}_i$ *mutually commute*:

$$\{\varphi_k, \varphi_l\} = 0 \qquad \textit{for all } k, l > 0.$$

PROOF. The proof is left to the reader as an exercise.   □

Note that the operator $\mathcal{R} = A_2 \circ A_1^{-1} = D_x^2 - 4u - 2u_x D_x^{-1}$ takes the symmetry φ_k to the symmetry φ_{k+1}.

EXERCISE 4.8. Prove that $\mathcal{R}$ is a recursion operator (see Remark 4.3 of Ch. 4) for the Korteweg–de Vries equation (the Lenard recursion operator).

EXERCISE 4.9. The Harry Dym equation has the form

$$u_t = \left(\frac{1}{\sqrt{u}} \right)_{xxx}.$$

Show that this equation possesses two Hamiltonian structures:

$$u_t = A_1\frac{\delta}{\delta u}\mathcal{H}_1 = A_2\frac{\delta}{\delta u}\mathcal{H}_0,$$

where $A_1 = 2uD_x + u_x$, $\mathcal{H}_1 = \int \left(\dfrac{u_{xx}}{2\sqrt{u^3}} - \dfrac{5u_x^2}{8\sqrt{u^5}} \right) dx$, $A_2 = D_x^3 = \Gamma_{0,0}$, $\mathcal{H}_0 = \int 2\sqrt{u}\, dx$. Study the symmetries and conservation laws of this equation and prove the analog of Proposition 4.1 for it.

CHAPTER 6

Nonlocal symmetries

1. Coverings

In preceding chapters we dealt with local objects (symmetries, conservation laws, etc.), i.e., with the objects depending on unknown functions $u^1, \ldots, u^m$ in a differential way. For example, generating functions of symmetries, being the elements of the module $\mathcal{F}(\pi, \pi)$, are differential operators in the bundle π. A natural way of generalization is to consider "integro-differential" dependencies, similar to extension of the base field in the theory of algebraic equation. This formal, at first glance, step leads to rather general geometric constructions, which are called *coverings* over infinitely prolonged differential equations [**142, 61, 62**]. Translating to the language of coverings the main elements of the above constructed theory, we arrive to the theory of nonlocal symmetries and nonlocal conservation laws for differential equations. It so happens that many well-known constructions (various differential substitutions, Bäcklund transformations, recursion operators, etc.) are elements of the covering theory [**62**].

1.1. First examples. Consider several examples clarifying how nonlocal objects arise.

EXAMPLE 1.1. Consider the Burgers equation $\mathcal{E} = \{u_t = uu_x + u_{xx}\}$. The algebra $\operatorname{sym} \mathcal{E}$ of local symmetries for this equation is identified with the kernel of the operator $\bar{\ell}_F = \bar{D}_x^2 + p_0 \bar{D}_x + p_1 - \bar{D}_t$ (see Ch. 4), where

$$\bar{D}_x = \frac{\partial}{\partial x} + \sum_k p_{k+1} \frac{\partial}{\partial p_k}, \qquad \bar{D}_t = \frac{\partial}{\partial t} + \sum_k \bar{D}_x^k (p_0 p_1 + p_2) \frac{\partial}{\partial p_k}.$$

Let us try to find a simplest "integro-differential" symmetry of the Burgers equation, whose generating function depends on the "integral" variable $p_{-1} = \int p_0 \, dx = \bar{D}_x^{-1}(p_0)$. From formal point of view, this means that we must extend the manifold $\mathcal{E}^\infty$ up to a manifold $\widetilde{\mathcal{E}}$ by adding a new coordinate to internal coordinates on $\mathcal{E}^\infty$. The total derivative of this new coordinate functions along x should be equal to p_0. The total derivative along t is natural to be defined as follows:

$$\bar{D}_t(p_{-1}) = \bar{D}_t(\bar{D}_x^{-1}(p_0)) = \bar{D}_x^{-1}(\bar{D}_t(p_0)) = \bar{D}_x^{-1}(p_2 + p_0 p_1) = p_1 + \frac{1}{2} p_0^2 + c,$$

where $c = c(t)$ is the "constant of integration", which may be set to be equal to zero. In other words, simultaneously with extending the manifold $\mathcal{E}^\infty$ we have to extend the total derivatives operators $\bar{D}_x$ and $\bar{D}_t$ by setting

$$\widetilde{D}_x = \bar{D}_x + p_0 \frac{\partial}{\partial p_{-1}}, \qquad \widetilde{D}_t = \bar{D}_t + \left(p_1 + \frac{1}{2} p_0^2 \right) \frac{\partial}{\partial p_{-1}}.$$

Obviously, the integrability condition $[\widetilde{D}_x, \widetilde{D}_t] = 0$ must be preserved.

The operator $\bar{\ell}_F$ is naturally extended up to the operator

$$\widetilde{\ell}_F = \widetilde{D}_x^2 + p_0\widetilde{D}_x + p_1 - \widetilde{D}_t.$$

in this case. It is natural to call a solution of the equation $\widetilde{\ell}_F(\varphi) = 0$, a *nonlocal symmetry* of the Burgers equation depending on p_{-1}.

If the function φ depends on the variables x, t, p_{-1}, and p_0 only, then the equation $\widetilde{\ell}_F(\varphi) = 0$ is easily to solve explicitly and to obtain the solution

$$\varphi_a = \left(ap_0 - 2\frac{\partial a}{\partial x}\right)e^{-\frac{1}{2}p_{-1}},$$

where $a = a(x,t)$ is an arbitrary solution of the heat equation $a_t = a_{xx}$.

To show possible applications of the above computations, let us find the solutions of the Burgers equation invariant with respect to some symmetry of the form φ_a. By general theory (see Ch. 4), such solutions are determined by the following system of equations:

$$\begin{cases} u_t = u_{xx} + uu_x, \\ \left(au - 2\dfrac{\partial a}{\partial x}\right)e^{-\frac{1}{2}\int u\,dx} = 0. \end{cases}$$

From this system we obtain, in particular, that $u = 2a_x/a$, i.e., we obtain the Cole–Hopf transformation reducing the Burgers equation to the heat equation.

"Integro-differential" symmetries naturally arise in studies of equation admitting recursion operator.

EXAMPLE 1.2. Consider the Korteweg–de Vries equation[1]

$$\mathcal{E} = \{u_t = u_{xxx} + uu_x\}.$$

The operator

$$R = \bar{D}_x^2 + \frac{2}{3}p_0 + \frac{1}{3}p_1\bar{D}_x^{-1}$$

commutes with the universal linearization operator

$$\bar{\ell}_F = \bar{D}_x^3 + p_0\bar{D}_x + p_1 - \bar{D}_t.$$

Consequently, if $\bar{\ell}_F(\varphi) = 0$, then $\bar{\ell}_F(R(\varphi)) = R(\bar{\ell}_F(\varphi)) = 0$. This means that if $\varphi \in \mathrm{sym}\,\mathcal{E}$ and $R(\varphi) \in \mathcal{F}(\mathcal{E}^\infty)$, then $R(\varphi)$ is also a local symmetry of the equation $\mathcal{E}$. Such operators are called *recursion operators* (cf. Ch. 4; see also, e.g., [**91**]).

The Korteweg–de Vries equation possesses an infinite series of local symmetries $\varphi_k = R^k(p_1)$, where p_1 is the translation along x. It also possesses another two symmetries, Galilean and scale ones, and the recursion operator takes the former to the latter. These two symmetries can be considered as the starting point for the second series. But one can easily see that applying the recursion operator to the

---

[1]We use here a representation of the KdV equation different from that considered in Ch. 3 and 5. Evidently, both forms are equivalent and reduce to each other by the scale transformation.

scale symmetry, we shall obtain the expression containing the term $\bar{D}_x^{-1}(p_0)$.

$$
\begin{array}{ccc}
\vdots & \vdots & \\
\bullet & \circ & ? \\
R\uparrow & \uparrow & \\
\varphi_2=p_5+p_0p_3+p_1p_2+p_0^2p_1 \quad \bullet & \circ & t\varphi_2+x\varphi_1+\frac{4}{3}p_2+\frac{4}{9}p_0^2+\frac{1}{9}p_1\bar{D}_x^{-1}p_0 \ ? \\
R\uparrow & R\uparrow & \\
\varphi_1=p_3+p_0p_1 \quad \bullet & \bullet & t\varphi_1+\frac{1}{3}xp_1+\frac{2}{3}p_0 \\
R\uparrow & R\uparrow & \\
\varphi_0=p_1 \quad \bullet & \bullet & tp_1+1
\end{array}
$$
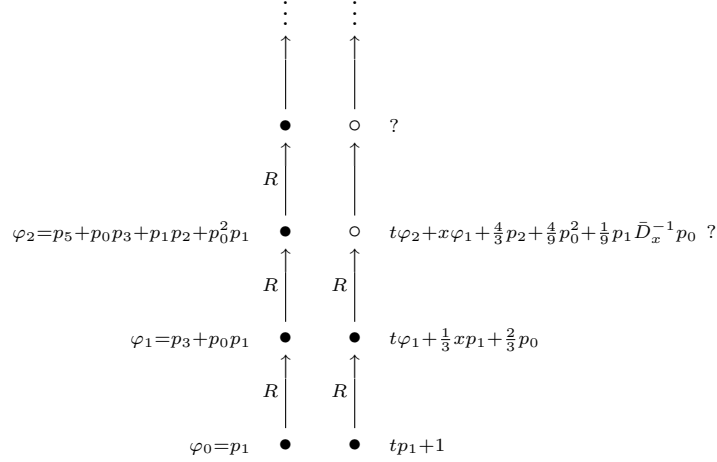
Nevertheless, introducing the additional variable p_{-1} and extending the operators $\bar{D}_x$, $\bar{D}_t$, and $\bar{\ell}_F$ to the operators

$$
\widetilde{D}_x = \bar{D}_x + p_0\frac{\partial}{\partial p_{-1}}, \ \widetilde{D}_t = \bar{D}_t + \left(p_2+\frac{1}{2}p_0^2\right)\frac{\partial}{\partial p_{-1}}, \ \widetilde{\ell}_F = \widetilde{D}_x^3 + p_0\widetilde{D}_x + p_1 - \widetilde{D}_t
$$

respectively, we shall see that the function $t\varphi_2 + x\varphi_1 + \frac{4}{3}p_2 + \frac{4}{9}p_0^2 + \frac{1}{9}p_1p_{-1}$ is a solution of the equation $\bar{\ell}_F(\varphi) = 0$ and consequently may be called a nonlocal symmetry of the Korteweg–de Vries equation.

EXERCISE 1.1. Show that the operator $R = \bar{D}_x + \frac{1}{2}p_0 + \frac{1}{2}p_1\bar{D}_x^{-1}$ is a recursion operator for the Burgers equation.

EXAMPLE 1.3. Consider the potential Korteweg–de Vries equation

$$
\mathcal{E} = \left\{u_t = u_{xxx} + \frac{1}{2}u_x^2\right\}.
$$

The procedure of introducing a new variable $p_{-1} = \bar{D}^{-1}(p_0)$ to this equation is more complicated. Namely, let us try to compute $\bar{D}_t(p_{-1})$:

$$
\bar{D}_t(p_{-1}) = \bar{D}_t(\bar{D}_x^{-1}(p_0)) = \bar{D}_x^{-1}(\bar{D}_t(p_0)) = \bar{D}_x^{-1}\left(p_3+\frac{1}{2}p_1^2\right) = p_2 + \frac{1}{2}\bar{D}_x^{-1}(p_1^2).
$$

Here we meet the necessity to introduce another variable w whose total derivative along x equals p_1^2. Then

$$
\bar{D}_t(w) = \bar{D}_t(\bar{D}_x^{-1}(p_1^2)) = \bar{D}_x^{-1}(\bar{D}_t(p_1^2)) = 2\bar{D}_x^{-1}(p_1(p_4+p_1p_2))
$$

$$
= 2\left(p_1p_3 - \frac{1}{2}p_2^2 + \frac{1}{3}p_1^3\right).
$$

Thus the operators $\bar{D}_x$ and $\bar{D}_t$ are to be extended to the operators

$$
\widetilde{D}_x = \bar{D}_x + p_0\frac{\partial}{\partial p_{-1}} + p_1^2\frac{\partial}{\partial w},
$$

$$
\widetilde{D}_t = \bar{D}_t + \left(p_2+\frac{1}{2}w\right)\frac{\partial}{\partial p_{-1}} + 2\left(p_1p_3 - \frac{1}{2}p_2^2 + \frac{1}{3}p_1^3\right)\frac{\partial}{\partial w}.
$$

EXERCISE 1.2. Try to introduce the variable p_{-1} for the equation $\mathcal{E} = \{u_t = u_{xxx} + u_x^3\}$.

The last two examples show that attempts to apply the above constructions in a general situation may meet technical difficulties. In fact, we have no reasons, except for a formal parallel with the local picture, to treat "nonlocal" solutions of the equation $\bar{\ell}_F(\varphi) = 0$ as nonlocal symmetries (see, e.g., [**50, 30**]). Obviously, we need a conceptual analysis of the notion of a nonlocal symmetry itself. By this reasons, it seems to be natural to analyze a definition of nonlocal symmetries by studying geometrical structures on manifolds $\widetilde{\mathcal{E}}$: a similar way has led us to the definition of local symmetries of equations $\mathcal{E}^\infty$.

1.2. Definition of coverings. Let $\mathcal{E}$ be a differential equation in the bundle $\pi\colon E^{n+m} \to M^n$, $\mathcal{E}^\infty$ be its infinite prolongation. Recall that at any point $\theta \in \mathcal{E}^\infty$ the n-dimensional plane $\mathcal{C}_\theta \subset T_\theta(\mathcal{E}^\infty)$ is defined (the Cartan plane). The Cartan distribution $\mathcal{C} = \{\mathcal{C}_\theta\}_{\theta \in \mathcal{E}^\infty}$ on $\mathcal{E}^\infty$ is completely integrable, i.e., it satisfies the assumptions of the classical Frobenius theorem. In local coordinates, the Cartan distribution on $\mathcal{E}^\infty$ is given by the system of n vector fields $\bar{D}_1, \dots, \bar{D}_n$, where $\bar{D}_i$ is the restriction to $\mathcal{E}^\infty$ of the total derivative operator along the i-th independent variable. The extension procedure of the manifold $\mathcal{E}^\infty$ considered in the previous examples may be formalized by introducing the concept of a covering.

DEFINITION 1.1. We shall say that a *covering* $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ of the equation $\mathcal{E}$ is given, if the following objects are fixed:

1. A smooth manifold $\widetilde{\mathcal{E}}$, infinite-dimensional in general[2].
2. An n-dimensional integrable distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}}$.
3. A regular mapping τ of the manifold $\widetilde{\mathcal{E}}$ onto $\mathcal{E}^\infty$ such that for any point $\theta \in \widetilde{\mathcal{E}}$ the tangent mapping $\tau_{*,\theta}$ is an isomorphism of the plane $\widetilde{\mathcal{C}}_\theta$ to the Cartan plane $\mathcal{C}_{\tau(\theta)}$ of the equation $\mathcal{E}^\infty$ at the point $\tau(\theta)$.

The dimension of the bundle τ is called the dimension of the corresponding covering. Below by $\mathcal{F}(\widetilde{\mathcal{E}})$ the ring of smooth functions on $\widetilde{\mathcal{E}}$ is denoted.

From the definition of coverings it follows that the mapping τ takes any n-dimensional integral manifold $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{E}}$ of the distribution $\widetilde{\mathcal{C}} = \{\widetilde{\mathcal{C}}_\theta\}_{\theta \in \widetilde{\mathcal{E}}}$ to an n-dimensional integral manifold $\mathcal{U} = \tau(\widetilde{\mathcal{U}}) \subset \mathcal{E}^\infty$ of the Cartan distribution on $\mathcal{E}^\infty$, i.e., to a solution of the equation $\mathcal{E}$. Conversely, if $\mathcal{U} \subset \mathcal{E}^\infty$ is a solution of the equation $\mathcal{E}$, then the restriction of the distribution $\widetilde{\mathcal{C}}$ to the inverse image $\widetilde{\mathcal{U}} = \tau^{-1}(\mathcal{U}) \subset \widetilde{\mathcal{E}}$, as it is easily seen, is an integrable n-dimensional distribution. If, in particular, $\dim \tau^{-1}(\theta) = N < \infty$, $\theta \in \mathcal{U}$, then the manifold $\widetilde{\mathcal{U}}$ is locally fibered by N-parameter family of integral manifolds of the distribution $\widetilde{\mathcal{C}}\big|_{\widetilde{\mathcal{U}}}$.

Thus, to any solution of the equation $\mathcal{E}$ there corresponds a family of integral manifolds lying in $\widetilde{\mathcal{E}}$. The elements of this family are convenient to be understood as solutions of the equation $\mathcal{E}$ parametrized by some nonlocal quantities. In Examples 1.1 and 1.2 the integration constant can be taken for such a parameter.

---

[2]All infinite-dimensional manifolds considered below are inverse limits of chains of finite-dimensional ones. Differential geometry for such manifolds is constructed in the same way as it was done for the manifolds of the form $\mathcal{E}^\infty$ (see Ch. 4); this allows to overcome usual topological difficulties.

1.3. Coverings in the category of differential equations. To explain the term "covering" for the above introduced objects, we use a parallel between the category of differential equation (called the DE category below) and the category of smooth manifolds.

In what follows, we take a very simple approach to the DE category suitable for our purposes, though it should be noted that there exists a more thorough approach to definition of this category (see, for example, [**60**]).

The *objects* of the DE category are (infinite-dimensional) manifolds $\mathcal{O}$ endowed with completely integrable finite-dimensional distributions $\mathcal{P}$. Obviously, infinitely prolonged equations $\mathcal{E}^\infty$ together with the Cartan distribution $\mathcal{C}$ are objects of this category. The *morphisms* in the DE category are smooth mappings $\varphi\colon \mathcal{O} \to \mathcal{O}'$ such that $\varphi_*(\mathcal{P}_\theta) \subset \mathcal{P}'_{\varphi(\theta)}$ at all points $\theta \in \mathcal{O}$, where $\mathcal{P}$ and $\mathcal{P}'$ are the distributions on $\mathcal{O}$ and $\mathcal{O}'$ respectively. *Dimension* of the object $(\mathcal{O}, \mathcal{P})$ is the dimension of the distribution $\mathcal{P}$.

Using as an analogy the notion of a covering in the category of smooth manifolds, we can give the following definition:

DEFINITION 1.2. A surjective map $\varphi\colon \mathcal{O} \to \mathcal{O}'$ is called a *covering* in the DE category, if it preserves dimensions, i.e., if $\operatorname{Dim} \mathcal{O} = \operatorname{Dim} \mathcal{O}'$ and for any point $\theta \in \mathcal{O}$ one has $\varphi_*(\mathcal{P}_\theta) = \mathcal{P}'_{\varphi(\theta)}$.

1.4. Examples of coverings. Let us consider some examples of coverings arising in various situations.

EXAMPLE 1.4 (the covering associated to a differential operator). Let us consider a differential operator $\Delta\colon \Gamma(\pi) \to \Gamma(\pi')$ of order k acting from sections of the bundle $\pi\colon E \to M$ to sections of $\pi'\colon E' \to M$. The operator Δ and its prolongations $\Delta_s = j_s \circ \Delta$ determine the family of smooth mappings $\varphi_s = \Phi_{\Delta_s}\colon J^{k+s}(\pi) \to J^s(\pi')$ (see §§1.2 and 1.3 of Ch. 4), which can be included in the following commutative diagram:

$$
\begin{array}{ccccccc}
\cdots \longrightarrow J^{k+s+1}(\pi) & \xrightarrow{\pi_{k+s+1,k+s}} & J^{k+s}(\pi) & \longrightarrow \cdots \longrightarrow & J^k(\pi) & & \\
\varphi_{s+1} \downarrow & & \varphi_s \downarrow & & \varphi_0 \downarrow & \searrow^{\pi_k} & \\
\cdots \longrightarrow J^{s+1}(\pi') & \xrightarrow{\pi'_{s+1,s}} & J^s(\pi') & \longrightarrow \cdots \longrightarrow & J^0(\pi') = E' & \xrightarrow{\pi'} & M
\end{array}
$$

It is easily seen that if the operator Δ is regular (i.e., if all the mappings φ_s are surjections), then the system of mappings $\varphi = \{\varphi_s\}$ determines the covering $\varphi\colon J^\infty(\pi) \to J^\infty(\pi')$. We shall say that this is the covering *associated to the operator* Δ.

If an equation $\mathcal{E} \subset J^k(\pi')$ is given, then, using the above construction, one can construct the covering $\varphi\colon \varphi^{-1}(\mathcal{E}^\infty) \to \mathcal{E}^\infty$. Such coverings correspond to differential changes of variables

$$v^i = \varphi^i(x, u^1, \ldots, u^m, \ldots, u^j_\sigma),$$

where v^i and u^j are dependent variables in the bundles π' and π respectively. We shall consider this type of coverings in the next two examples.

EXAMPLE 1.5 (Laplace transformation). Consider an equation of the form

$$u_{xy} + A u_x + B u_y + C u = 0. \tag{1.1}$$

Recall (see [**92**]) that the functions $h = \partial A/\partial x + AB - C$ and $l = \partial B/\partial y + AB - C$ are called *Laplace invariants* of the equation (1.1).

Assume that $h, l \neq 0$ and consider one-dimensional coverings over the equation $\mathcal{E}^\infty$ determined by the operators

$$\tau_h \colon u = \frac{w_x^1 + w^1}{h} \qquad\qquad (x \text{ Laplace transformation}),$$

$$\tau_l \colon u = \frac{w_y^2 + w^2}{l} \qquad\qquad (y \text{ Laplace transformation}).$$

It is easily checked that the spaces of these coverings are isomorphic to an equation $(\mathcal{E}')^\infty$, where $\mathcal{E}'$ is of the form (1.1) for some new functions A, B, and C. Let us denote thus obtained equations by $\mathcal{E}_h$ and $\mathcal{E}_l$ respectively. If the Laplace invariant l of the equation $\mathcal{E}_h$ does not vanish, one can consider the equation $(\mathcal{E}_h)_l$ (or, similarly, the equation $(\mathcal{E}_l)_h$), and this equation, as it happens, is equivalent to the equation $\mathcal{E}$. In such a way, we obtain a two-dimensional covering of the equation $\mathcal{E}^\infty$ over itself. If, at some step of this procedure, we obtain an equation, for which either h or l vanishes, we shall be able to construct the formula for general solution of the equation under consideration (further details can be found in [**92**]).

EXAMPLE 1.6 (the covering associated to a local symmetry). Let an equation $\mathcal{E} \subset J^k(\pi)$ be given and suppose that φ is the generating function of some symmetry of this equation. Then, as it was shown in Example 1.4, the covering $\varphi \colon \varphi^{-1}(\mathcal{E}^\infty) \to \mathcal{E}^\infty$ arises. If the equation $\mathcal{E}$ is determined by an operator Δ, i.e., $\mathcal{E} = \{\varphi_\Delta = 0\}$, then the function φ satisfies the equation $\bar{\ell}_\Delta(\varphi) = 0$, where $\bar{\ell}_\Delta$ is the universal linearization operator for Δ restricted to $\mathcal{E}^\infty$. Thus the space of the covering under consideration is determined by the system of equations

$$\begin{cases} \bar{\ell}_\Delta(\varphi) = 0, \\ \varphi^j = u^j, \qquad j = 1, \ldots, \dim \pi. \end{cases}$$

If $\mathcal{E}$ is a linear equation, we may assume that $\bar{\ell}_\Delta = \Delta$. This means that the covering space is the equation $\mathcal{E}^\infty$ itself.

EXAMPLE 1.7 (factorization of differential equations). Let us consider a differential equation $\mathcal{E} \subset J^k(\pi)$ and let F be a finite-dimensional subgroup in the group of its classical symmetries. It is clear that the factorization mapping (see §6 of Ch. 3) $\pi_F \colon \mathcal{E}^\infty \to \mathcal{E}^\infty/F$ is a covering.

For example, the heat equation $u_t = u_{xx}$ admits the one-parameter group of scale symmetries $u \mapsto \varepsilon u$, $\varepsilon \in \mathbb{R}$. The corresponding quotient equation is the Burgers equation. Thus the covering of the Burgers equation by the heat equation arises.

1.5. Coordinates. Consider a coordinate interpretation of the covering concept.

The manifold $\widetilde{\mathcal{E}}$ and the mapping $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$, by regularity of τ, can be locally realized as the direct product $\mathcal{E}^\infty \times W$, where $W \subseteq \mathbb{R}^N$ is an open set, $0 < N \leq \infty$, and as the natural projection $\mathcal{E}^\infty \times W \to \mathcal{E}^\infty$ respectively. Then the distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}} = \mathcal{E}^\infty \times W$ can be described by the system of vector fields (see Figure 6.1)

$$\widetilde{D}_i = \bar{D}_i + \sum_{j=1}^N X_{ij} \frac{\partial}{\partial w_j}, \qquad i = 1, \ldots, n, \tag{1.2}$$
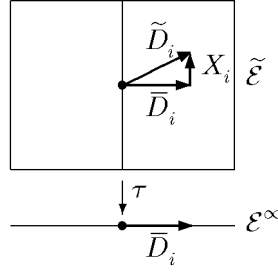
FIGURE 6.1

where $X_i = \sum_{j=1}^{N} X_{ij} \partial/\partial w_j$, $X_{ij} \in \mathcal{F}(\widetilde{\mathcal{E}})$ are τ-vertical fields on $\widetilde{\mathcal{E}}$ and $w_1, w_2, \ldots$ are standard coordinates in $\mathbb{R}^N$. In these terms, the Frobenius condition is equivalent to the equations $[\widetilde{D}_i, \widetilde{D}_j] = 0$, $i, j = 1, \ldots, n$, or, which is the same, to

$$\widetilde{D}_i(X_{jk}) = \widetilde{D}_j(X_{ik}) \tag{1.3}$$

for all $i, j = 1, \ldots, n$, $0 \le k \le N$ (since $[\bar{D}_i, \bar{D}_j] = 0$).

Relations (1.3) constitute a system of differential equations in functions X_{ij} describing all possible N-dimensional coverings over the equation $\mathcal{E}$.

The coordinates w_i will be called *nonlocal*.

1.6. Basic concepts of the covering theory. In this subsection, we introduce some concepts of the covering theory needed below.

DEFINITION 1.3. Two coverings $\tau_i \colon \widetilde{\mathcal{E}}_i \to \mathcal{E}^\infty$, $i = 1, 2$, are called *equivalent*, if there exists a diffeomorphism $\alpha \colon \widetilde{\mathcal{E}}_1 \to \widetilde{\mathcal{E}}_2$ such that the diagram

$$\widetilde{\mathcal{E}}_1 \xrightarrow{\ \ \alpha\ \ } \widetilde{\mathcal{E}}_2$$
$$\tau_1 \searrow \qquad \swarrow \tau_2$$
$$\mathcal{E}^\infty$$

is commutative and $\alpha_*(\widetilde{\mathcal{C}}_y^1) = \widetilde{\mathcal{C}}_{\alpha(y)}^2$ for all points $y \in \widetilde{\mathcal{E}}_1$.

EXERCISE 1.3. Show that if a diffeomorphism α constitutes an equivalence of coverings τ_1 and τ_2 over the equation $\mathcal{E}^\infty$ and the coverings are given by the fields

$$\widetilde{D}_i^{(1)} = \bar{D}_i + \sum_k X_{ik}^{(1)} \frac{\partial}{\partial w_k^{(1)}}, \qquad \widetilde{D}_i^{(2)} = \bar{D}_i + \sum_l X_{ik}^{(2)} \frac{\partial}{\partial w_k^{(2)}},$$

respectively, $i = 1, \ldots, n$, then $\alpha_* \widetilde{D}_i^{(1)} = \widetilde{D}_i^{(2)}$, i.e.,

$$\alpha_* \widetilde{D}_i^{(1)} = \bar{D}_i + \sum_k (\alpha^{-1})^* (\widetilde{D}_i^{(1)}(\alpha_k)) \frac{\partial}{\partial w_k^{(2)}}, \qquad i = 1, \ldots, n, \tag{1.4}$$

where $w_k^{(i)}$ are nonlocal variables in the covering τ_i, $i = 1, 2$, and

$$\alpha(x_i, p_\sigma^j, w_1^{(1)}, w_2^{(1)}, \ldots)$$
$$= (x_i, p_\sigma^j, \alpha_1(x_i, p_\sigma^j, w_1^{(1)}, w_2^{(1)}, \ldots), \alpha_2(x_i, p_\sigma^j, w_1^{(1)}, w_2^{(1)}, \ldots), \ldots).$$

Relations (1.4) can be also rewritten in the form

$$\widetilde{D}_i^{(1)}(\alpha_k) = \alpha^*(X_{ik}^{(2)}). \tag{1.5}$$

Let now $\mathcal{E}^\infty$ be a differential equation with the Cartan distribution determined by the fields $\bar{D}_1, \ldots, \bar{D}_n$. Consider one particular covering over $\mathcal{E}^\infty$. Let us set $\widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}^N$ and let $\tau = \mathrm{pr}_N \colon \widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}^N \to \mathcal{E}^\infty$ be the projection to the first factor. Assume that the distribution on $\widetilde{\mathcal{E}}$ is determined by the fields $\bar{D}_1, \ldots, \bar{D}_n$.

DEFINITION 1.4. A covering $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ is called *trivial*, if it is equivalent to the covering pr_N for some $0 < N \le \infty$.

EXERCISE 1.4. Show that if a covering $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ is trivial, then the covering space $\widetilde{\mathcal{E}}$ is fibered by integral submanifolds isomorphic to $\mathcal{E}^\infty$.

EXAMPLE 1.8. Let $\mathcal{E} = \{u_t = u_{xx} + uu_x\}$ be the Burgers equation. Consider the one-dimensional covering $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$, where $\widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}^1$, τ is the projection to the first factor, v is a nonlocal variable, while the distribution on $\widetilde{\mathcal{E}}$ is given by the fields

$$\widetilde{D}_x = \bar{D}_x + p_1 \frac{\partial}{\partial v}, \qquad \widetilde{D}_t = \bar{D}_t + (p_2 + p_0 p_1) \frac{\partial}{\partial v}.$$

Let us show that this covering is equivalent to the one-dimensional trivial covering of the Burgers equation. Define the mapping $\alpha \colon \mathcal{E}^\infty \times \mathbb{R}^1 \to \mathcal{E}^\infty \times \mathbb{R}^1 = \widetilde{\mathcal{E}}$, by setting $\alpha(x, t, p_k, w) = (x, t, p_k, w + p_0)$. Then α is a fiber-wise diffeomorphism and

$$\alpha_* \bar{D}_x = \bar{D}_x + \bar{D}_x(w + p_0) \frac{\partial}{\partial v} = \bar{D}_x + p_1 \frac{\partial}{\partial v} = \widetilde{D}_x,$$

$$\alpha_* \bar{D}_t = \bar{D}_t + \bar{D}_t(w + p_0) \frac{\partial}{\partial v} = \bar{D}_x + (p_2 + p_0 p_1) \frac{\partial}{\partial v} = \widetilde{D}_t.$$

Thus the mapping α establishes equivalence of the coverings pr_1 and τ.

Let us now describe another construction playing an important role in the theory of coverings. Let $\tau_i \colon \widetilde{\mathcal{E}}_i \to \mathcal{E}^\infty$, $i = 1, 2$, be two coverings over the equation $\mathcal{E}^\infty$. Consider the direct product $\widetilde{\mathcal{E}}_1 \times \widetilde{\mathcal{E}}_2$ and take the subset $\widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$ consisting of the points (y_1, y_2), $y_1 \in \widetilde{\mathcal{E}}_1$, $y_2 \in \widetilde{\mathcal{E}}_2$, such that $\tau_1(y_1) = \tau_2(y_2)$. Then the projection $\tau_1 \oplus \tau_2 \colon \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2 \to \mathcal{E}^\infty$ is defined, for which $(\tau_1 \oplus \tau_2)(y_1, y_2) = \tau_1(y_1) = \tau_2(y_2)$. Obviously, we have the commutative diagram

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2 & \\
{\scriptstyle \tau_1(\tau_2)} \swarrow & \downarrow {\scriptstyle \tau_1 \oplus \tau_2} & \searrow {\scriptstyle \tau_2(\tau_1)} \\
\widetilde{\mathcal{E}}_1 & & \widetilde{\mathcal{E}}_2 \\
{\scriptstyle \tau_1} \searrow & \downarrow & \swarrow {\scriptstyle \tau_2} \\
 & \mathcal{E}^\infty &
\end{array}
$$

where the mappings $\tau_1(\tau_2)$ and $\tau_2(\tau_1)$ are induced by the projections to the left and right factors respectively. In other words, $\tau_1 \oplus \tau_2 \colon \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2 \to \mathcal{E}^\infty$ is the Whitney product of the bundles $\tau_i \colon \widetilde{\mathcal{E}}_i \to \mathcal{E}^\infty$, $i = 1, 2$. Let us now define a distribution $\mathcal{C}^\oplus$ on the manifold $\widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$ by setting $\mathcal{C}^\oplus_{(y_1, y_2)} = (\tau_1(\tau_2))_*^{-1}(\mathcal{C}^{(1)}_{y_1}) \cap (\tau_2(\tau_1))_*^{-1}(\mathcal{C}^{(2)}_{y_2})$, $(y_1, y_2) \in \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$. Obviously, this is an integrable distribution

and $(\tau_1 \oplus \tau_2)_*(\mathcal{C}^{\oplus}_{(y_1,y_2)}) = \mathcal{C}_{(\tau_1 \oplus \tau_2)(y_1,y_2)}$, i.e., the projection $\tau_1 \oplus \tau_2$ and the distribution $\mathcal{C}^{\oplus}$ determine a covering structure over $\mathcal{E}^{\infty}$ on the manifold $\widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$. This covering is called the *Whitney product* of the coverings τ_1 and τ_2. Note also that the projections $\tau_1(\tau_2)$ and $\tau_2(\tau_1)$ are coverings over the objects $\widetilde{\mathcal{E}}_1$ and $\widetilde{\mathcal{E}}_2$ respectively.

A coordinate interpretation of the covering $\tau_1 \oplus \tau_2$ is as follows. Let $\mathbb{R}^{N_\alpha}$ be the fiber of the bundle τ_α; $w_1^{(1)}, w_2^{(1)}, \ldots$ be the coordinates in the fiber of the projection τ_1; $w_1^{(2)}, w_2^{(2)}, \ldots$ be the coordinates in the fiber of the projection τ_2. Let the coverings τ_1 and τ_2 be locally determined by the fields $\widetilde{D}_i^{(1)} = \bar{D}_i + X_i^{(1)}$ and $\widetilde{D}_i^{(2)} = \bar{D}_i + X_i^{(2)}$, $i = 1, \ldots, n$, respectively, $X_i^{(\alpha)} = \sum_k X_{ik}^{(\alpha)} \partial/\partial w_k^{(\alpha)}$, $\alpha = 1, 2$. Then the space $\mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$ is the fiber of the bundle $\tau_1 \oplus \tau_2$ with the coordinates $w_1^{(1)}, w_2^{(1)}, \ldots, w_1^{(2)}, w_2^{(2)}, \ldots$, while the distribution on $\widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$ is determined by the fields

$$\widetilde{D}_i^{\oplus} = \bar{D}_i + \sum_k X_{ik}^{(1)} \frac{\partial}{\partial w_k^{(1)}} + \sum_l X_{il}^{(2)} \frac{\partial}{\partial w_l^{(2)}}, \qquad i = 1, \ldots, n.$$

The Whitney product of an arbitrary number of coverings is defined in an obvious way.

The Whitney product construction is convenient to analyze equivalence of coverings. Namely, the following proposition is valid:

PROPOSITION 1.1. *Let τ_1 and τ_2 be two coverings of finite dimension $N = \dim \tau_1 = \dim \tau_2$ over the equation $\mathcal{E}^{\infty}$. These coverings are equivalent if and only if there exists a manifold $\mathcal{X} \subset \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$, $\operatorname{codim} \mathcal{X} = N$, invariant[3] of the distribution $\widetilde{\mathcal{C}}^{\oplus}$ such that the restrictions $\tau_1(\tau_2)|_{\mathcal{X}}$ and $\tau_2(\tau_1)|_{\mathcal{X}}$ are surjective.*

PROOF. The proof consists in a straightforward check of definitions. Indeed, let the coverings τ_1 be τ_2 equivalent and α be a diffeomorphism realizing this equivalence. Let us set $\mathcal{X} = \{ (y, \alpha(y)) \mid y \in \widetilde{\mathcal{E}}_1 \} \subset \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$. Obviously, $\operatorname{codim} \mathcal{X} = N$ and $\tau_1(\tau_2)|_{\mathcal{X}}$ and $\tau_2(\tau_1)|_{\mathcal{X}}$ are surjections. Since the mapping α_* takes the distribution $\widetilde{\mathcal{C}}_1$ to $\widetilde{\mathcal{C}}_2$, the submanifold $\mathcal{X}$ is invariant with respect to the distribution $\widetilde{\mathcal{C}}^{\oplus}$.

Conversely, let $\mathcal{X} \subset \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$ be a submanifold satisfying the assumptions of the proposition. Then the correspondence $y \mapsto \tau_2(\tau_1)\left((\tau_1(\tau_2))^{-1}(y) \cap \mathcal{X}\right)$, $y \in \widetilde{\mathcal{E}}_1$, determines an isomorphism of the bundles τ_1 and τ_2. By invariance of $\mathcal{X}$, this is an equivalence of coverings. $\qquad\square$

Let us introduce another notion.

DEFINITION 1.5. A covering $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^{\infty}$ over the equation $\mathcal{E}^{\infty}$ is called *reducible*, if it is equivalent to a covering of the form $\tau_1 \oplus \operatorname{pr}_N$, where τ_1 is some covering over the equation $\mathcal{E}^{\infty}$ and $N > 0$. Otherwise, the covering is called *irreducible*.

A local analog of this definition is obvious.

If $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^{\infty}$ is a reducible covering, then for any point $\theta \in \widetilde{\mathcal{E}}$ there exists an invariant manifold $\mathcal{X}$ of the distribution $\widetilde{\mathcal{C}}$ passing through this point and such that the restriction $\tau|_{\mathcal{X}} : \mathcal{X} \to \mathcal{E}^{\infty}$ of the projection τ is surjective and this restriction, together with the restriction of the distribution $\widetilde{\mathcal{C}}$ to $\mathcal{X}$, determine in $\mathcal{X}$ a covering structure over $\mathcal{E}^{\infty}$.

---

[3]A manifold $\mathcal{X} \subset \widetilde{\mathcal{E}}$ is called *invariant* with respect for the distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}}$, if at any point $x \in \mathcal{X}$ one has the embedding $T_x\mathcal{X} \supset \widetilde{\mathcal{C}}_x$.

Using Proposition 1.1, one can easily prove the following result:

PROPOSITION 1.2. *Let τ_1 and τ_2 be irreducible coverings of finite dimension $N = \dim \tau_1 = \dim \tau_2$ over the equation $\mathcal{E}^\infty$. Then, if the Whitney product $\widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$ of these coverings is reducible and codimension of invariant manifolds of the distribution $\widetilde{\mathcal{C}}^\oplus$ equals N, then almost at all points $\theta \in \widetilde{\mathcal{E}}$ the covering τ_1 is equivalent to the covering τ_2.*

PROOF. Consider invariant manifolds of the distribution $\widetilde{\mathcal{C}}^\oplus$. Almost all of them project surjectively to $\widetilde{\mathcal{E}}_1$ and $\widetilde{\mathcal{E}}_2$ under the mappings $\tau_1(\tau_2)$ and $\tau_2(\tau_1)$ respectively. In fact, otherwise their images under projections $\tau_1(\tau_2)$ and $\tau_2(\tau_1)$ would be invariant manifolds of the distributions $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ respectively and it contradicts to irreducibility of the coverings τ_1 and τ_2. Choosing one of these manifolds, we arrive to Proposition 1.1, and this concludes the proof. $\qquad\square$

COROLLARY 1.3. *If τ_1 and τ_2 are one-dimensional coverings over the equation $\mathcal{E}$ and their Whitney product is reducible, then almost at all points $\theta \in \widetilde{\mathcal{E}}$ the covering τ_1 is equivalent to the covering τ_2.*

In local coordinates, the problem of reducibility is equivalent to studying a certain system of linear differential equations. Namely, the following statement is valid:

PROPOSITION 1.4. *Let $\mathcal{U} \subseteq \mathcal{E}^\infty$ be a domain such that the manifold $\widetilde{\mathcal{U}} = \tau^{-1}(\mathcal{U})$ is representable in the form of the direct product $\mathcal{U} \times \mathbb{R}^N$, $N = 1, 2, \ldots, \infty$, the mapping $\tau|_{\widetilde{\mathcal{U}}} \colon \widetilde{\mathcal{U}} \to \mathcal{U}$ is the projection to the first factor, and the fields $\widetilde{D}_1, \ldots, \widetilde{D}_n$ determine the distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{U}}$. Then the covering τ is locally irreducible if and only if the system*

$$\widetilde{D}_1(\varphi) = 0, \ldots, \widetilde{D}_n(\varphi) = 0, \tag{1.6}$$

where $\varphi \in \mathcal{F}(\widetilde{\mathcal{E}})$, possesses constant solutions only.

PROOF. Assume that there exists a solution $\varphi \neq \text{const}$ of system (1.6). Then, since the only solutions of the system

$$\bar{D}_1(\varphi) = 0, \ldots, \bar{D}_n(\varphi) = 0$$

are constants, the function φ depends at least on one nonlocal variable. Without loss of generality, we may assume that $\partial\varphi/\partial w_1 \neq 0$ in some neighborhood $\mathcal{U}' \times W$, $\mathcal{U}' \subseteq \mathcal{U}$, $W \subseteq \mathbb{R}^N$. Let us define the diffeomorphism $\alpha \colon \mathcal{U}' \times W \to \alpha(\mathcal{U}' \times W)$ by the formula $\alpha(x_i, p_\sigma^j, w_1, \ldots, w_N) = (x_i, p_\sigma^j, \varphi, w_2, \ldots, w_N)$. Since $\alpha_* \widetilde{D}_i = \bar{D}_i + \sum_{j>1} X_{ij}\partial/\partial w_j$, the covering τ is locally equivalent to the covering $\text{pr}_1 \oplus \tau_1$ for some covering τ_1. In other words, τ is a reducible covering.

Assume now that the covering τ is locally reducible, i.e., is locally equivalent to a covering of the form $\tau_1 \oplus \text{pr}_{N'}$. Let a mapping α be the corresponding equivalence. Then, if f is a smooth function on the space of the covering $\text{pr}_{N'}$, the function $\alpha^*\left((\text{pr}_{N'}(\tau_1))^*(f)\right) = f^* \in \mathcal{F}(\widetilde{\mathcal{E}})$ is a solution of system (1.6). Obviously, there exists a function f on the space of the covering $\text{pr}_{N'}$ such that the function $f^* \in \mathcal{F}(\widetilde{\mathcal{E}})$ is not constant. $\qquad\square$

1.7. Coverings and connections. Let $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be an arbitrary covering and let $v \in \mathcal{C}_\theta$. Then for any point $\widetilde{\theta} \in \widetilde{\mathcal{E}}$ projecting to θ by the mapping φ, a vector $\widetilde{v} \in \mathcal{C}_{\widetilde{\theta}}$ satisfying $\tau_* \widetilde{v} = v$ is uniquely determined.

Let $\mathcal{E} \subset J^k(\pi)$. Recall that there exists the Cartan connection in the bundle $\pi_\infty \colon J^\infty(\pi) \to M$. This connection takes a vector field $X \in \mathrm{D}(M)$ to the vector field $\widehat{X} \in \mathrm{D}(J^\infty(\pi))$ (see §1.4 of Ch. 4). All vector fields $\widehat{X}$ constructed in such a way are tangent to $\mathcal{E}^\infty$ and span the Cartan distribution on this manifold. Consequently, one can put into correspondence to any field $X \in \mathrm{D}(M)$ a uniquely defined field $\widetilde{X} \in \mathrm{D}(\widetilde{\mathcal{E}})$, such that $\tau_*(\widetilde{X}) = \widehat{X}$. Besides, it is obvious that if the fields $\widetilde{X}, \widetilde{Y} \in \mathrm{D}(\widetilde{\mathcal{E}})$ are the liftings of the fields $X, Y \in \mathrm{D}(M)$ respectively, then $[\widetilde{X}, \widetilde{Y}] = [\widetilde{X, Y}]$. Thus the covering $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ determines a flat connection in the bundle $\widetilde{\pi} = \pi_\infty \circ \tau \colon \widetilde{\mathcal{E}} \to M$ compatible with the Cartan connection. One can easily see that the converse statement is also valid. More precisely, the following theorem takes place:

THEOREM 1.5. *Let $\mathcal{E} \subset J^k(\pi)$ be an equation and $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a fiber bundle. Then the following statements are equivalent*:

1. *A covering structure exists in τ.*
2. *A connection $\nabla^\tau \colon \mathrm{D}(M) \to \mathrm{D}(\widetilde{\mathcal{E}})$, $\nabla^\tau(X) = \widetilde{X}$, exists in the bundle $\widetilde{\pi} = \pi_\infty \circ \tau \colon \widetilde{\mathcal{E}} \to M$ possessing the following properties*:
   (a) $[\widetilde{X}, \widetilde{Y}] = [\widetilde{X, Y}]$ *for any vector fields $X, Y \in \mathrm{D}(M)$, i.e., the connection ∇^τ is flat*;
   (b) $\tau_*(\widetilde{X}) = \widehat{X}$, *i.e., ∇^τ is compatible with the Cartan connection.*

Note that for any point $\widetilde{\theta} \in \widetilde{\mathcal{E}}$ the plane $\widetilde{\mathcal{C}}_{\widetilde{\theta}}$ is spanned by vectors of the form $\widetilde{X}_{\widetilde{\theta}}$, $X \in \mathrm{D}(M)$.

In local coordinates, the above considered connection is determined in the covering $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ by the correspondence $\bar{D}_i \mapsto \widetilde{D}_i$, $i = 1, \dots, n$, which makes it possible to lift $\mathcal{C}$-differential operators $\Delta\colon \mathcal{F}(\mathcal{E}^\infty) \to \mathcal{F}(\mathcal{E}^\infty)$ to $\widetilde{\mathcal{E}}$. Namely, if $\Delta = \sum_\sigma a_\sigma \bar{D}_\sigma \colon \mathcal{F}(\mathcal{E}^\infty) \to \mathcal{F}(\mathcal{E}^\infty)$ is a $\mathcal{C}$-differential operator, then it can be extended up to an operator $\widetilde{\Delta}\colon \mathcal{F}(\widetilde{\mathcal{E}}) \to \mathcal{F}(\widetilde{\mathcal{E}})$ by setting

$$\widetilde{\Delta} = \sum_\sigma a_\sigma \widetilde{D}_\sigma.$$

In particular, if $\mathcal{E}^\infty = J^\infty(\pi)$ and $\pi'\colon E' \to M$ is another vector bundle, then for any element $F \in \mathcal{F}(\pi, \pi')$ the lifting $\widetilde{\ell}_F$ of the corresponding universal linearization operator exists. Similar to Ch. 4, let us set $\widetilde{\Im}_\varphi(F) = \widetilde{\ell}_F(\varphi)$, where φ is a section of the pullback $\pi^*(\pi)$. In coordinates, the field $\widetilde{\Im}_\varphi$ is of the form

$$\widetilde{\Im}_\varphi = {\sum_{\sigma, j}}' \widetilde{D}_\sigma(\varphi^j) \frac{\partial}{\partial p_\sigma^i},$$

where $\varphi_i \in \mathcal{F}(\widetilde{\mathcal{E}})$ and $\sum'$ denotes summation over internal coordinates.

1.8. The horizontal de Rham complex and nonlocal conservation laws. Let a covering $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ over the equation $\mathcal{E}^\infty$ be given. Then the horizontal de Rham complex on $\mathcal{E}^\infty$ can also be lifted to $\widetilde{\mathcal{E}}$, since the corresponding differentials are $\mathcal{C}$-differential operators. Locally this is described in the following

way. Horizontal k-forms on $\widetilde{\mathcal{E}}$ are represented as

$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \qquad a_{i_1 \ldots i_k} \in \mathcal{F}(\widetilde{\mathcal{E}}). \qquad (1.7)$$

The differential $\widetilde{d}$ acts in the following way:

$$\widetilde{d}\omega = \sum_{i_1 < \cdots < i_k} \left(\sum_{j=1}^{n} \widetilde{D}_j(a_{i_1 \ldots i_k})\, dx_j \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \qquad (1.8)$$

The cohomology of the horizontal de Rham complex on $\widetilde{\mathcal{E}}$ will be denoted by $\widetilde{H}^k(\widetilde{\mathcal{E}})$. The group $\widetilde{H}^{n-1}(\widetilde{\mathcal{E}})$, similar to the local situation, will be called the *group of nonlocal conservation laws* of the equation $\mathcal{E}^\infty$.

Note that the horizontal de Rham complex on $\widetilde{\mathcal{E}}$ can be introduced directly, as it was done with the horizontal de Rham complex on $\mathcal{E}^\infty$ (i.e., skipping the procedure of lifting of $\mathcal{C}$-differential operators to the covering). To do this, let us introduce two notions.

A vector field X on $\widetilde{\mathcal{E}}$ is called *vertical*, if $X((\tau \circ \pi_\infty)^* f) = 0$ for any function $f \in C^\infty(M^n)$. A differential form $\omega \in \Lambda^k(\widetilde{\mathcal{E}})$ is called *horizontal*, if $\omega(X) = 0$ for any vertical vector field X on $\widetilde{\mathcal{E}}$. Horizontal forms of degree k constitute a submodule in $\Lambda^k(\widetilde{\mathcal{E}})$, which will be denoted by $\Lambda_0^k(\widetilde{\mathcal{E}})$.

EXERCISE 1.5. Show that in local coordinates a form $\omega \in \Lambda_0^k(\widetilde{\mathcal{E}})$ is written down as (1.7).

A differential form $\omega \in \Lambda^k(\widetilde{\mathcal{E}})$ is called a *Cartan form*, if $\omega(X) = 0$ for any vector field X lying in the distribution $\widetilde{\mathcal{C}}$. Cartan forms of degree k constitute a submodule in $\Lambda^k(\widetilde{\mathcal{E}})$, which will be denoted by $\mathcal{C}\Lambda^k(\widetilde{\mathcal{E}})$. One has the decomposition

$$\Lambda^k(\widetilde{\mathcal{E}}) = \sum_{\alpha + \beta = k} \Lambda_0^\alpha(\widetilde{\mathcal{E}}) \wedge \mathcal{C}\Lambda^\beta(\widetilde{\mathcal{E}}). \qquad (1.9)$$

The de Rham differential $d\colon \Lambda^k(\widetilde{\mathcal{E}}) \to \Lambda^{k+1}(\widetilde{\mathcal{E}})$ acts on the form $\omega \in \Lambda_0^\alpha(\widetilde{\mathcal{E}}) \wedge \mathcal{C}\Lambda^\beta(\widetilde{\mathcal{E}})$ as follows:

$$d\omega = \omega_0 + \omega_C,$$

where $\omega_0 \in \Lambda_0^{\alpha+1}(\widetilde{\mathcal{E}}) \wedge \mathcal{C}\Lambda^\beta(\widetilde{\mathcal{E}})$, $\omega_C \in \Lambda_0^\alpha(\widetilde{\mathcal{E}}) \wedge \mathcal{C}\Lambda_0^{\beta+1}(\widetilde{\mathcal{E}})$. Using the decomposition (1.9), one can introduce the mapping

$$\widehat{d}\colon \Lambda_0^\alpha(\widetilde{\mathcal{E}}) \wedge \mathcal{C}\Lambda^\beta(\widetilde{\mathcal{E}}) \longrightarrow \Lambda_0^{\alpha+1}(\widetilde{\mathcal{E}}) \wedge \mathcal{C}\Lambda^\beta(\widetilde{\mathcal{E}}),$$

by setting $\widehat{d}\omega = \omega_0$.

It is easily checked that $\widehat{d}_h^2 = 0$ and that in local coordinates the operator $\widehat{d}\big|_{\Lambda_0^*(\widetilde{\mathcal{E}})}$ act by the formula (1.8), i.e., the complex $\{\Lambda_0^*(\widetilde{\mathcal{E}}), \widehat{d}\}$ is the horizontal de Rham complex on $\widetilde{\mathcal{E}}$.

1.9. Covering equations. Consider a covering $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$. Let us find out when the covering manifold $\widetilde{\mathcal{E}}$ is of the form $(\mathcal{E}')^\infty$ for some equation $\mathcal{E}' \subset J^l(\pi')$. To answer this question, we shall restrict ourselves with the following situation.

Assume that two bundles, $\pi\colon E \to M$ and $\xi\colon W \to M$, over the manifold M are given. Then the following commutative diagram takes place

$$
\begin{array}{ccccccc}
W^\infty & \longrightarrow & \cdots & \longrightarrow & W^1 & \longrightarrow & W^0 & \longrightarrow & W \\
{\scriptstyle \pi_\infty^*(\xi)=\tau}\Big\downarrow & & & & {\scriptstyle \pi_1^*(\xi)}\Big\downarrow & & {\scriptstyle \pi^*(\xi)}\Big\downarrow & & {\scriptstyle \xi}\Big\downarrow \\
J^\infty(\pi) & \longrightarrow & \cdots & \longrightarrow & J^1(\pi) & \longrightarrow & E & \longrightarrow & M
\end{array}
$$

where $\pi_i^*(\xi)\colon W^i \to J^k(\pi)$, $i = 1, 2, \ldots, \infty$, denote the pullbacks. Let us denote by τ the projection $\pi_\infty^*(\xi)$ and assume that a covering structure exists in the bundle $\tau\colon W^\infty \to J^\infty(\pi)$. Then, as it was proved in Theorem 1.5, it is equivalent to existence of a flat connection ∇^τ in the bundle $\pi_\infty \circ \tau$. The coefficients of this connection are functions on $J^\infty(\pi)$, i.e., differential operators acting from $\Gamma(\pi)$ to $C^\infty(M)$. Let $f \in \Gamma(\pi)$ be a section of the bundle π. Then a flat connection is induced in the bundle $j_\infty(f)^*(\tau) \approx \xi$ in a natural way. Denote this connection by $\nabla^\tau(f)$. In other words, the covering structure in τ determines a differential operator, which takes a section of the bundle π to a flat connection in the bundle ξ. Let us denote by $\deg\tau$ the order of this operator and call this number the *order of the covering τ*.

Consider now a point $\theta \in W^k$, $\theta = ([f]_x^k, w_0)$, where $x \in M$, $f \in \Gamma(\pi)$, and $w_0 \in \xi^{-1}(x) \subset W$. Since the bundle $\xi\colon W \to M$ possesses the flat connection $\nabla^\tau(f)$, we can locally consider its integral section g passing through the point w_0. The pair (f, g) determines the local section $\pi \oplus \xi$ of the Whitney product of the bundles π and ξ. Let us set

$$
\rho_k(\theta) = [(f, g)]_x^k \in J^k(\pi \oplus \xi).
$$

The family $\rho = \{\rho_k\}$ of the mappings $\rho_k\colon W^k \to J^k(\pi \oplus \xi)$ possesses the following properties:

1. The mapping ρ_k is an immersion for any k.
2. For any k, there exists the following commutative diagram:

$$
\begin{array}{ccc}
W^{k+1} & \xrightarrow{\ \rho_{k+1}\ } & J^{k+1}(\pi \oplus \xi) \\
\Big\downarrow & & \Big\downarrow {\scriptstyle (\pi\oplus\xi)_{k+1,k}} \\
W^k & \xrightarrow{\ \ \rho_k\ \ } & J^k(\pi \oplus \xi)
\end{array}
$$

3. If $k_0 = \max(1, \deg\tau)$ and $\mathcal{E}_\tau = \rho_{k_0}(W^{k_0}) \subset J^{k_0}(\pi \oplus \xi)$, then for all $k > k_0$ the equality $\rho_k(W^k) = (\mathcal{E}_\tau)^{(k-k_0)}$ holds.
4. The mapping $\rho_{\infty,*}$ takes the distribution $\widetilde{\mathcal{C}}$ on W^∞ to the Cartan distribution of the equation $(\mathcal{E}_\tau)^\infty$.

In other words, the space of the covering $\tau\colon W^\infty \to J^\infty(\pi)$, as a manifold with distribution, is isomorphic to the infinitely prolonged equation $(\mathcal{E}_\tau)^\infty$. In the case, when the base of the covering is an equation $\mathcal{E}^\infty \subset J^\infty(\pi)$, we can show by similar reasoning that the space of the covering $\widetilde{\mathcal{E}}$ is isomorphic to the infinitely prolonged equation $(\mathcal{E} \cap \mathcal{E}_\tau)^\infty = \mathcal{E}^\infty \cap (\mathcal{E}_\tau)^\infty$. We shall say that the equation $\mathcal{E}_\tau$ *covers* the equation $\mathcal{E}$. The equation $\mathcal{E}_\tau$ will also be called a *covering equation* for $\mathcal{E}$.

If in local coordinates, the distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}}$ is determined by the fields $\widetilde{D}_i = \bar{D}_i + \sum_{j=1}^{N} X_{ij}(x_\alpha, p_\sigma^\beta, w_\gamma)\partial/\partial w_j$, $i = 1, \ldots, n$, then the covering equation $\mathcal{E}_\tau$

is locally described by the relations

$$\frac{\partial w_j}{\partial x_i} = X_{ij}(x_\alpha, u_\sigma^\beta, w_\gamma), \qquad i = 1, \ldots, n, \quad j = 1, \ldots, N. \qquad (1.10)$$

The fact that the connection ∇^τ in the bundle $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ is flat means that the compatibility conditions for system (1.10) are differential consequences of the equation $\mathcal{E}$. To any solution of this equation, there corresponds an N-parameter family of solutions of the covering equation and the parameters for this family are analogs of integration constants for system (1.10).

EXAMPLE 1.9. Consider the one-dimensional covering of the Korteweg–de Vries equation $\mathcal{E} = \{u_t = uu_x + u_{xxx}\}$ determined by the fields

$$\begin{cases} \widetilde{D}_x = \bar{D}_x + \left(p_0 + \frac{1}{6} w^2 \right) \frac{\partial}{\partial w}, \\[2mm] \widetilde{D}_t = \bar{D}_t + \left(p_2 + \frac{1}{3} w p_1 + \frac{1}{3} p_0^2 + \frac{1}{18} w^2 p_0 \right) \frac{\partial}{\partial w}. \end{cases}$$

Eliminating u from the system

$$\begin{cases} w_x = u + \frac{1}{6} w^2, \\[2mm] w_t = u_{xx} + \frac{1}{3} w u_x + \frac{1}{3} u^2 + \frac{1}{18} w^2 u, \end{cases}$$

we shall see that the covering equation for the Korteweg–de Vries equation is the modified Korteweg–de Vries equation

$$w_t = w_{xxx} - \frac{1}{6} w^2 w_x.$$

The relation between the variables u and w given by the equation $w_x = u + \frac{1}{6} w^2$, is exactly the Miura–Gardner transformation [**86, 88**].

EXERCISE 1.6. Show that if the one-dimensional covering over the Korteweg–de Vries equation is given by the fields

$$\begin{cases} \widetilde{D}_x = \bar{D}_x + p_0 \frac{\partial}{\partial w}, \\[2mm] \widetilde{D}_t = \bar{D}_t + \left(p_2 + \frac{1}{2} p_0^2 \right) \frac{\partial}{\partial w}, \end{cases}$$

then the covering equation is the potential Korteweg–de Vries equation $w_t = w_{xxx} + \frac{1}{2} w_x^2$.

1.10. Horizontal de Rham cohomology and coverings. In this subsection, we shall establish relation between the group $\bar{H}^1(\mathcal{E}^\infty)$ of $(n-1)$-dimensional horizontal de Rham cohomology and special coverings over the equation $\mathcal{E}^\infty$.

Let first $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a one-dimensional covering over the equation $\mathcal{E}^\infty$, $\widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}$, τ be the projection to the first factor and the distribution on $\widetilde{\mathcal{E}}$ be given by the fields

$$\widetilde{D}_i = \bar{D}_i + X_i \frac{\partial}{\partial w}, \qquad X_i \in \mathcal{F}(\mathcal{E}^\infty). \qquad (1.11)$$

Let us put into correspondence the horizontal form

$$\omega = \sum_{i=1}^{n} X_i \, dx_i.$$

to this covering. The conditions (1.3) acquire the form $\bar{D}_i(X_j) = \bar{D}_j(X_i)$ in this case, or $\widehat{d}(\omega) = 0$, i.e., the form ω is closed in the horizontal de Rham complex. Conversely, to any closed horizontal 1-form $\omega = \sum_{i=1}^{n} X_i \, dx_i$ there corresponds a covering over $\mathcal{E}^\infty$ locally described by the fields (1.11).

The following two statements are easily proved:

PROPOSITION 1.6. *Let $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a one-dimensional covering given by equations (1.11) and ω be the corresponding 1-form. The covering τ is trivial if and only if the form ω is exact, i.e., it generates the trivial element in the cohomology group $\bar{H}^1(\mathcal{E}^\infty)$.*

PROPOSITION 1.7. *Let $\tau_i\colon \widetilde{\mathcal{E}}_i \to \mathcal{E}^\infty$, $i=1,2$, be two one-dimensional coverings over $\mathcal{E}^\infty$ given by (1.11) and let the forms ω_1 and ω_2 correspond to these coverings. The coverings τ_1 and τ_2 are equivalent if and only if the corresponding cohomology classes $[\omega_1]$ and $[\omega_2]$ coincide.*

EXERCISE 1.7. Prove Propositions 1.6 and 1.7.

Thus, we have a one-to-one correspondence between the elements of the group $\bar{H}^1(\mathcal{E}^\infty)$ and classes of equivalence of one-dimensional coverings (1.11) over $\mathcal{E}^\infty$.

Let now $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be an N-dimensional covering over the equation $\mathcal{E}^\infty$, where $\widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}^N$, τ is the projection to the first factor, and the distribution on $\widetilde{\mathcal{E}}$ is determined by the fields

$$\widetilde{D}_i = \bar{D}_i + \sum_{j=1}^{N} X_{ij} \frac{\partial}{\partial w_j}, \qquad X_{ij} \in \mathcal{F}(\mathcal{E}^\infty), \quad i=1,\dots,n.$$

It is easily seen that the horizontal 1-forms

$$\omega_j = \sum_{i=1}^{n} X_{ij} dx_i, \qquad j=1,\dots,N$$

are closed. Therefore, they generate some elements $[\omega_1],\dots,[\omega_N]$ of the group $\bar{H}^1(\mathcal{E}^\infty)$. Let $V(\tau)$ be the subspace of the linear space $\bar{H}^1(\mathcal{E}^\infty)$ spanned by the classes $[\omega_1],\dots,[\omega_N]$. The following statement is valid:

PROPOSITION 1.8. *Let $\tau_i\colon \widetilde{\mathcal{E}}_i \to \mathcal{E}^\infty$, $i=1,2$, be a locally irreducible N-dimensional, $N < \infty$, covering over the equation $\mathcal{E}^\infty$. Let $\widetilde{\mathcal{E}}_i = \mathcal{E}^\infty \times \mathbb{R}^N$, τ_i be the projections to the first factor, while the distribution $\widetilde{\mathcal{E}}_i$ be described by the fields*

$$\widetilde{D}_j^{(i)} = \bar{D}_j + \sum_{k} X_{jk}^{(i)} \frac{\partial}{\partial w_k^{(i)}}, \qquad X_{jk}^{(i)} \in \mathcal{F}(\mathcal{E}^\infty), \quad j=1,\dots,n, \quad i=1,2.$$

Then the coverings τ_1 and τ_2 are equivalent if and only if $V(\tau_1) = V(\tau_2)$.

PROOF. Let the coverings τ_1 and τ_2 be equivalent and let a fiber-wise diffeomorphism $\alpha\colon \widetilde{\mathcal{E}}_1 \to \widetilde{\mathcal{E}}_2$ establish their equivalence. Then $\alpha_*(\widetilde{D}_j^{(1)}) = \widetilde{D}_j^{(2)}$, i.e., $\alpha^*(X_{jk}^{(2)}) = \widetilde{D}_j^{(1)}(\alpha_k)$, where

$$\alpha(x_i, p_\sigma^j, w_k^{(1)}) = (x_i, p_\sigma^j, \alpha_1(x_i, p_\sigma^j, w_k^{(1)}), \alpha_2(x_i, p_\sigma^j, w_k^{(1)}), \dots)$$

(see equation (1.5)). Since $X_{jk}^{(i)} \in \mathcal{F}(\mathcal{E}^\infty)$, we have the identities $[\widetilde{D}_j^{(1)}, \partial/\partial w_s^{(1)}] = 0$ and $\widetilde{D}_j^{(1)}(\partial\alpha_k/\partial w_s^{(1)}) = 0$ for any j, k, s. Since the covering τ_1 is irreducible, from Proposition 1.4 it follows that $\partial\alpha_k/\partial w_s^{(1)} = c_{ks} = \mathrm{const}$, i.e., $\alpha_k = \sum_{s=1}^N c_{ks} w_s^{(1)} + \varphi_k(x, p_\sigma^i)$. From here we obtain that

$$\alpha^*(X_{jk}^{(2)}) = \widetilde{D}_j^{(1)}(\alpha_k) = \sum_{s=1}^N c_{ks} X_{js}^{(1)} + \bar{D}_j(\varphi_k)$$

and

$$\omega_k^{(2)} = \sum_{j=1}^n X_{jk}^{(2)} dx_j = \sum_{s=1}^N c_{ks} \omega_s^{(1)} + \widehat{d}(\varphi_k).$$

Thus $\omega_k^{(2)} \in V(\tau_1)$ for all $k = 1, \dots, N$. Similar calculations for the diffeomorphism α^{-1} show that $\omega_k^{(1)} \in V(\tau_2)$ for all $k = 1, \dots, N$. Consequently, $V(\tau_1) = V(\tau_2)$.

Let now $V(\tau_1) = V(\tau_2) = V$. Since the covering τ_1 is irreducible, we can consider the forms $\omega_1^{(1)}, \dots, \omega_N^{(1)}$, determining the covering τ_1, to constitute a basis of the space V. Then the representation

$$[\omega_k^{(2)}] = \sum_{s=1}^N c_{ks}[\omega_s^{(1)}], \qquad c_{ks} \in \mathbb{R},$$

is valid, or

$$\omega_k^{(2)} = \sum_{s=1}^N c_{ks}\omega_s^{(1)} + \widehat{d}(\varphi_k), \qquad \varphi_k \in \mathcal{F}(\mathcal{E}^\infty).$$

It is easily seen that the submanifold in the space $\widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$ determined by the equations

$$w_k^{(2)} - \sum_{s=1}^N c_{ks} w_s^{(1)} - \varphi_k = 0, \qquad k = 1, \dots, N,$$

is an invariant submanifold of the distribution $\widetilde{\mathcal{C}}^\oplus$ and is of codimension N. From Proposition 1.2 it follows that the coverings τ_1 and τ_2 are equivalent. $\qquad\square$

EXERCISE 1.8. Prove that a finite-dimensional covering τ of the form under consideration is irreducible if and only if $\dim V(\tau) = \dim \tau$.

REMARK 1.1. From the above said it follows that in the case $\dim M = 2$ (i.e., when the equation $\mathcal{E}$ is in two independent variables), there exists a one-to-one correspondence between one-dimensional coverings of the above considered form and conservation laws of the equation $\mathcal{E}$.

1.11. Bäcklund transformations. The above considered examples show that the theory of coverings is a convenient and adequate language to describe various nonlocal effects arising in studies of differential equations. In particular, it was shown that the constructions such as the Cole–Hopf substitution and the Miura–Gardner transformation are of a common nature and can be interpreted in terms of coverings in a natural way. In §2 we shall see that the Wahlquist–Estabrook prolongation structures are particular cases of coverings, while in this subsection we give a definition of Bäcklund transformations using the language of coverings.

Let us recall [**65, 93**] that a Bäcklund transformation between equations $\mathcal{E}$ and $\mathcal{E}'$ is a system of differential relations in unknown functions u and u' possessing the following property: if a function u is a solution of the equation $\mathcal{E}$ and u and u' satisfy the relations at hand, then the function u' is a solution of $\mathcal{E}'$.

Using the language of coverings, this definition reads as follows [**62**].

DEFINITION 1.6. A *Bäcklund transformation* between equations $\mathcal{E}$ and $\mathcal{E}'$ is the diagram

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{E}} & \\
 {\scriptstyle\tau}\swarrow & & \searrow{\scriptstyle\tau'} \\
\mathcal{E}^\infty & & (\mathcal{E}')^\infty
\end{array}
$$

in which the mappings τ and τ' are coverings.

If $\mathcal{E}^\infty = (\mathcal{E}')^\infty$, then the Bäcklund transformation of the equation $\mathcal{E}$ is sometimes called a *Bäcklund autotransformation*.

EXAMPLE 1.10. Consider the following system $\widetilde{\mathcal{E}}$ of equations

$$
\widetilde{w}_x = a \sin w, \quad w_y = \frac{1}{a} \sin \widetilde{w}, \qquad a \in \mathbb{R}, \quad a \neq 0,
$$

in the two-dimensional bundle $\pi\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$. Then the zero order operator (i.e., a morphism of vector bundles) $\Delta^+\colon v = w + \widetilde{w}$ determines the covering $\tau^+\colon J^\infty(\pi) \to J^\infty(\xi)$, where $\xi\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ (see §1.4). The restriction of this covering to $\widetilde{\mathcal{E}}^\infty$ determines a covering of the equation $\mathcal{E} = \{v_{xy} = \sin(v)\}$ by the equation $\widetilde{\mathcal{E}}^\infty$. In a similar way, the operator $\Delta^-\colon v = w - \widetilde{w}$ determines another covering $\tau^-\colon \widetilde{\mathcal{E}}^\infty \to \mathcal{E}^\infty$. Thus we obtain the Bäcklund autotransformation of the sine-Gordon equation

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{E}}^\infty & \\
 {\scriptstyle\tau^+}\swarrow & & \searrow{\scriptstyle\tau^-} \\
\mathcal{E}^\infty & & \mathcal{E}^\infty
\end{array}
$$

coinciding with the one discovered by Bäcklund [**87**].

Let us describe a natural way to construct Bäcklund transformations. Consider the diagram

$$
\begin{array}{ccc}
\widetilde{\mathcal{E}}_1 & \xrightarrow{\ \varphi\ } & \widetilde{\mathcal{E}}_2 \\
{\scriptstyle\tau_1}\downarrow & & \downarrow{\scriptstyle\tau_2} \\
\mathcal{E}_1^\infty & & \mathcal{E}_2^\infty
\end{array}
$$

where the vertical arrows are coverings and the mapping φ is an isomorphism. Then obviously $\tau_2 \circ \varphi\colon \widetilde{\mathcal{E}} \to \mathcal{E}_2^\infty$ is a covering and thus we see that the equations $\mathcal{E}_1^\infty$ and $\mathcal{E}_2^\infty$ are related by the Bäcklund transformation.

The next examples show that some known Bäcklund transformation can be obtained in the this way.

EXAMPLE 1.11. Consider the modified Korteweg–de Vries equation

$$
u_t = u_{xxx} + 6u^2 u_x \tag{1.12}
$$

The vector fields $\widetilde{D}_x = \bar{D}_x + X$, $\widetilde{D}_t = \bar{D}_t + T$, where

$$X = p_0 \frac{\partial}{\partial w} + (p_0 + \alpha \sin(w + \widetilde{w})) \frac{\partial}{\partial \widetilde{w}},$$

$$T = (p_2 + p_0^3) \frac{\partial}{\partial w} + (p_2 + 2\alpha p_1 \cos(w + \widetilde{w}) + 2p_0^3$$

$$+ 2\alpha p_0^2 \sin(w + \widetilde{w}) + 2\alpha^2 p_0 + \alpha^3 \sin(w + \widetilde{w})) \frac{\partial}{\partial \widetilde{w}},$$

determine the two-dimensional covering $\tau \colon \widetilde{\mathcal{E}}^\infty = \mathbb{R}^2 \times \mathcal{E}^\infty \to \mathcal{E}^\infty$. Using the change of coordinates $w \mapsto \widetilde{w}$, $\widetilde{w} \mapsto w$, we obtain a new covering. Consequently, if $u(t, x)$ is a solution of equation (1.12) and the functions $w(t, x)$ and $\widetilde{w}(t, x)$ satisfy the equations

$$u = w_x, \qquad \widetilde{w}_x - w_x = \alpha \sin(w + \widetilde{w}),$$

then the function $\widetilde{u} = \widetilde{w}_x$ is also a solution of equation (1.12). This is a well-known Bäcklund transformations for the modified Korteweg–de Vries equation [**70**].

EXAMPLE 1.12. Consider the Korteweg–de Vries equation

$$u_t = u_{xxx} + 12uu_x$$

There exists a two-dimensional covering over this equation determined by the field X of the form

$$X = p_0 \frac{\partial}{\partial w} + (\alpha - p_0 - (w - \widetilde{w})^2) \frac{\partial}{\partial \widetilde{w}}$$

(we leave to the reader as an exercise computation of the field T). As in the preceding example, by the change of variables $w \mapsto \widetilde{w}$, $\widetilde{w} \mapsto w$ [**70**] we obtain the Bäcklund transformation

$$u = w_x, \qquad \widetilde{w}_x + w_x = \alpha - (w - \widetilde{w})^2, \qquad \widetilde{u} = \widetilde{w}_x.$$

2. Examples of computations: coverings

In this section, we shall find coverings of a special type for some well-known equations of mathematical physics in two independent variables.

The process of constructing coverings for particular differential equations is a computational procedure similar to that used to compute symmetries. Namely, the covering space $\widetilde{\mathcal{E}}$ of any covering is locally diffeomorphic to the direct product $\mathcal{E}^\infty \times \mathbb{R}^N$ while the mapping $\tau \colon \widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}^N \to \mathcal{E}^\infty$ is the projection to the first factor. Therefore, a covering is locally determined by a distribution $\widetilde{\mathcal{C}}$, which, in the case of two independent variables, is given by two fields $\widetilde{D}_1 = \bar{D}_1 + X_1$ and $\widetilde{D}_2 = \bar{D}_2 + X_2$, where $X_i = \sum_{j=1}^N X_{ij} \partial / \partial w_j$, $X_{ij} \in \mathcal{F}(\widetilde{\mathcal{E}})$, $i = 1, 2$. Here $w_1, w_2, \ldots, w_N$ are coordinates in the fiber $\mathbb{R}^N$ (nonlocal variables) and $0 < N \leq \infty$. The integrability condition $[\widetilde{D}_1, \widetilde{D}_2] = 0$ for this distribution is of the form

$$[\bar{D}_1, X_2] + [X_1, \bar{D}_2] + [X_1, X_2] = 0. \tag{2.1}$$

This relation is a system of differential equations for the functions $X_{ij} \in \mathcal{F}(\widetilde{\mathcal{E}})$ describing all N-dimensional coverings over the equation $\mathcal{E}^\infty$. Solving this system, we shall find all coverings over the equation under consideration. Establishing equivalence of coverings, we use Corollary 1.3.

2.1. Coverings over the Burgers equation. Let us consider the Burgers equation (see also [**62**])

$$\mathcal{E} = \{u_t = uu_x + u_{xx}\}.$$

Let us choose the functions $x, t, p_1, \ldots, p_k, \ldots$ for internal coordinates on $\mathcal{E}^\infty$ (see §4.1 of Ch. 4).

The total derivatives $\bar{D}_x = \bar{D}_1$ and $\bar{D}_t = \bar{D}_2$ are of the form

$$\bar{D}_x = \frac{\partial}{\partial x} + \sum_k p_{k+1} \frac{\partial}{\partial p_k}, \qquad \bar{D}_t = \frac{\partial}{\partial t} + \sum_k \bar{D}_x^k (p_0 p_1 + p_2) \frac{\partial}{\partial p_k}.$$

Let the distribution on $\widetilde{\mathcal{E}}$ be given by the fields

$$\widetilde{D}_x = \bar{D}_x + X, \qquad \widetilde{D}_t = \bar{D}_t + T,$$

where

$$X = \sum_j X_j \frac{\partial}{\partial w_j}, \qquad T = \sum_k T_k \frac{\partial}{\partial w_k}, \qquad X_j, T_k \in \mathcal{F}(\widetilde{\mathcal{E}}).$$

Then the integrability condition (2.1) acquires the form

$$\frac{\partial T}{\partial x} + \sum_{i \geq 0} p_{i+1} \frac{\partial T}{\partial p_i} - \frac{\partial X}{\partial t} - \sum_{j \geq 0} \bar{D}_x^j (p_0 p_1 + p_2) \frac{\partial X}{\partial p_j} + [X, T] = 0, \qquad (2.2)$$

where the operators $\partial/\partial p_k$ act on the fields X and T component-wise (the coefficients of the fields X and T depend on $x, t, p_1, \ldots, p_k, \ldots, w_1, \ldots, w_N$). We restrict ourselves with the case when all functions X_j, T_k are independent of the variables x, t, and p_i, $i > 1$. In this case, equation (2.2) is written down in the form

$$p_1 \frac{\partial T}{\partial p_0} + p_2 \frac{\partial T}{\partial p_1} - (p_0 p_1 + p_2) \frac{\partial X}{\partial p_0} - -(p_1^2 + p_0 p_2 + p_3) \frac{\partial X}{\partial p_1} + [X, T] = 0. \quad (2.3)$$

The left-hand side of (2.3) is polynomial in p_2 and p_3. Therefore, the coefficients at p_2 and p_3 vanish, from where it follows that the function X is independent of p_1 while $\partial T/\partial p_1 = \partial X/\partial p_0$, i.e.,

$$T = p_1 \frac{\partial X}{\partial p_0} + R, \qquad (2.4)$$

where the coefficients of the field R are independent of p_1. Substituting now relations (2.4) to equation (2.3), we obtain

$$p_1^2 \frac{\partial^2 X}{\partial p_0^2} + p_1 \left(\frac{\partial R}{\partial p_0} - p_0 \frac{\partial X}{\partial p_0} + \left[X, \frac{\partial X}{\partial p_0} \right] \right) + [X, R] = 0.$$

Since the fields X and R are independent of p_1, the last equation is equivalent to the system

$$\frac{\partial^2 X}{\partial p_0^2} = 0, \qquad \frac{\partial R}{\partial p_0} = p_0 \frac{\partial X}{\partial p_0} + \left[X, \frac{\partial X}{\partial p_0} \right], \qquad [X, R] = 0. \qquad (2.5)$$

From the first equation of this system it follows that

$$X = p_0 A + B, \qquad (2.6)$$

where the coefficients of the fields A and B now depend on the variables w_k only. In other words, A and B are the fields in the fiber $\mathbb{R}^N$ of the covering. Using (2.6),

we see that the second equation of system (2.5) acquires the form $\partial R/\partial p_0 = p_0 A + [A, B]$, or

$$R = \frac{1}{2}\, p_0^2 A + p_0[A, B] + C, \tag{2.7}$$

where C is a field on $\mathbb{R}^N$. Finally, substituting expressions (2.6) and (2.7) to the last equation of system (2.5), we obtain the equation

$$p_0^2\left([A, [A, B]] + \frac{1}{2}\,[B, A]\right) + p_0([A, C] + [B, [A, B]]) + [B, C] = 0,$$

which is equivalent to the relations

$$[A, [A, B]] = \frac{1}{2}[A, B], \qquad [B, [B, A]] = [A, C], \qquad [B, C] = 0. \tag{2.8}$$

Thus, we proved the following result

THEOREM 2.1. *Any covering of the Burgers equation, where the coefficients of the fields X and T are independent of the variables t, x, and p_k, $k > 1$, are determined by the fields of the form*

$$\widetilde{D}_x = \bar{D}_x + p_0 A + B,$$

$$\widetilde{D}_t = \bar{D}_t + \left(p_1 + \frac{1}{2}p_0^2\right)A + p_0[A, B] + C,$$

where A, B, and C are arbitrary fields in the fiber of the covering satisfying relations (2.8).

REMARK 2.1. Let us stress here a remarkable fact. Consider the free Lie algebra $\mathcal{G}$ with generators a, b, and c, satisfying the relations

$$[a, [a, b]] = \frac{1}{2}[a, b], \qquad [b, [b, a]] = [a, c], \qquad [b, c] = 0.$$

Then the Lie algebra generated by the fields A, B, and C is the image of the algebra $\mathcal{G}$ when it is represented in the Lie algebra of vector fields on the manifold $\mathbb{R}^N$. Thus, Theorem 2.1 shows that coverings over the Burgers equation satisfying the above formulated conditions are uniquely determined by representations of the algebra $\mathcal{G}$ in vector fields. By this reason, we shall call the Lie algebra $\mathcal{G}$ *universal*.

REMARK 2.2. Universal algebras first aroused in the context of the so-called *prolongation structures* of Wahlquist–Estabrook [**145, 51**]. Examples of universal algebras for various evolution equations, as well as some particular representations of these algebras, can be found in [**23**]. Note that Wahlquist–Estabrook prolongation structures are particular cases of coverings.

Let us now describe one-dimensional coverings of the type under consideration over the Burgers equation. Let w be a nonlocal variable. Then $A = \alpha\partial/\partial w$, $B = \beta\partial/\partial w$, $C = \gamma\partial/\partial w$, where α, β, $\gamma \in C^\infty(\mathbb{R})$ and (2.8) reduces to a system of ordinary differential equations in α, β, and γ.

Consider the case $A = 0$ first. Then system (2.8) reduces to the relation $[B, C] = 0$, or $\beta'\gamma - \beta\gamma' = 0$, where the "prime" denotes the derivative with respect to w. If the field B does not vanish, the coordinate w in $\mathbb{R}$ can be chosen in such a way that $B = \partial/\partial w$, from where it follows that $\gamma = \mathrm{const}$. Note that choosing another coordinate in the fiber, we pass to a new covering equivalent to the initial one. If $B = 0$, then $C \neq 0$ and this case is considered in a similar way. Thus,

the coordinate w can be always locally be chosen in such a way that $\beta, \gamma = \text{const}$. Therefore, the system of fields

$$\widetilde{D}_x = \bar{D}_x + \beta \frac{\partial}{\partial w}, \qquad \widetilde{D}_t = \bar{D}_t + \gamma \frac{\partial}{\partial w}$$

possesses a nontrivial kernel consisting of functions $\psi(w - \beta x - \gamma t)$. In other words, the manifold $\widetilde{\mathcal{E}}$ is fibered by invariant manifolds $\mathcal{X}_\lambda = \{w - \beta x - \gamma t = \lambda\}$, $\lambda \in \mathbb{R}$, isomorphic to the manifold $\mathcal{E}^\infty$, i.e., the covering under consideration is trivial.

Let us pass now to the case $A \neq 0$ and study the structure of the coverings in a neighborhood of a generic point of the field A. Let choose a coordinate w in this neighborhood such that $A = \partial/\partial w$. Then system (2.8) acquires the form

$$\beta'' = \frac{1}{2}\beta', \qquad (\beta')^2 - \beta\beta'' = \gamma', \qquad \beta'\gamma = \gamma'\beta$$

and possesses solutions of two types:

$$\beta = \text{const}, \qquad \gamma = \text{const}$$

and

$$\beta = \mu e^{w/2} + \nu, \quad \gamma = -\frac{\nu}{2}(\mu e^{w/2} + \nu), \quad \mu = \text{const} \neq 0, \quad \nu = \text{const}. \qquad (2.9)$$

All coverings of this type are pair-wise nonequivalent and by the coordinate change $w \mapsto w - \beta x - \gamma t$ reduce to the form

$$\widetilde{D}_x = \bar{D}_x + p_0 \frac{\partial}{\partial w}, \qquad \widetilde{D}_t = \bar{D}_t + \left(p_1 + \frac{1}{2}p_0^2\right)\frac{\partial}{\partial w}.$$

Consider coverings of the second type and denote by $\tau_{\mu,\nu}$ the covering, determined by equation (2.9). Then the coverings τ_{μ_1,ν_1} and τ_{μ_2,ν_2} are equivalent if and only if $\text{sign}\,\mu_1 = \text{sign}\,\mu_2$ and $\nu_1 = \nu_2$.

Thus, we proved the following statement:

PROPOSITION 2.2. *Any nontrivial one-dimensional covering over the Burgers equation* $\mathcal{E} = \{u_t = uu_x + u_{xx}\}$, *where the coefficients of the fields* X *and* T *are independent of the variables* t, x, *and* p_k, $k > 1$, *are locally equivalent to one of the following*:

$$\tau^0 : \widetilde{D}_x = \bar{D}_x + p_0 \frac{\partial}{\partial w},$$

$$\widetilde{D}_t = \bar{D}_t + \left(p_1 + \frac{1}{2}p_0^2\right)\frac{\partial}{\partial w},$$

$$\tau_\nu^+ : \widetilde{D}_x = \bar{D}_x + (p_0 + e^{w/2} + \nu)\frac{\partial}{\partial w},$$

$$\widetilde{D}_t = \bar{D}_t + \left(p_1 + \frac{1}{2}p_0^2 + \frac{1}{2}e^{w/2}p_0 - \frac{\nu}{2}e^{w/2} - \frac{\nu^2}{2}\right)\frac{\partial}{\partial w},$$

$$\tau_\nu^- : \widetilde{D}_x = \bar{D}_x + (p_0 - e^{w/2} + \nu)\frac{\partial}{\partial w},$$

$$\widetilde{D}_t = \bar{D}_t + \left(p_1 + \frac{1}{2}p_0^2 - \frac{1}{2}e^{w/2}p_0 + \frac{\nu}{2}e^{w/2} - \frac{\nu^2}{2}\right)\frac{\partial}{\partial w},$$

where $\nu = \text{const}$. *All these coverings are pair-wise nonequivalent.*

In each of the above listed cases, the manifold $\widetilde{\mathcal{E}}$ is contact equivalent to a manifold of the form $(\mathcal{E}')^\infty$, where $\mathcal{E}'$ is a second order evolution equation. The corresponding change of variables is:

$u = w_x$ for the covering τ^0;
$u = w_x - e^{w/2} - \nu$ for the coverings τ_ν^+;
$u = w_x + e^{w/2} - \nu$ for the coverings τ_ν^-.

EXERCISE 2.1. Write down the covering equations in each of the above listed cases.

REMARK 2.3. Similar, but more cumbersome calculations, allow one to describe higher dimensional coverings over the Burgers equation. For example, there exist six 3-parameter families, one 4-parameter, and one 1-parameter family of nontrivial pair-wise nonequivalent two-dimensional coverings.

2.2. Coverings over the Korteweg–de Vries equation. Consider the Korteweg–de Vries equation

$$\mathcal{E} = \{u_t = uu_x + u_{xxx}\}$$

now.

Let us take the functions x, t, $p_1, \ldots, p_k, \ldots$ for internal coordinates on $\mathcal{E}^\infty$. The total derivatives on $\mathcal{E}^\infty$ are of the form

$$\bar{D}_x = \frac{\partial}{\partial x} + \sum_k p_{k+1}\frac{\partial}{\partial p_k}, \quad \bar{D}_t = \frac{\partial}{\partial t} + \sum_k \bar{D}_x^k(p_0 p_1 + p_3)\frac{\partial}{\partial p_k}.$$

Let the distribution on $\widetilde{\mathcal{E}}$ be described by the fields

$$\widetilde{D}_x = \bar{D}_x + X, \qquad \widetilde{D}_t = \bar{D}_t + T,$$

where

$$X = \sum_j X_j\frac{\partial}{\partial w_j}, \quad T = \sum_k T_k\frac{\partial}{\partial w_k}, \qquad X_j, T_k \in \mathcal{F}(\widetilde{\mathcal{E}}).$$

We shall solve equation (2.1) assuming that all the functions X_j, T_k are independent of the variables x, t, and p_i, $i > 2$.

Then computation similar to those exposed in §2.1 lead to the following result:

THEOREM 2.3. *Any covering of the Korteweg–de Vries equation such that the coefficients of the fields X and T are independent of the variables t, x, and p_k $k > 2$, are determined by the fields of the form*

$$\widetilde{D}_x = \bar{D}_x + p_0^2 A + p_0 B + C,$$

$$\widetilde{D}_t = \bar{D}_t + \left(2p_0 p_2 - p_1^2 + \frac{2}{3}p_0^3 A\right) + \left(p_2 + \frac{1}{2}p_0^2\right)B$$

$$+ p_1[B, C] + \frac{1}{2}p_0^2[B, [C, B]] + p_0[C, [C, B]] + D,$$

where A, B, C, D are vector fields on the fiber of the covering satisfying the relations

$$[A, B] = [A, C] = [C, D] = 0,$$
$$[B, [B, [B, C]]] = 0, \quad [B, D] + [C, [C, [C, B]]] = 0, \tag{2.10}$$
$$[A, D] + \frac{1}{2}[C, B] + \frac{3}{2}[B, [C, [C, B]]] = 0.$$

REMARK 2.4. The free Lie algebra with four generators and relations (2.10) is the universal algebra of the Korteweg–de Vries equation. It first aroused in studies of prolongation structures [**145**]. It is known that this algebra is infinite-dimensional [**107**].

Let us describe one-dimensional coverings over the Korteweg–de Vries equation. It is easily seen that if a covering is nontrivial, then at least one of the fields A or B must not vanish. Assume that $A \neq 0$. Then a nonlocal variable w can be chosen in such a way that $A = \partial/\partial w$. In this case, from (2.10) it easily follows that

$$B = \beta \frac{\partial}{\partial w}, \qquad C = \gamma \frac{\partial}{\partial w}, \qquad D = \delta \frac{\partial}{\partial w},$$

where β, γ, δ are constants. Thus, any covering over the KdV equation with $A \neq 0$ is equivalent to the covering determined by the fields

$$
\begin{aligned}
X &= (p_0^2 + \beta p_0 + \gamma) \frac{\partial}{\partial w}, \\
T &= \left(2p_0 p_2 - p_1^2 + \frac{2}{3} p_0^3 + \beta \left(p_2 + \frac{1}{2} p_0^2 \right) + \delta \right) \frac{\partial}{\partial w}.
\end{aligned}
\tag{2.11}
$$

From the results of §1.6 it easily follows that the coverings of the form (2.11) are nontrivial. To check whether they are equivalent to each other, consider the Whitney product of two coverings (2.11),

$$
\begin{aligned}
\widetilde{D}_x &= \bar{D}_x + (p_0^2 + \beta_1 p_0 + \gamma_1) \frac{\partial}{\partial w_1} + (p_0^2 + \beta_2 p_0 + \gamma_2) \frac{\partial}{\partial w_2}, \\
\widetilde{D}_t &= \bar{D}_t + \left(2p_0 p_2 - p_1^2 + \frac{2}{3} p_0^3 + \beta_1 \left(p_2 + \frac{1}{2} p_0^2 \right) + \delta_1 \right) \frac{\partial}{\partial w_1} \\
&\qquad + \left(2p_0 p_2 - p_1^2 + \frac{2}{3} p_0^3 + \beta_2 \left(p_2 + \frac{1}{2} p_0^2 \right) + \delta_2 \right) \frac{\partial}{\partial w_2},
\end{aligned}
$$

and the linear system

$$\widetilde{D}_x(f) = 0, \qquad \widetilde{D}_t(f) = 0,$$

where f is a function on the space of the Whitney product. It is easily seen that this system is equivalent to the following one:

$$
\begin{aligned}
&\frac{\partial f}{\partial x} + \gamma_1 \frac{\partial f}{\partial w_1} + \gamma_2 \frac{\partial f}{\partial w_2} = 0, \quad \frac{\partial f}{\partial t} + \delta_1 \frac{\partial f}{\partial w_1} + \delta_2 \frac{\partial f}{\partial w_2} = 0, \\
&\beta_1 \frac{\partial f}{\partial w_1} + \beta_2 \frac{\partial f}{\partial w_2} = 0, \quad \frac{\partial f}{\partial w_1} + \frac{\partial f}{\partial w_2} = 0,
\end{aligned}
\tag{2.12}
$$

where $f = f(x, t, w_1, w_2)$. Solutions of (2.12) are different from constants if and only if $\beta_1 = \beta_2$. Using (2.12), it is easy to show that the equivalence classes of coverings with $A \neq 0$ are parametrized by β and their representatives can be written as

$$
\begin{aligned}
X &= (p_0^2 + \beta p_0) \frac{\partial}{\partial w}, \\
T &= \left(2p_0 p_2 - p_1^2 + \frac{2}{3} p_0^3 + \beta \left(p_2 + \frac{1}{2} p_0^2 \right) \right) \frac{\partial}{\partial w}.
\end{aligned}
\tag{2.13}
$$

Assume now that $A = 0$. Then $B \neq 0$ and in a suitable coordinate system the field B can be written in the form $\partial/\partial w$. Then relations (2.10) are transformed to

$$\frac{\partial^3 C}{\partial w^3} = 0, \quad \frac{\partial C}{\partial w} = 3\left[\frac{\partial C}{\partial w}, C\right], \quad \frac{\partial D}{\partial w} = \left[C, \left[C, \frac{\partial C}{\partial w}\right]\right], \quad [C, D] = 0.$$

This system possesses solutions of the two types

$$C = \left(\frac{1}{6}w^2 + \beta w + \gamma\right)\frac{\partial}{\partial w}, \qquad D = \left(\beta^2 - \frac{2}{3}\gamma\right)\frac{\partial}{\partial w}$$

and

$$C = \gamma\frac{\partial}{\partial w}, \qquad D = \delta\frac{\partial}{\partial w},$$

where β, γ, δ are constants. Respectively, we have

$$X = \left(p_0 + \frac{1}{6}w^2 + \beta w + \gamma\right)\frac{\partial}{\partial w},$$
$$T = \left(p_2 + p_1\left(\frac{1}{3}w + \beta\right) + \frac{1}{3}p_0^2 + \left(\frac{1}{18}w^2 + \frac{1}{3}\beta w + \beta^2 - \frac{1}{3}\gamma\right)p_0\right.$$
$$\left. + \left(\beta^2 - \frac{2}{3}\gamma\right)\left(\frac{1}{6}w^2 + \beta w + \gamma\right)\right)\frac{\partial}{\partial w},$$

and

$$X = \left(p_0 + \gamma\right)\frac{\partial}{\partial w}, \qquad T = \left(p_2 + \frac{1}{2}p_0^2 + \delta\right)\frac{\partial}{\partial w}.$$

In the first case, by the change of variables $w \mapsto w - \frac{1}{3}\beta$, we can transform the covering to

$$X = \left(p_0 + \frac{1}{6}w^2 + \alpha\right)\frac{\partial}{\partial w},$$
$$T = \left(p_2 + \frac{1}{3}wp_1 + \frac{1}{3},p_0^2 + \left(\frac{1}{18}w^2 - \frac{1}{3}\alpha\right)p_0 - \frac{2}{3}\alpha\left(\frac{1}{6}w^2 + \alpha\right)\right)\frac{\partial}{\partial w}, \tag{2.14}$$

where α is an arbitrary constant. The coverings belonging to this family are mutually nonequivalent.

In the second case, all coverings are equivalent to each other. As a representative of this class, one can take, for example, the covering

$$X = p_0\frac{\partial}{\partial w}, \qquad T = \left(p_2 + \frac{1}{2}p_0^2\right)\frac{\partial}{\partial w}. \tag{2.15}$$

It is easily seen that all the coverings (2.13)–(2.15) are pair-wise nonequivalent. Thus, we obtained a complete description of coverings over KdV of the type under consideration. Let us summarize the results.

PROPOSITION 2.4. *Any nontrivial covering of the Korteweg–de Vries equation $\mathcal{E} = \{u_t = uu_x + u_{xxx}\}$, such that the coefficient of the fields X and t are independent*

of t, x, and p_k, $k > 2$, are locally equivalent to one of the following:

$$\tau^0\colon \widetilde{D}_x = \bar{D}_x + p_0 \frac{\partial}{\partial w},$$

$$\widetilde{D}_t = \bar{D}_t + \left(p_2 + \frac{1}{2}p_0^2\right)\frac{\partial}{\partial w},$$

$$\tau^1_\alpha\colon \widetilde{D}_x = \bar{D}_x + \left(p_0 + \frac{1}{6}w^2 + \alpha\right)\frac{\partial}{\partial w},$$

$$\widetilde{D}_t = \bar{D}_t + \left(p_2 + \frac{1}{3}wp_1 + \frac{1}{3}p_0^2 + \left(\frac{1}{18}w^2 - \frac{1}{3}\alpha\right)p_0\right.$$

$$\left.- \frac{2}{3}\alpha\left(\frac{1}{6}w^2 + \alpha\right)\right)\frac{\partial}{\partial w}, \quad \alpha \in \mathbb{R},$$

$$\tau^2_\beta\colon \widetilde{D}_x = \bar{D}_x + \left(p_0^2 + \beta p_0\right)\frac{\partial}{\partial w},$$

$$\widetilde{D}_t = \bar{D}_t + \left(2p_0p_2 - p_1^2 + \frac{2}{3}p_0^3 + \beta\left(p_2 + \frac{1}{2}p_0^2\right)\right)\frac{\partial}{\partial w}, \quad \beta \in \mathbb{R}.$$

All these coverings are pair-wise nonequivalent.

In each of the listed cases, the manifold $\widetilde{\mathcal{E}}$ is contact diffeomorphic to a manifold of the form $(\mathcal{E}')^\infty$, where $\mathcal{E}'$ is a third order evolution equation. The corresponding change of variables is of the form

$u = w_x$ for the covering τ^0;
$u = w_x - \frac{1}{6}w^2 - \frac{3}{2}\alpha$ for the coverings τ^1_α;
$u = -\alpha \pm \sqrt{\beta^2 + w_x}$ for the coverings τ^2_β.

As we already saw (see §1.9), the covering equation for the covering τ^0 is the potential Korteweg–de Vries equation $w_t = w_{xxx} + \frac{1}{2}w_x^2$, while for the covering τ^1_α, $\alpha = 0$, the covering equation is the modified Korteweg–de Vries equation $w_t = w_{xxx} - \frac{1}{6}w^2w_x$.

2.3. Coverings over the equation $u_t = (B(u)u_x)_x$. Let us describe, without proof, another result of covering computations [**54**].

PROPOSITION 2.5. *Any nontrivial one- or two-dimensional covering of the equation $\mathcal{E} = \{u_t = (B(u)u_x)_x\}$, where the coefficients of the fields X and T are independent of the variables t, x, and p_k, $k > 1$, is locally equivalent to one of the*

following:

$$\tau^1 \colon \widetilde{D}_x = \bar{D}_x + p_0 \frac{\partial}{\partial w},$$

$$\widetilde{D}_t = \bar{D}_t + B(p_0)p_1 \frac{\partial}{\partial w},$$

$$\tau^2_1 \colon \widetilde{D}_x = \bar{D}_x + p_0 \frac{\partial}{\partial w_1} + w_1 \frac{\partial}{\partial w_2},$$

$$\widetilde{D}_t = \bar{D}_t + B(p_0), p_1 \frac{\partial}{\partial w_1} + \widetilde{B}(p_0) \frac{\partial}{\partial w_2},$$

$$\tau^2_2 \colon \widetilde{D}_x = \bar{D}_x + \frac{\partial}{\partial w_1} - p_0 w_1 \frac{\partial}{\partial w_2},$$

$$\widetilde{D}_t = \bar{D}_t + (\widetilde{B}(p_0) - w_1 B(p_0) \, p_1) \frac{\partial}{\partial w_2},$$

There are no irreducible coverings of dimension greater than two.

To conclude this section, let us consider two examples, where the equation under consideration is not an evolution one.

2.4. Covering over the f-Gordon equation. Consider the equation $\mathcal{E}_f = \{u_{xy} = f\}$, where $f = f(u)$ is an arbitrary function depending on u. Let us choose the functions x, y, u, p_1, $q_1, \ldots, p_k$, $q_k, \ldots$, where $x = x_1$, $y = x_2$, $u = p_{(0,0)}$, $p_k = p_{(k,0)}$, $q_k = p_{(0,k)}$ for internal coordinates on $\mathcal{E}_f^\infty$. Consider the coverings $\tau \colon \widetilde{\mathcal{E}}_f = \mathcal{E}_f^\infty \times \mathbb{R}^N \to \mathcal{E}_f^\infty$, $\widetilde{D}_x = \bar{D}_x + X$, $\widetilde{D}_y = \bar{D}_y + Y$, $\bar{D}_x = \bar{D}_1$, $\bar{D}_y = \bar{D}_2$, such that the coefficients of the fields X and Y are independent of the variables x, y and p_i, q_i, $i > 1$. In this case conditions (2.1) are transformed to

$$\begin{cases} \dfrac{\partial X}{\partial q_1} = \dfrac{\partial Y}{\partial p_1} = 0, \\[2mm] p_1 \dfrac{\partial Y}{\partial u} + f\left(\dfrac{\partial Y}{\partial q_1} - \dfrac{\partial X}{\partial p_1} \right) - q_1 \dfrac{\partial X}{\partial u} + [X, Y] = 0. \end{cases}$$

Analyzing this system for the case of one-dimensional coverings, we obtain the following result:

PROPOSITION 2.6. *Any nontrivial one-dimensional covering of the equation $\mathcal{E} = \{u_{xy} = f\}$, $\partial f/\partial u \neq 0$, where the coefficients of the fields X and Y are independent of the variables x, y, and p_i, q_i, $i > 1$, are locally (in a neighborhood of any point, where $\partial^2 X/\partial p_1^2 \neq 0$ and $\partial Y^2/\partial q_1^2 \neq 0$), is equivalent to the following one*:

$$\tau_\alpha^f \colon \qquad \widetilde{D}_x = \bar{D}_x + (\alpha p_1^2 + 2F) \frac{\partial}{\partial w}, \quad \widetilde{D}_y = \bar{D}_y + (q_1^2 + 2\alpha F) \frac{\partial}{\partial w},$$

where $\alpha \in \mathbb{R}$, $F(u) = \int_0^u f(u)\,du$. These coverings are pair-wise nonequivalent for different values of the parameter α.

Consider the equations $\mathcal{E}_f$ and $\mathcal{E}_g$ and let us equate the right-hand sides of the equations determining the covering τ_β^g. As a result, we shall obtain the system

$$\begin{aligned} \alpha u_x^2 + 2F(u) &= \beta v_x^2 + 2G(v), \\ u_y^2 + 2\alpha F(u) &= v_y^2 + 2\beta G(v). \end{aligned} \tag{2.16}$$

A simple check shows that if $v = v(x, y)$ is a solution of the equation $\mathcal{E}_g$ such that system (2.16) is compatible, then any solution $u = u(x, y)$ of equation (2.16), such that $\alpha u_x + u_y \neq 0$, is also a solution of $\mathcal{E}_f$ for a given v. Thus, system (2.16) determines a "partial Bäcklund transformation" between equations $\mathcal{E}_f$ and $\mathcal{E}_g$.

REMARK 2.5. Prolongation structures for the equation $\mathcal{E}_f$ were studied in [**106**] The result obtained can be reformulated as follows: the equation $\mathcal{E}_f$ possesses nontrivial one-dimensional coverings, such that the coefficients of the fields X and Y are linear in p_1 and q_1 respectively, if and only if the function f satisfies the equation $d^2 f/du^2 = af$, $a = \text{const}$. In particular, if $f = \sin u$, then the manifold $\widetilde{\mathcal{E}}$ is contact diffeomorphic to $(\mathcal{E}_{\sin u})^\infty$ and the corresponding covering is the classical Bäcklund transformation of the sine-Gordon equation. If $f = e^u$, then $\widetilde{\mathcal{E}}_f$ is contact diffeomorphic to the manifold $(\mathcal{E}_0)^\infty$ and the corresponding covering determines the Bäcklund transformation between the Liouville equation and the wave equation.

2.5. Coverings of the equation $u_{xx} + u_{yy} = \varphi(u)$. Consider the equation $\mathcal{E}_\varphi = \{u_{xx} + u_{yy} = \varphi(u)\}$, where φ is a smooth function depending on u. Let us choose for coordinates on $\mathcal{E}^\infty$ the functions x, y, u, $p_1, q_1, \ldots, p_k, q_k, \ldots$, where $p_k = p_{(k,0)}$, $q_k = p_{(k-1,1)}$ (see also [**62**]). Then the total derivatives $\bar{D}_x$ and $\bar{D}_y$ on $\mathcal{E}^\infty$ are of the form

$$
\begin{cases}
\bar{D}_x = \dfrac{\partial}{\partial x} + p_1 \dfrac{\partial}{\partial u} + \displaystyle\sum_k p_{k+1} \dfrac{\partial}{\partial p_k} + \sum_k q_{k+1} \dfrac{\partial}{\partial q_k}, \\[3mm]
\bar{D}_y = \dfrac{\partial}{\partial y} + q_1 \dfrac{\partial}{\partial u} + \displaystyle\sum_k q_{k+1} \dfrac{\partial}{\partial p_k} + \sum_k \bar{D}_x^{k-1}(\varphi - p_2) \dfrac{\partial}{\partial q_k}.
\end{cases}
$$

Let us set $\widetilde{D}_x = \bar{D}_x + X$, $\widetilde{D}_y = \bar{D}_y + Y$ and assume that the coefficients of the fields X and Y depend on the variables $w_1, \ldots, w_s$, u, p_1, q_1 only. In this case, conditions (2.1) transform to

$$
\frac{\partial Y}{\partial p_1} + \frac{\partial X}{\partial q_1} = 0, \quad \frac{\partial Y}{\partial q_1} - \frac{\partial X}{\partial p_1} = 0,
$$
$$
p_1 \frac{\partial Y}{\partial u} - q_1 \frac{\partial X}{\partial u} - \varphi \frac{\partial X}{\partial q_1} + [X, Y] = 0. \tag{2.17}
$$

Let us introduce new complex variables $z = p_1 + iq_1$, $H = X + iY$. Then from the first two equations (2.17) it follows that H is an analytic function in z:

$$
H(z) = \sum_{k=0}^\infty H_k z^k, \qquad H_k = H_k(w, u) = X_k + iY_k, \tag{2.18}
$$

while the third equation in this system acquires the form

$$
\bar{z} \frac{\partial H}{\partial u} - z \frac{\partial \bar{H}}{\partial u} + \varphi \left(\frac{\partial H}{\partial z} - \frac{\partial \bar{H}}{\partial z} \right) - [H, \bar{H}] = 0, \tag{2.19}
$$

Substituting (2.18) to (2.19), we obtain

$$
\sum_{k=0}^\infty \left(\bar{z} z^k \frac{\partial H_k}{\partial u} - z \bar{z}^k \frac{\partial \bar{H}_k}{\partial u} + k\varphi(z^{k-1} H_k - \bar{z}^{k-1} \bar{H}_k) \right) = \sum_{k,j=0}^\infty z^k \bar{z}^j [H_k, \bar{H}_j],
$$

from where it follows that

$$\varphi(H_1 - \bar{H}_1) = [H_0, \bar{H}_0], \quad \frac{\partial H_0}{\partial u} - 2\varphi H_2 = [H_0, \bar{H}_1],$$

$$\frac{\partial H_1}{\partial u} - \frac{\partial \bar{H}_1}{\partial u} = [H_1, \bar{H}_1], \quad (k+1)\varphi H_{k+1} = [H_k, \bar{H}_0], \tag{2.20}$$

$$\frac{\partial H_k}{\partial u} = [H_k, \bar{H}_1], \quad [H_k, \bar{H}_j] = 0, \qquad k, j > 1, \quad k \geq j.$$

Let us note now that by a coordinate change in the fiber of the covering, the field $\partial/\partial u + X_1$ may be transformed to $\partial/\partial u$. Then equation (2.20) will take the form

$$2i\varphi Y_1 = [H_0, \bar{H}_0], \quad 2\varphi H_2 = \left[\frac{\partial}{\partial u} + iY_1, \bar{H}_0\right],$$

$$\frac{\partial Y_1}{\partial u} = 0, \quad (k+1)\varphi H_{k+1} = [H_k, \bar{H}_0], \tag{2.21}$$

$$[H_k, \bar{H}_j] = 0, \qquad k, j > 1, \quad k \geq j.$$

There are two options for the field Y_1: (a) $Y_1 = 0$ and (b) $Y_1 \neq 0$. For the first case we have

$$2i[H_0, \bar{H}_0] = 0, \quad 2\varphi H_2 = \frac{\partial H_0}{\partial u},$$

$$\frac{\partial H_k}{\partial u} = 0, \quad (k+1)\varphi H_{k+1} = [H_k, \bar{H}_0], \tag{2.22}$$

$$[H_k, \bar{H}_j] = 0, \qquad k, j > 1, \quad k \geq j.$$

In the case (b), after a suitable coordinate change in the fiber, the field Y_1 is transformed to the field $\partial/\partial w$, where $w = w_1$, and system (2.21) acquires the form

$$2i\varphi \frac{\partial}{\partial w} = [H_0, \bar{H}_0], \quad 2\varphi H_2 = i\frac{\partial \bar{H}_0}{\partial \xi},$$

$$\frac{\partial H_k}{\partial \bar{\xi}} = 0, \quad (k+1)\varphi H_{k+1} = [H_k, \bar{H}_0], \tag{2.23}$$

$$[H_k, \bar{H}_j] = 0, \qquad k, j > 1, \quad k \geq j,$$

where $\xi = w + iu$. Solving systems (2.22) and (2.23) in the case $\dim \tau = 1$, for an arbitrary function φ we obtain the solution

$$H = (2\mu_2\Phi + \mu_0 + \bar{\mu}_2 z^2)\frac{\partial}{\partial w}, \qquad \Phi = \int_{u_0}^{u} \varphi\, du,$$

where $\mu_0, \mu_2 \in \mathbb{C}$.

For $\varphi = 0$ (and, of course, for equations transforming to this value of φ) we have

$$H = \left(\exp(\gamma\xi)\sum_{k \neq 1}\mu_k z^k + iz\right)\frac{\partial}{\partial w}, \qquad \tau \in \mathbb{R}$$

and

$$H = \sum_{k \neq 1}\mu_k z^k \frac{\partial}{\partial w}.$$

If $\varphi = u$, then

$$H = \left(\mu_0 + \mu_1 \xi + iz \right) \frac{\partial}{\partial w}.$$

For

$$\varphi = \frac{1}{2} \left(\frac{1}{\gamma} + \gamma \right) \sinh(2\gamma u), \qquad \gamma \in \mathbb{R}, \quad \gamma \neq 0, 1,$$

we have

$$H = \left(\cos(\gamma\xi) + \frac{1}{\gamma} \sin(\gamma\xi) + iz \right) \frac{\partial}{\partial w}$$

and for

$$\varphi = \frac{1}{2} \left(\frac{1}{\gamma} - \gamma \right) \sin(2\gamma u), \qquad \gamma \in \mathbb{R}, \quad \gamma \neq 0, 1,$$

the solution is

$$H = \left(\cosh(\gamma\xi) + \frac{1}{\gamma} \sin(\gamma\xi) + iz \right) \frac{\partial}{\partial w}.$$

In the first case (an arbitrary φ) the classes of nontrivial coverings are represented by the following fields:

$$H = \left(2\exp(i\theta)\Phi + \exp(-i\theta)z^2 \right) \frac{\partial}{\partial w}, \qquad \theta \in [0, 2\pi).$$

We leave to the reader to analyze other cases.

3. Nonlocal symmetries

3.1. Definition of nonlocal symmetries. Let $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a covering over the equation $\mathcal{E}^\infty$. A *nonlocal symmetry* of the equation $\mathcal{E}$ will be called a local symmetry of the object $\widetilde{\mathcal{E}}$. In other words, a nonlocal symmetry of the equation $\mathcal{E}$ is a transformation (finite or infinitesimal) of the object $\widetilde{\mathcal{E}}$, which preserves the distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}}$. Note that the definition of nonlocal symmetries assumes existence of some covering over the equation $\mathcal{E}^\infty$. To underline this fact, nonlocal symmetries in the covering $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ will be called *symmetries of type τ*, or *nonlocal τ-symmetries*. In what follows, we shall consider *infinitesimal* symmetries only and shall call them just "nonlocal symmetries".

Since general differential-geometric structure of covering manifolds $\widetilde{\mathcal{E}}$ is quite similar to that of infinitely prolonged equations, definition of nonlocal infinitesimal symmetries corresponds exactly to definition of higher symmetries for differential equations.

DEFINITION 3.1. The *algebra of nonlocal symmetries* of type τ (or of *nonlocal τ-symmetries*) of the equation $\mathcal{E}$ is the quotient Lie algebra

$$\operatorname{sym}_\tau \mathcal{E} = \mathrm{D}_{\mathcal{C}}(\widetilde{\mathcal{E}}) \big/ \mathcal{C}\mathrm{D}(\widetilde{\mathcal{E}}),$$

where

$$\mathcal{C}\mathrm{D}(\widetilde{\mathcal{E}}) = \left\{ \sum_{i=1}^n \varphi_i \widetilde{D}_i \mid \varphi_i \in \mathcal{F}(\widetilde{\mathcal{E}}) \right\},$$

while $\mathrm{D}_{\mathcal{C}}(\widetilde{\mathcal{E}})$ consists of vector fields X on $\widetilde{\mathcal{E}}$ such that $[X, \mathcal{C}\mathrm{D}(\widetilde{\mathcal{E}})] \subset \mathcal{C}\mathrm{D}(\widetilde{\mathcal{E}})$.

EXERCISE 3.1. Prove that when coverings τ_1 and τ_2 are equivalent, then the Lie algebras of nonlocal symmetries $\mathrm{sym}_{\tau_1}\,\mathcal{E}$ and $\mathrm{sym}_{\tau_2}\,\mathcal{E}$ are isomorphic.

3.2. How to seek for nonlocal symmetries? The procedure of finding nonlocal symmetries for a particular differential equation splits in a natural way into two steps: first, it is necessary to construct a covering τ over the equation under consideration and then to find τ-symmetries.

The problem of covering construction was discussed in details in §2. Let us note here that in search for nonlocal symmetries, one has to choose the corresponding covering in a special way: in general, it may happen that the algebra $\mathrm{sym}_\tau\,\mathcal{E}$ diminishes in comparison with $\mathrm{sym}\,\mathcal{E}$. For example, this is the case, when we take the covering over the Korteweg–de Vries equation by the modified Korteweg–de Vries equation (see Example 3.1 below). Besides, some nonlocal symmetries are not interesting in applications (for example, symmetries of trivial coverings).

Consider now the problem of computation of nonlocal symmetry algebra in a given covering $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$. If the covering object $\widetilde{\mathcal{E}}$ is of the form $\widetilde{\mathcal{E}} = (\mathcal{E}')^\infty$, then the problem reduces to computation of local symmetries for the equation $(\mathcal{E}')^\infty$. When $\widetilde{\mathcal{E}}$ is not represented in this form explicitly, the computational procedure may be based on one of the theorems proved below. Examples of computations of nonlocal symmetries will be exposed in the next section.

THEOREM 3.1. *The algebra* $\mathrm{sym}_\tau\,\mathcal{E}$ *is isomorphic to the Lie algebra of vector fields X on $\widetilde{\mathcal{E}}$ satisfying the following conditions*:

1. X *is a vertical field, i.e.,* $X(\tau^*(f)) = 0$ *for any function* $f \in C^\infty(M) \subset \mathcal{F}(\mathcal{E}^\infty)$.
2. $[X, \widetilde{D}_i] = 0$, $i = 1, \ldots, n$.

PROOF. Note that the first condition means that in local coordinates the coefficients of the field X at $\partial/\partial x_i$, $i = 1, \ldots, n$, vanish. Therefore, the intersection of the set of vertical fields on $\widetilde{\mathcal{E}}$ with the algebra $\mathcal{CD}(\widetilde{\mathcal{E}})$ is trivial. On the other hand, in every coset $[X] \in \mathrm{sym}_\tau\,\mathcal{E}$ there exists one and only one vertical representative X^v. Indeed, let X' be an arbitrary representative of the class $[X]$. Then $X^v = X' - \sum_{i=1}^n a_i \widetilde{D}_i$, where a_i is the coefficients of the of the field X' at $\partial/\partial x_i$. $\square$

THEOREM 3.2. *Let the covering* $\tau\colon \widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}^N \to \mathcal{E}^\infty$ *be locally determined by the fields* $\widetilde{D}_i = \bar{D}_i + \sum_{j=1}^N X_{ij}\partial/\partial w_j$, $i = 1, \ldots, n$, $X_{ij} \in \mathcal{F}(\widetilde{\mathcal{E}})$, *where* $w_1, w_2, \ldots$ *are nonlocal variables. Then any* τ-*symmetry of the equation* $\mathcal{E} = \{F = 0\}$ *is of the form*

$$\widetilde{\eth}_{\varphi,A} = \widetilde{\eth}_\varphi + \sum_{j=1}^N a_j \frac{\partial}{\partial w_j},$$

where $\varphi = (\varphi_1, \ldots, \varphi_m)$, $A = (a_1, \ldots, a_N)$, $\varphi_i, a_j \in \mathcal{F}(\widetilde{\mathcal{E}})$, *and the functions* φ_i, a_j *satisfy the equations*

$$\widetilde{\ell}_F(\varphi) = 0, \tag{3.1}$$

$$\widetilde{D}_i(a_j) = \widetilde{\eth}_{\varphi,A}(X_{ij}). \tag{3.2}$$

PROOF. Let $S \in \mathrm{sym}_\tau \mathcal{E}$. Using Theorem 3.1, let us write down the vertical field S in local coordinates in the form

$$S = \sideset{}{'}\sum_{\sigma,k} b_\sigma^k \frac{\partial}{\partial p_\sigma^k} + \sum_{j=1}^N a_j \frac{\partial}{\partial w_j}, \qquad b_\sigma^k, a_j \in \mathcal{F}(\widetilde{\mathcal{E}}),$$

where the "prime" denotes summation on all internal coordinates p_σ^k of the equation $\mathcal{E}$. Equating the coefficients at $\partial/\partial p_\sigma^k$ in the commutator $[S, \widetilde{D}_i]$ to zero, we shall obtain the equations $\widetilde{D}_i(b_\sigma^k) = b_{\sigma+1_i}^k$, if $p_{\sigma+1_i}^k$ is an internal coordinate on $\mathcal{E}^\infty$, or $\widetilde{D}_i(b_\sigma^k) = S(\bar{p}_{\sigma+1_i}^k)$ otherwise.

Solving these equations in the same way as it was done for local symmetries, we shall obtain the relations

$$b_\sigma^k = \widetilde{D}_\sigma(b_\varnothing^k), \qquad \widetilde{\ell}_F(b_\varnothing) = 0, \qquad b_\varnothing = (b_\varnothing^1, \ldots, b_\varnothing^N).$$

In particular, $S = \widetilde{\eth}_{\varphi,A}$, where $\varphi = b_\varnothing$. Then equations (3.2) are obtained by equating to zero the coefficients at $\partial/\partial w_j$ in the commutator $[S, \widetilde{D}_i]$. $\qquad\square$

Note that for finite N the covering equation is locally of the form $(\mathcal{E}')^\infty$, where $\mathcal{E}' = \{F' = 0\}$ and

$$F' = \left(F, \frac{\partial w_j}{\partial x_i} - X_{ij} \right), \qquad i = 1, \ldots, n, \ j = 1, \ldots, N,$$

and the system of equations on its local symmetry with the generating function $\psi = (\varphi, A)$ coincides exactly with system (3.1)–(3.2).

EXERCISE 3.2. Prove the following identity:

$$[\widetilde{\eth}_{\varphi,A}, \widetilde{\eth}_{\psi,B}] = \widetilde{\eth}_{\chi,C},$$

where $\chi = \widetilde{\eth}_{\varphi,A}(\psi) - \widetilde{\eth}_{\psi,B}(\varphi)$, $C = \widetilde{\eth}_{\varphi,A}(B) - \widetilde{\eth}_{\psi,B}(A)$.

REMARK 3.1. If the covering is zero-dimensional (naturally, in this case $X_{ij} = 0$), i.e., τ is a local isomorphism, then equations (3.2) fulfill in a trivial way while system (3.1)–(3.2) reduces to the equation $\widetilde{\ell}_F(\varphi) = 0$. Thus, the local symmetry theory embeds to the nonlocal one in a natural way. More precise, the algebra of local symmetries of the equation $\mathcal{E}$ coincides with the algebra of nonlocal ones in the covering $\mathrm{id} \colon \mathcal{E}^\infty \to \mathcal{E}^\infty$.

If a covering $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ of the equation $\mathcal{E}^\infty$ is given and a function $\varphi \in \mathcal{F}(\widetilde{\mathcal{E}})$ satisfies equation (3.1), then, in general, there may be no symmetry of the form $\widetilde{\eth}_{\varphi,A}$: system of equations (3.2) may have no solution for a given φ. In particular, not every local symmetry $\bar{\eth}_\varphi$, $\varphi \in \mathcal{F}(\mathcal{E}^\infty)$, can be extended to a symmetry $\widetilde{\eth}_{\varphi,A}$ in the covering $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$.

EXAMPLE 3.1. Consider the one-dimensional coveting $\tau \colon \widetilde{\mathcal{E}} = \mathcal{E}^\infty \times \mathbb{R}^1 \to \mathcal{E}^\infty$ over the Korteweg–de Vries equation $\mathcal{E} = \{u_t = u_{xxx} + uu_x\}$ given by the fields

$$\widetilde{D}_x = \bar{D}_x + \left(p_0 + \frac{1}{6} w^2 \right) \frac{\partial}{\partial w},$$

$$\widetilde{D}_t = \bar{D}_t + \left(p_2 + \frac{1}{3} w p_1 + \frac{1}{3} p_0^2 + \frac{1}{18} w^2 p_0 \right) \frac{\partial}{\partial w}.$$

As we saw already (see Example 1.9), $\widetilde{\mathcal{E}} = (\mathcal{E}')^\infty$, where $\mathcal{E}'$ is the modified Korteweg–de Vries equation

$$w_t = w_{xxx} - \frac{1}{6}w^2 w_x.$$

Consider the local symmetry of the KdV equation with the generating function $\varphi = tp_1 + 1$ (the Galilean symmetry) and try to extend it to a symmetry $\widetilde{\eth}_{\varphi,a} = \widetilde{\eth}_\varphi + a\partial/\partial w$, $a \in \mathcal{F}(\widetilde{\mathcal{E}})$, in the covering τ.

To do this, we have to solve the following system of equations for the function $a \in \mathcal{F}(\widetilde{\mathcal{E}})$:

$$\widetilde{D}_x(a) = \widetilde{\eth}_{\varphi,a}\left(p_0 + \frac{1}{6}w^2\right) = \varphi + \frac{1}{3}aw,$$

$$\widetilde{D}_t(a) = \widetilde{\eth}_{\varphi,a}\left(p_2 + \frac{1}{3}wp_1 + \frac{1}{3}p_0^2 + \frac{1}{18}w^2 p_0\right)$$

$$= \bar{D}_x^2(\varphi) + \frac{1}{3}w\bar{D}_x(\varphi) + \left(\frac{2}{3}p_0 + \frac{1}{18}w^2\right)\varphi + a\left(\frac{1}{3}p_1 + \frac{1}{9}wp_0\right).$$

It is easily seen that this system has no solution. Thus, there is no symmetry $\widetilde{\eth}_{\varphi,a}$, where φ is the Galilean symmetry, in the covering under consideration and the nonlocal symmetry algebra diminishes in comparison with the local one. Nevertheless, one can consider different coverings and may hope that in some of them the symmetry algebra will be extended. We consider this problem in §5.

4. Examples of computation: nonlocal symmetries of the Burgers equation

Consider the problem of finding nonlocal symmetries for the Burgers equation in the coverings described by Theorem 2.1 (see [**61**]). Using Theorem 3.1, we shall identify elements of the algebra $\mathrm{sym}_\tau \mathcal{E}$ with the fields S on $\widetilde{\mathcal{E}}$ such that $S = P + \Phi$, where

$$P = \sum_{i\geq 0} P_i \frac{\partial}{\partial p_i}, \quad \Phi = \sum_{j\geq 0} \Phi_j \frac{\partial}{\partial w_j},$$

$P_i, \Phi_j \in \mathcal{F}(\widetilde{\mathcal{E}})$, and $[S, \widetilde{D}_x] = [S, \widetilde{D}_t] = 0$.

Using equalities (2.8), these relations can be transformed to

$$[S, \widetilde{D}_x] = \sum_{i\geq 0}\left(P_{i+1} - \widetilde{D}_x(P_i)\right)\frac{\partial}{\partial p_i} + P_0 A + [\Phi, \widetilde{D}_x] = 0,$$

$$[S, \widetilde{D}_t] = \sum_{i\geq 0}\left(\sum_{k\geq 0}\frac{\partial}{\partial p_k}\left(\widetilde{D}_x^i(p_0 p_1 + p_2)\right) - \widetilde{D}_t(P_i)\right)\frac{\partial}{\partial p_i}$$

$$+ (P_1 + p_0 P_0)A + P_0[A, B] + [\Phi, \widetilde{D}_t] = 0.$$

The coefficients of the fields $[\Phi, \widetilde{D}_x]$ and $[\Phi, \widetilde{D}_t]$ at $\partial/\partial p_i$ vanish for all $i \geq 0$, i.e., these fields are vertical with respect to the projection τ. Therefore, the first of the above obtained equations is equivalent to

$$P_0 A + [\Phi, \widetilde{D}_x] = 0, \quad P_{i+1} = \widetilde{D}_x(P_i), \qquad i = 0, 1, \ldots,$$

which implies that $P_i = \widetilde{D}_x^i(\psi)$, where ψ denotes the function P_0. In other words, $P = \widetilde{\mathfrak{D}}_\psi = \sum_{i\geq 0} \widetilde{D}_x^i(\psi)\partial/\partial p_i$. In a similar way, the second equation is equivalent to the equalities

$$(\widetilde{D}_x(\psi) + p_0\psi)A + \psi[A, B] + [\Phi, \widetilde{D}_t] = 0, \tag{4.1}$$

$$\mathfrak{D}_\psi\left(\widetilde{D}_x^i(p_0p_1 + p_2)\right) = \widetilde{D}_t(\widetilde{D}_x^i(\psi)), \qquad i \geq 0. \tag{4.2}$$

Since $[\widetilde{\mathfrak{D}}_\psi, \widetilde{D}_x] = 0$, equation (4.2), for $i > 0$, is obtained by applying the operator $\widetilde{D}_x^i$ to equation (4.2), where $i = 0$, i.e.,

$$\mathfrak{D}_\psi(p_0p_1 + p_2) = \widetilde{D}_t(\psi),$$

or, which is the same,

$$\widetilde{D}_x^2(\psi) + p_0\widetilde{D}_x(\psi) + p_1\psi = \widetilde{D}_t(\psi).$$

Thus, we proved the following result:

PROPOSITION 4.1. *Any nonlocal symmetry of the Burgers equation in the covering (2.8) is of the form $\widetilde{\mathfrak{D}}_\psi + \Phi$, where $\Phi = \sum_{j\geq 0} \Phi_j\partial/\partial w_j$, ψ, $\Phi_j \in \mathcal{F}(\widetilde{\mathcal{E}})$, while the function ψ and the field Φ satisfy the following equations*:

$$\psi A = [\widetilde{D}_x, \Phi],$$
$$(\widetilde{D}_x(\psi) + p_0\psi)A + \psi[A, B] + [\Phi, \widetilde{D}_t] = 0, \tag{4.3}$$
$$\widetilde{\ell}_F(\psi) = \widetilde{D}_x^2(\psi) + p_0\widetilde{D}_x(\psi) + p_1\psi - \widetilde{D}_t(\psi) = 0,$$

where F is the function determining the Burgers equation.

Subsequent analysis of system (4.3) is to be based on particular realizations of the Burgers equation universal algebra. Let us start with studying nonlocal symmetries in the coverings described in Proposition 2.4. Consider the covering τ^0. Let $\Phi = \varphi\partial/\partial w$. Then for the covering τ^0 system (4.3) acquires the form

$$\psi = \widetilde{D}_x(\varphi),$$
$$\widetilde{D}_x(\psi) + p_0\psi = \widetilde{D}_t(\varphi), \tag{4.4}$$
$$\widetilde{\ell}_F(\psi) = 0.$$

Note that using Theorem 3.2, we would obtain the same system of equations.

It is easily seen that the third equation of system (4.4) is a consequence of the first two ones. Therefore, the system can be transformed to the following one:

$$\psi = \widetilde{D}_x(\varphi),$$
$$\widetilde{D}_x^2(\varphi) + p_0\widetilde{D}_x(\varphi) = \widetilde{D}_t(\varphi).$$

Let us now pay attention to the fact that the second equation is of the form $\widetilde{\ell}_H(\varphi) = 0$, where $\widetilde{\ell}_H$ is the universal linearization operator for the equation $\mathcal{E}' = \{v_t = v_{xx} + \frac{1}{2}v_x^2\}$. This agrees with the fact that the equation $\widetilde{\mathcal{E}}$ is contact diffeomorphic to the manifold $(\mathcal{E}')^\infty$. The diffeomorphism is established by the following coordinate change:

$$x = x', \quad w = p_0', \quad t = t', \quad p_k = p_{k+1}', \quad k = 0, 1, \ldots,$$

where x', t', p_k' are the standard coordinates on $(\mathcal{E}')^\infty$. From here it follows that $\text{sym}_{\tau^0}\mathcal{E} = \text{sym}\,\mathcal{E}'$. Computing the algebra $\text{sym}\,\mathcal{E}'$ using the scheme described in

Ch. 4 and applying the above discussed diffeomorphism, we shall obtain that the algebra $\operatorname{sym}\mathcal{E}'$ is additively generated by elements of the form

$$\varphi_{-\infty} = \eta(x,t)e^{-w/2}, \quad \eta_{xx} = \eta_t, \quad \varphi_{-1}^0 = 1,$$

$$\varphi_k^i = t^i p_k + \frac{1}{2}((k+1)t^i p_0 + ixt^{i-1})p_{k-1} + O(k-2),$$

$$k = 0, 1, \ldots, \quad i = 0, 1, \ldots, k+1,$$

where the function $O(r)$ depends on the variables x, t, $p_0, \ldots, p_r$ only.

Stress the fact that the form of system (4.3) essentially depends on choice of a coordinate in $\mathbb{R}$, i.e., on the form of a field A representation. If $A = \alpha(w)\partial/\partial w$, while B and C are still equal to zero, then relations (4.3) reduce to the form

$$\psi = \frac{1}{a}\left(tD_x(\varphi) - p_0\alpha'\varphi\right),$$

$$\widetilde{D}_x^2(\varphi) + (1 - 2\alpha')p_0\widetilde{D}_x(\varphi) + \left((\alpha')^2 - \alpha\alpha'' - \frac{1}{2}\alpha'\right)p_0^2\varphi = \widetilde{D}_t(\varphi).$$

Let us rewrite the last equality in the form $L(\varphi) = 0$, where

$$L = \widetilde{D}_x^2 + (1 - 2\alpha')p_0\widetilde{D}_x - \widetilde{D}_t + \left((\alpha')^2 - \alpha\alpha'' - \frac{1}{2}\alpha'\right).$$

The operator L acquires a simpler form, if one sets $\alpha = \frac{1}{2}w$. Then $L = \widetilde{D}_x^2 - \widetilde{D}_t$.

In this case, $\widetilde{\mathcal{E}} = (\mathcal{E}'')^\infty$, where $\mathcal{E}''$ is the heat equation $w_{xx} = w_t$. Thus, with a suitable choice of a coordinate in the fiber, the covering τ_0 reduces to the form $(\mathcal{E}'')^\infty \to \mathcal{E}^\infty$ and, moreover, the Burgers equation represents as a quotient equation of the heat equation $\mathcal{E}''$ over the one-parameter group of its symmetries $w \mapsto \varepsilon w$, where ε is a parameter. Note that such a transformation group is admitted by all linear equations. In the chosen coordinate system on $\widetilde{\mathcal{E}}$ (i.e., when $A = \frac{1}{2}w\partial/\partial w$) the generators of the algebra $\operatorname{sym}_{\tau^0}\mathcal{E} = \operatorname{sym}\mathcal{E}''$ are of the form

$$\varphi_{-\infty} = \eta(x,t)e^{-w/2}, \quad \eta_{xx} = \eta_t, \quad \varphi_{-1}^0 = w,$$

$$\varphi_k^i = w\left(t^i p_k + \frac{1}{2}\left((k+1)t^i p_0 + ixt^{i-1}\right)p_{k-1} + O(k-2)\right),$$

$$k = 0, 1, \ldots, \quad i = 0, 1, \ldots, k+1.$$

Consider the coverings $\tau_\nu^\pm$. System (4.3) for these coverings is easily transformed to

$$\square(\varphi) = 0,$$
$$\psi = \widetilde{D}_x(\varphi) - \beta'\varphi, \tag{4.5}$$

where $\beta = \pm e^{w/2} + \nu$ and

$$\square = \widetilde{D}_x^2 + p_0\widetilde{D}_x - \widetilde{D}_t - \beta'(p_0 + \beta). \tag{4.6}$$

System (4.5) reduces to the sole equation $\square(\varphi) = 0$ and is solved in the same way as the equation $\bar{\ell}_F(\varphi) = 0$ determining higher symmetries of the Burgers equation (see Ch. 4). The result shows that the algebras $\operatorname{sym}_{\tau_\nu^\pm}\mathcal{E}$ are isomorphic to the algebra $\operatorname{sym}\mathcal{E}$. The solutions φ of (4.5) depending on the variables x, t, w,

p_0, and p_1 are linear combinations of the following ones:

$$
\begin{aligned}
\varphi_1^0 &= p_0 + \beta, \\
\varphi_1^1 &= t(p_0 + \beta) - (\beta')^{-1}, \\
\varphi_2^0 &= p_1 + \frac{1}{2}p_0^2 + \beta' p_0 - \frac{1}{2}\nu\beta, \\
\varphi_2^1 &= t\left(p_1 + \frac{1}{2}p_0^2\right) + \left(t\beta' + \frac{1}{2}x\right)p_0 + \frac{1}{2}\beta(x - \nu t) + \frac{1}{2}\nu(\beta')^{-1} + 1, \\
\varphi_2^2 &= t^2\left(p_1 + \frac{1}{2}p_0^2\right) + (t^2\beta' + xt)p_0 + \frac{1}{2}\beta(2xt - \nu t^2) + (\nu t - x)(\beta')^{-1} + 2t.
\end{aligned}
\tag{4.7}
$$

The corresponding functions ψ are

$$
\begin{aligned}
\psi_1^0 &= p_1, \\
\psi_1^1 &= tp_1 + \frac{1}{2}(p_0 + \nu)(\beta')^{-1} + 2, \\
\psi_2^0 &= p_2 + p_0 p_1, \\
\psi_2^1 &= t(p_2 + p_0 p_1) + \frac{1}{2}(xp_1 + p_0) - \frac{1}{4}\nu(p_0 + \beta)(\beta')^{-1}, \\
\psi_2^2 &= t^2(p_2 + p_0 p_1) + t(xp_1 + p_0) - \frac{1}{2}\nu(p_0 + \beta)(\beta')^{-1} \\
&\quad + \left(\frac{1}{2}x(p_0 + \beta + 2\beta') - 1\right)(\beta')^{-1}.
\end{aligned}
$$

Let S_j^i be a symmetry corresponding to the function φ_j^i. Then from (4.7) it follows that the symmetries S_1^0 and S_2^0 are local, the symmetry S_1^2 is essentially nonlocal for $\nu \neq 0$, and the symmetries S_1^1 and S_2^2 are essentially nonlocal.

The isomorphism between the algebras $\mathrm{sym}_{\tau_\nu^\pm} \mathcal{E}$ and $\mathrm{sym}\, \mathcal{E}$ implies the hypothesis that the space $\widetilde{\mathcal{E}}_{\tau_\nu^\pm}$ of the covering $\tau_\nu^\pm$ is contact diffeomorphic to the infinite prolongation $\mathcal{E}^\infty$ of the Burgers equation $\mathcal{E}$. To check it, let us try to represent the manifold $\widetilde{\mathcal{E}}_{\tau_\nu^\pm}$ in the form $(\mathcal{E}')^\infty$, where $\mathcal{E}'$ is a differential equation. Assuming x and t to be independent variables in $\mathcal{E}$ and w to be the dependent one, let us introduce in $\widetilde{\mathcal{E}}_{\tau_\nu^\pm}$ the coordinates x, t, $y_0 = w, \ldots, y_i = \widetilde{D}_x^i(w), \ldots$ Then, as it is easily seen, $\widetilde{D}_t(w) = y_2 + \frac{1}{2}y_1^2 - \beta y_1$, i.e., the desired equation $\mathcal{E}'$ is of the form

$$
w_{xx} + \frac{1}{2}w_x^2 - \beta w_x = w_t. \tag{4.8}
$$

As it was expected, the universal linearization operator for equation (4.8) coincides with the operator (4.6).

Further, the change of variables $v = -\beta \mp e^{w/2} - \nu$ transforms equation (4.8) to the Burgers equation $v_t = v_{xx} + vv_x$. Thus, choosing the coordinates on $\widetilde{\mathcal{E}}_{\tau_\nu^\pm}$ in the form

$$
x, \ t, \ y_0' = v, \ \ldots, \ y_i' = \widetilde{D}_x^i(v), \ \ldots,
$$

we shall find out that $\widetilde{\mathcal{E}}_{\tau_\nu^\pm} \approx \mathcal{E}^\infty$ and, by this reason, the mapping $\tau_\nu^\pm$ can be considered as a mapping of the Burgers equation to itself. The explicit formula for this mapping is obtained from the relation $v_x = \widetilde{D}_x(v) = -\beta'(u + \beta) = \frac{1}{2}(v +$

$\nu)(u - \nu)$, from which we have

$$u = \frac{2v_x}{v + \nu} + v \tag{4.9}$$

Thus, we arrived to the following remarkable result. If v is a solution of the Burgers equation, then the function u determined by formula (4.9) is also a solution of the Burgers equation for all values of the parameter ν.

To conclude this section, we shall consider one infinite-dimensional covering of the Burgers equation. Let us set

$$
\begin{aligned}
A &= \frac{\partial}{\partial w_1}, \\
B &= e^{w_1/2} \frac{\partial}{\partial w_2} + w_2 \frac{\partial}{\partial w_3} + w_3 \frac{\partial}{\partial w_4} + \dots, \\
C &= e^{w_1/2} \frac{\partial}{\partial w_3} + w_2 \frac{\partial}{\partial w_4} + w_3 \frac{\partial}{\partial w_5} + \dots
\end{aligned} \tag{4.10}
$$

Obviously, the fields A, B, and C satisfy the relations (2.8) and thus determine an infinite-dimensional covering over the Burgers equation. Let $S = \widetilde{\mathcal{Z}}_\psi + \sum_i \Phi_i \partial/\partial w_i$ be a nonlocal symmetry. Then, since $[A, B] = \frac{1}{2} e^{w_1/1} \partial/\partial w_2$, system (4.3) in the case under consideration is of the form

$$p_1 \psi + p_0 \widetilde{D}_x(\psi) + \widetilde{D}_x^2(\psi) = \widetilde{D}_t(\psi),$$

$$\psi = \widetilde{D}_x(\Phi_1), \qquad\qquad p_0 \psi + \widetilde{D}_x(\psi) = \widetilde{D}_t(\Phi_1),$$

$$\Phi_1 = 2 e^{-w_1/2} \widetilde{D}_x(\Phi_2), \qquad \psi + \tfrac{1}{2} p_0 \Phi_1 = 2 e^{-w_1/2} \widetilde{D}_x(\Phi_2),$$

$$\Phi_2 = \widetilde{D}_x(\Phi_3), \qquad\qquad \Phi_1 = 2 e^{-w_1/2} \widetilde{D}_x(\Phi_3),$$

$$\Phi_3 = \widetilde{D}_x(\Phi_4), \qquad\qquad \Phi_2 = \widetilde{D}_t(\Phi_4),$$

$$\dots\dots\dots\dots \qquad\qquad \dots\dots\dots\dots$$

$$\Phi_{k-1} = \widetilde{D}_x(\Phi_k), \qquad\qquad \Phi_{k-2} = \widetilde{D}_t(\Phi_k),$$

$$\dots\dots\dots\dots\dots \qquad\qquad \dots\dots\dots\dots\dots$$

It can be easily checked that this system is equivalent to the following one:

$$\psi = \widetilde{D}_x(\Phi_1), \qquad\qquad p_1 \psi + p_0 \widetilde{D}_x(\psi) + \widetilde{D}_x^2(\psi) = \widetilde{D}_t(\psi),$$

$$\Phi_1 = 2 e^{-w_1/2} \widetilde{D}_x(\Phi_2), \qquad p_0 \widetilde{D}_x(\Phi_1) + \widetilde{D}_x^2(\Phi_1) = \widetilde{D}_t(\Phi_1),$$

$$\Phi_2 = \widetilde{D}_x(\Phi_3), \qquad\qquad \widetilde{D}_x^2(\Phi_2) = \widetilde{D}_t(\Phi_2),$$

$$\Phi_3 = \widetilde{D}_x(\Phi_4), \qquad\qquad \widetilde{D}_x^2(\Phi_3) = \widetilde{D}_t(\Phi_3), \tag{4.11}$$

$$\dots\dots\dots\dots \qquad\qquad \dots\dots\dots\dots\dots$$

$$\Phi_k = \widetilde{D}_x(\Phi_{k+1}), \qquad\qquad \widetilde{D}_x^2(\Phi_k) = \widetilde{D}_t(\Phi_k),$$

$$\dots\dots\dots\dots\dots \qquad\qquad \dots\dots\dots\dots\dots$$

Before solving this system of differential equations, let us introduce a filtration to the algebra $\mathcal{F}(\widetilde{\mathcal{E}})$. Let $k > 0$ and $\varphi \in \mathcal{F}(\widetilde{\mathcal{E}})$. Then we set $\deg \varphi = k$, if $\partial \varphi/\partial p_i = 0$ for all $i > k$; $\deg \varphi = -k$, if $\partial \varphi/\partial w_j = 0$ for all $j < k$; $\deg \varphi = 0$, if $\partial \varphi/\partial p_i = 0$ for all $i \geq 0$. Note further that if an element ψ satisfies all equations of the left column

in system (4.11) and if ψ satisfies the k-th equation in the right column, then ψ satisfies all equations in (4.11) for $i < k$.

For subsequent arguments, it is convenient to consider two cases: (a) $\deg \Phi_2 \geq 4$ and (b) $\deg \Phi_2 < 4$. In the first case, since the operator $\widetilde{D}_x$ increases filtration by 1, from the equations of the first column it follows that there exists a number k such that starting from this number the functions Φ_i will depend on the variables t, x, $w_4, w_5, \ldots$ only. Let $(\psi, \Phi_1, \Phi_2, \ldots, \Phi_k)$ be a solution of $(k+1)$ equations of system (4.11) and k be such that

$$\Phi_k = \Phi_k(t, x, w_4, w_5, \ldots, w_r), \quad \frac{\partial \Phi_k}{\partial w_4} = 0, \qquad k \geq 2. \tag{4.12}$$

For functions of the form (4.12) equations (4.11) on Φ_k are reduced to the form

$$\frac{\partial^2 \Phi_k}{\partial x^2} + 2 \sum_{i=4}^{r} w_{i-1} \frac{\partial^2 \Phi_k}{\partial x \partial w_i} + \sum_{i,j=4}^{r} w_{i-1} w_{j-1} \frac{\partial^2 \Phi_k}{\partial w_i \partial w_i} = \frac{\partial \Phi_k}{\partial t}.$$

Using induction on r, one can show that any solution of this equation linear depends on the variables w_i, i.e.,

$$\Phi_k = \varphi_{k4} w_4 + \varphi_{k5} w_5 + \cdots + \varphi_{kr} w_r + \varphi_{k0},$$

where the function φ_{k0} is a solution of the heat equation $\partial^2 \varphi_{k0}/\partial x^2 = \partial \varphi_{k0}/\partial t$ while the functions φ_{ki}, $i = 4, \ldots, r$ are to satisfy the following system of differential equations:

$$
\begin{aligned}
&\frac{\partial \varphi_{k4}}{\partial x} = 0, \\
&\frac{\partial^2 \varphi_{k4}}{\partial x^2} + 2 \frac{\partial \varphi_{k5}}{\partial x} = \frac{\partial \varphi_{k4}}{\partial t}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\frac{\partial^2 \varphi_{kr-1}}{\partial x^2} + 2 \frac{\partial \varphi_{kr}}{\partial x} = \frac{\partial \varphi_{kr-1}}{\partial t}, \\
&\frac{\partial^2 \varphi_{kr}}{\partial x^2} = \frac{\partial \varphi_{kr}}{\partial t}.
\end{aligned}
\tag{4.13}
$$

It is easy to show that all solution of system (4.13) are polynomial in x and t, and φ_{k4} is an arbitrary polynomial of degree $r - 4$ depending on t only. To compute nonlocal symmetries in our covering, we need to find solutions of system (4.13) defined up to solutions with $\varphi_{k4} = 0$. This space is of dimension $r - 3$ and its basis is determined by the functions $\varphi_{k4} = 1, t, \ldots, t^{r-4}$.

Let $\Phi_k = \varphi_{k4} w_4 + \varphi_{k5} w_5 + \cdots + \varphi_{kr} w_r + \varphi_{k0}$ be a solution of the k-th pair in (4.11). To find the corresponding nonlocal symmetry of the Burgers equation it needs to construct Φ_{k+i}, $i = 1, 2, \ldots$, satisfying the $(k+i)$-the pair in (4.11) and such that $\Phi_k = \widetilde{D}_x(\Phi_{k+1})$, $\Phi_{k+1} = \widetilde{D}_x(\Phi_{k+2}), \ldots$ Let us set

$$\Phi_{k+1} = \sum_{i=4}^{r} \sum_{\alpha > 0} (-1)^{\alpha+1} \frac{\partial^{\alpha-1} \varphi_{ki}}{\partial x^{\alpha-1}} w_{i+\alpha}. \tag{4.14}$$

Since, as it was said above, all functions φ_{ki} are polynomial in x, the function Φ_{k+1} is well defined by equations (4.11). Calculating $\widetilde{D}_x^2(\Phi_{k+1})$ and $\widetilde{D}_t(\Phi_{k+1})$ and using relations (4.13), it is easy to see that Φ_{k+1} satisfies the $(k+1)$-st pair in (4.11). The functions $\Phi_{k+2}, \ldots, \Phi_{k+i}, \ldots$ are constructed in the same way and it is proved that they satisfy the corresponding equations of (4.11).

The above arguments, with insignificant corrections, are carried on to the case (b), when $\Phi_2 = \Phi_2(t, x, w_5, \ldots, w_r)$.

Let us denote by Φ_k^i, where $k \geq -2$, $i \geq 0$, the symmetry for which the function Φ_{k+4} is of the form $\Phi_{k+4} = t^i w_4 + \Psi(t, x, w_5, \ldots, w_r)$ and by Φ_k^i, where $k < -2$, $i \geq 0$, the symmetry, for which the function Φ_2 is $\Phi_2 = t^i w_{2-k} + \Psi(t, x, w_{2-k+1}, \ldots, w_r)$. Then the following theorem is valid:

THEOREM 4.2. *The Lie algebra of nonlocal symmetries of the Burgers equation in the covering τ determined by the fields* (4.10) *is additively generated by the functions Φ_k^i, where $k = 0, \pm1, \pm2, \ldots$, $i = 0, 1, 2, \ldots$, and by the symmetries, for which $\Phi_2 = \Phi_2(x, t)$ is an arbitrary solution of the heat equation $\partial^2 \Phi_2 / \partial x^2 = \partial \Phi_2 / \partial t$.*

Note that local symmetries of the form $\psi_k^i = t^i p_k + O(k-1)$ correspond to the generators Φ_k^i for $k > 0$, $i < k$; no local symmetry corresponds to other generators. For example, the generating function $\psi = (2w_2 - w_3 p_0)e^{-w_1/2}$ corresponds to Φ_{-2}^0; $\psi = 1 - w_2 p_0 e^{-w_1/2}$ corresponds to Φ_{-1}^0; $\psi = tp_1 - 2w_2 p_0 e^{-w_1/2} + 5$ corresponds to Φ_1^1. If Φ_2 is a solution of the heat equation, then $\psi = (2\partial\varphi/\partial t - \partial\varphi/\partial x p_0)e^{-w_1/2}$.

5. The problem of symmetry reconstruction

Let $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be an arbitrary covering over an equation $\mathcal{E}^\infty$ and $\varphi \in \mathcal{F}(\widetilde{\mathcal{E}}, \pi)$ be a solution of the equation $\widetilde{\ell}_F(\varphi) = 0$, $\mathcal{E} = \{F = 0\}$. As it was shown in Ch. 4, if $\varphi \in \mathcal{F}(\mathcal{E}^\infty, \pi)$, then $\bar{\partial}_\varphi$ is a local symmetry of the equation $\mathcal{E}^\infty$. Let us try to find a symmetry τ of the form $\widetilde{\partial}_{\varphi, A} = \widetilde{\partial}_\varphi + \sum_{i=1}^N a_i \partial/\partial w_i$ for the chosen function $\varphi \in \ker \widetilde{\ell}_F$. This problem will be called the *reconstruction problem* nonlocal for symmetries. As we saw already (see Example 3.1), for an arbitrary covering, this problem has no solution. Nevertheless, below, for an arbitrary differential equation, we shall construct a special covering, where this problem is always solvable. As it happens, additional series of symmetries possessing a recursion operator arise in this covering.

5.1. Universal Abelian covering. Among all coverings (1.2) over the equation $\mathcal{E}^\infty$, where $X_{ij} \in \mathcal{F}(\mathcal{E}^\infty)$, there is a distinguished one: the Whitney product of all coverings determined by basis elements of the space $\bar{H}^1(\mathcal{E}^\infty)$ (see [**52, 53**]). Denote this covering by $\tau_1 \colon \widetilde{\mathcal{E}}^{(1)} \to \mathcal{E}^\infty$. Let us now take the Whitney product of all one-dimensional coverings over $\widetilde{\mathcal{E}}^{(1)}$ determined by basis elements of the space $\widetilde{H}^1(\widetilde{\mathcal{E}}^{(1)})$. We shall obtain the covering $\tau_{2,1} \colon \widetilde{\mathcal{E}}^{(2)} \to \widetilde{\mathcal{E}}^{(1)}$ over $\widetilde{\mathcal{E}}^{(1)}$ and hence the covering $\tau_2 = \tau_1 \circ \tau_{2,1} \colon \widetilde{\mathcal{E}}^{(2)} \to \mathcal{E}^\infty$ over $\mathcal{E}^\infty$. Proceeding with this construction, we shall obtain the tower of coverings[4] over $\mathcal{E}^\infty$

$$\cdots \xrightarrow{\tau_{k+1,k}} \widetilde{\mathcal{E}}^{(k)} \xrightarrow{\tau_{k,k-1}} \widetilde{\mathcal{E}}^{(k-1)} \xrightarrow{\tau_{k-1,k-2}} \cdots \xrightarrow{\tau_{2,1}} \widetilde{\mathcal{E}}^{(1)} \xrightarrow{\tau} \mathcal{E}^\infty \qquad (5.1)$$

Let us denote the inverse limit of this chain of mappings by $\tau^* \colon \widetilde{\mathcal{E}}^* \to \mathcal{E}^\infty$. We shall call τ^* the *universal Abelian covering* over the equation $\mathcal{E}^\infty$.

---

[4]Infinite-dimensional manifolds arising in this procedure may not be inverse limits of finite-dimensional ones. For example, this is the case when the equation $\widetilde{\mathcal{E}}^{(k)}$, for some $k \geq 0$, possesses a family of conservation laws depending on functional parameter. Such objects and the corresponding theory, similar to that of Ch. 4, are not considered in this book.

REMARK 5.1. The condition $\widetilde{H}^1(\widetilde{\mathcal{E}}^*) = 0$ is convenient to reformulate in the following way. Let the covering τ^* be locally determined by the fields $\widetilde{D}_1, \ldots, \widetilde{D}_n$. Then, if $A_1, \ldots, A_n \in F(\widetilde{\mathcal{E}}^*)$ and

$$\widetilde{D}_i(A_j) = \widetilde{D}_j(A_i), \tag{5.2}$$

one can state that there exists a function $A \in F(\widetilde{\mathcal{E}}^*)$ such that $A_i = \widetilde{D}_i(A)$ for any i.

5.2. Symmetries in the universal Abelian covering. From now on, till the end of this section, the tilde sign over an operator notation will denote the lifting of this operator to the covering τ^*.

THEOREM 5.1. *Let $\tau^* \colon \widetilde{\mathcal{E}}^* \to \mathcal{E}^\infty$ be the universal Abelian covering over the equation $\mathcal{E} = \{F = 0\}$. Then for any vector function $\varphi = (\varphi_1, \ldots, \varphi_m)$, $\varphi_i \in \mathcal{F}(\widetilde{\mathcal{E}}^*)$, satisfying the equation $\widetilde{\ell}_F(\varphi) = 0$, there exists a set of functions $A = (a_{i\alpha})$, $a_{i\alpha} \in \mathcal{F}(\widetilde{\mathcal{E}}^*)$, such that $\eth_{\varphi,A}$ is a nonlocal symmetry of type τ^* for the equation $\mathcal{E}$.*

PROOF. Let us assume that locally the distribution on $\widetilde{\mathcal{E}}^*$ is given by the vector fields $\widetilde{D}_i = \bar{D}_i + \sum_{j,\alpha} X_{ij}^\alpha \partial/\partial w_{j\alpha}$, $X_{ij}^\alpha \in \mathcal{F}(\widetilde{\mathcal{E}}^*)$, where i enumerates independent variables, j enumerates levels of the tower, and α at the j-th level enumerates basis elements in the space $\widetilde{H}^1(\widetilde{\mathcal{E}}^{(j)})$. The functions X_{ij} satisfy the conditions

$$\widetilde{D}_i(X_{jk}^\alpha) = \widetilde{D}_j(X_{ik}^\alpha). \tag{5.3}$$

Note that by construction, X_{ij}^α is a function on $\widetilde{\mathcal{E}}^{(j-1)}$ (we set $\widetilde{\mathcal{E}}^{(0)} = \mathcal{E}^\infty$) for all i, α, i.e., is independent of $w_{k\alpha}$ for $k \geq j$. We are to prove solvability of the equations

$$\widetilde{D}_i(a_{j\alpha}) = \widetilde{\eth}_{\varphi,A}(X_{ij}^\alpha) \tag{5.4}$$

for any $\varphi \in \ker \widetilde{\ell}_F$.

By Remark 5.1, it suffices to prove existence of functions $(a_{j\alpha}) = A$ such that the functions $A_i = \widetilde{\eth}_{\varphi,A}(X_{ij}^\alpha)$, $1 \leq i \leq n$, for any fixed j and α satisfy (5.2). We shall prove this fact by induction on j.

Let $j = 1$. Since $[\widetilde{D}_i, \widetilde{\eth}_{\varphi,A}] = \sum_{j,\alpha}(\widetilde{D}_i(a_{j\alpha}) - \widetilde{\eth}_{\varphi,A}(X_{ij}^\alpha))\partial/\partial w_{j\alpha}$, where X_{i1}^α are functions on $\mathcal{E}^\infty$, for any set of functions $A = (a_{j\alpha})$ on $\widetilde{\mathcal{E}}$ one has

$$\widetilde{D}_i(\widetilde{\eth}_{\varphi,A}(X_{k1}^\alpha)) = \widetilde{\eth}_{\varphi,A}(\widetilde{D}_i(X_{k1}^\alpha)) = \widetilde{\eth}_{\varphi,A}(\widetilde{D}_k(X_{i1}^\alpha)) = \widetilde{D}_k(\widetilde{\eth}_{\varphi,A}(X_{i1}^\alpha))$$

(the second equality holds by (5.3)), and it implies solvability of equations (5.4) for $j = 1$.

Suppose now that solvability of equations (5.4) was proved for $j < s$ and let $a_{j\alpha}^0$, $j < s$, be arbitrary solutions. Then for any set $A = (a_{j\alpha})$, where $a_{j\alpha} = a_{j\alpha}^0$, if $j < s$, while other $a_{j\alpha}$ are arbitrary functions, we have $[\widetilde{D}_i, \widetilde{\eth}_{\varphi,A}]\big|_{\widetilde{\mathcal{E}}^{(s-1)}} = 0$. Since $X_{is}^\alpha \in \mathcal{F}(\widetilde{\mathcal{E}}^{s-1})$, similar to the case $j = 1$, we obtain equality (5.2) for $A_i = \widetilde{\eth}_{\varphi,A}(X_{is}^\alpha)$, $1 \leq i \leq n$. The theorem is proved. $\square$

5.3. Nonlocal symmetries for equations admitting a recursion operator. Consider now the situation when the equation $\mathcal{E} = \{F = 0\}$ possesses a recursion operator R. Recursion operators considered in literature are not, in general, differential operators. Therefore, the expression $R(\varphi)$, where φ is a symmetry of the given equation, may be undefined. There are various interpretations for this type of expressions (see, for example, [**91, 112, 58**]). Here we suggest, instead of

the operator R, to consider an operator $\widetilde{R}$, which in some cases is defined on the set $\ker \widetilde{\ell}_F$. Let us describe a class of equations, where recursion operators possess this property (other examples see in §5.6).

PROPOSITION 5.2. *Let an evolution equation*

$$\mathcal{E} = \{u_t = D_x(h(x, t, u, u_1, \ldots, u_k))\}$$

possess recursion operator $R = \sum_{i=-1}^{N} f_i D_x^i$, $f_i \in \mathcal{F}(\mathcal{E}^\infty)$. *Then for any symmetry* $\widetilde{\eth}_{\varphi,A}$ *in the covering* τ^* *there exists a symmetry* $\widetilde{\eth}_{\varphi',A'}$, *where* $\varphi' = R(\varphi)$.

PROOF. Existence of the function φ' easily follows from Remark 5.1. In fact, since $\widetilde{\eth}_{\varphi,A}$ is a symmetry, the function φ satisfies the equation $\widetilde{\ell}_F(\varphi) = 0$, which, in the situation under consideration, is of the form $\widetilde{D}_t(\varphi) = \widetilde{D}_x(\widetilde{\ell}_h(\varphi))$ and coincides with equations (5.2). Therefore, there exists a function φ' such that $\varphi = \widetilde{D}_x(\varphi')$, or $\varphi' = \widetilde{D}_x^{-1}(\varphi)$. Existence of the symmetry $\widetilde{\eth}_{\varphi',A'}$ is guaranteed by Theorem 5.1. $\square$

5.4. Example: nonlocal symmetries of the Korteweg–de Vries equation. Consider the KdV equation $\mathcal{E} = \{u_t = u_{xxx} + uu_x\}$. By Proposition 5.2, this equation possesses the series of nonlocal symmetries $\widetilde{\eth}_{\psi_n,A}$, where $\psi_n = \widetilde{R}^n(tp_1+1)$, $R = D_x^2 + \frac{2}{3}p_0 + \frac{1}{3}p_1 D_x^{-1}$. In fact, $\psi_n \in \mathcal{F}(\widetilde{\mathcal{E}}^{(1)})$ for any n. This fact is implied by the following statement:

PROPOSITION 5.3. *Let an equation* $\mathcal{E}$ *and its recursion operator* R *satisfy the assumptions of Proposition* 5.2. *Assume also that there exists a local symmetry* φ *of the equation* $\mathcal{E}$ *evolving by* R *to the infinite series of local symmetries* $\mathcal{S} = \{R^n(\varphi) \mid n = 0, 1, \ldots\}$ *and the operator* R *be such that* $f_{-1} \in \mathcal{S}$. *Let* $\widetilde{R}^{(1)}$ *be the lifting of the operator* R *to the covering* $\tau_1 \colon \widetilde{\mathcal{E}}^{(1)} \to \mathcal{E}^\infty$ *and* w_α *be nonlocal variables in the covering* τ_1. *Then, if* $\Phi = \sum \varphi_i w_i + \Psi \in \ker \widetilde{\ell}_F$, *where the number of summands is finite,* $\varphi_i \in \mathcal{S}$, *and* $\Psi \in \mathcal{F}(\mathcal{E}^\infty)$, *then the function* $\widetilde{R}^{(1)}(\Phi)$ *lies in* $\mathcal{F}(\widetilde{\mathcal{E}}^{(1)})$ *and has a similar form.*

PROOF. The subscript "loc" below denotes functions on $\mathcal{E}^\infty$. We shall not need to describe these functions in details.

One has $\widetilde{R}^{(1)}(\Phi) = \sum w_i R(\varphi_i) + f_{-1}\widetilde{D}_x^{-1}(\Psi - \sum \widetilde{D}_x(w_i)\widetilde{D}_x^{-1}(\varphi_i)) + \Psi_{\text{loc}}$, since by Green's formula (see [**132, 131**]),

$$\widetilde{D}_x^{-1}(w_i \varphi_i) = w_i \widetilde{D}_x^{-1}(\varphi_i) - \widetilde{D}_x^{-1}(\widetilde{D}_x(w_i)\widetilde{D}_x^{-1}(\varphi_i)).$$

Let us set $X_{\text{loc}} = \Psi - \sum \widetilde{D}_x(w_i)\widetilde{D}_x^{-1}(\varphi_i)$ and show that $\bar{D}_t(X_{\text{loc}}) = \bar{D}_x(T_{\text{loc}})$ for some function T_{loc}. Since this means that $X_{\text{loc}}\,dx + T_{\text{loc}}\,dt$ is locally a conserved current density, the statement will be proved. We have

$$\widetilde{D}_x^{-1}(\Phi) = \sum w_i \widetilde{D}_x^{-1}(\varphi_i) + \widetilde{D}_x^{-1}(X_{\text{loc}}) + A_{\text{loc}}.$$

From here and from the equality $\widetilde{D}_t(\Phi) = \widetilde{D}_x(\widetilde{\ell}_g(\Phi))$ it follows that

$$\widetilde{D}_t(\widetilde{D}_x^{-1}(\Phi)) = \widetilde{D}_x^{-1}(\widetilde{D}_t(\Phi)) = \widetilde{\ell}_g(\Phi) = \sum w_i \widetilde{\ell}_g(\varphi_i) + B_{\text{loc}}.$$

On the other hand, since φ_i is a symmetry, we have

$$\widetilde{D}_t(\widetilde{D}_x^{-1}(\Phi)) = \widetilde{D}_t\left(\sum w_i\widetilde{D}_x^{-1}(\varphi_i) + \widetilde{D}_x^{-1}(X_{\mathrm{loc}}) + A_{\mathrm{loc}}\right) =$$
$$= \sum w_i\widetilde{D}_t(\widetilde{D}_x^{-1}(\varphi_i)) + \widetilde{D}_t(\widetilde{D}_x^{-1}(X_{\mathrm{loc}})) + C_{\mathrm{loc}} =$$
$$= \sum w_i\widetilde{\ell}_g(\varphi_i) + \widetilde{D}_t(\widetilde{D}_x^{-1}(X_{\mathrm{loc}})) + C_{\mathrm{loc}}.$$

Comparing the last two equations, we obtain

$$\widetilde{D}_t(\widetilde{D}_x^{-1}(X_{\mathrm{loc}})) = B_{\mathrm{loc}} - C_{\mathrm{loc}} = T_{\mathrm{loc}}.$$

This completes the proof. □

Let us write down several symmetries of the nonlocal series for the Korteweg–de Vries equation. To this end, let us introduce the nonlocal variables defined by the equalities $\widetilde{D}_x w_0 = p_0$, $\widetilde{D}_x w_1 = \frac{1}{2}p_0^2$, $\widetilde{D}_x w_2 = \frac{1}{6}(p_0^3 - 3p_1^2), \ldots$, and set $\varphi_k = R^k p_1$, $k \geq 0$. Then

$$\psi_0 = t\varphi_0 + 1,$$
$$\psi_1 = t\varphi_1 + \frac{1}{3}x\varphi_0 + \frac{2}{3}p_0,$$
$$\psi_2 = t\varphi_2 + \frac{1}{3}x\varphi_1 + \frac{4}{3}p_2 + \frac{4}{9}p_0^2 + \frac{1}{9}\varphi_0 w_0,$$
$$\psi_3 = t\varphi_3 + \frac{1}{3}x\varphi_2 + 2\bar{D}_x(\varphi_1) + \frac{2}{3}p_0 p_2 + \frac{8}{27}p_0^3 + \frac{1}{9}\varphi_1 w_0 + \frac{1}{9}\varphi_0 w_1.$$

5.5. Master symmetries. Consider in the covering τ^* over the Korteweg–de Vries equation the symmetry $\widetilde{\eth}_{\psi_2,A}$ and take another τ^*-symmetry $\widetilde{\eth}_{\varphi,B}$, where φ is a local symmetry independent of x and t. Taking the commutator of these two symmetries, we shall obtain a new τ^*-symmetry $\widetilde{\eth}_{\chi,C} = [\widetilde{\eth}_{\psi_2,A}, \widetilde{\eth}_{\varphi,B}]$, where $\chi = \widetilde{\eth}_{\psi_2,A}(\varphi) - \widetilde{\eth}_{\varphi,B}(\psi_2) \in \ker \widetilde{\ell}_F$. By simple calculations, one can see that the function χ is a local symmetry of the Korteweg–de Vries equation independent of the variables x and t. In fact, since φ is a local symmetry and ψ_2 depends only on one nonlocal variable w_0 satisfying the relation $\widetilde{D}_x w_0 = p_0$, one has $\chi = \widetilde{\eth}_{\psi_2}(\varphi) - \widetilde{\eth}_\varphi(\psi_2) - \frac{1}{9}p_1\widetilde{D}_x^{-1}\varphi$, i.e., the function χ may depend on local variables and on w_0 only.

Further, since local symmetries of the Korteweg–de Vries equation independent of x and t mutually commute, we have in particular $\{\varphi, \varphi_1\} = \{\varphi, \varphi_2\} = 0$, from where it follows that

$$\chi = \frac{1}{3}\sum j\bar{D}_x^{j-1}\varphi_1\frac{\partial\varphi}{\partial p_j} + \widetilde{\eth}_\Psi(\varphi) - \bar{\eth}_\varphi(\Psi) - \frac{1}{9}p_1\widetilde{D}_x^{-1}\varphi, \qquad (5.5)$$

where $\Psi = \frac{4}{3}p_2 + \frac{4}{9}p_0^2 + \frac{1}{9}\varphi_0 w_0$. From this expression it follows that χ is independent of x and t. A more detailed analysis shows that χ is independent of w_0. Thus, χ is a local symmetry independent of the variables x and t.

Let $s = 2k + 1$ be the order of the symmetry φ, i.e., $\partial\varphi/\partial p_s = \mathrm{const} \neq 0$ and $\partial\varphi/\partial p_j = 0$ for all $j > s$. Then it is easily seen that the function χ is of the form

$$\chi = \frac{1}{3}sp_{s+2}\frac{\partial\varphi}{\partial p_s} + \text{lower order terms} = c\varphi_{k+1} + \text{lower order symmetries},$$

where $c = \frac{1}{3}s\partial\varphi/\partial p_s = \mathrm{const}$.

This means that the operator of commutation with the τ^*-symmetry $\widetilde{\eth}_{\psi_2,A}$ acts at the first component of the generating function of the τ^*-symmetry $\widetilde{\eth}_{\varphi,B}$,

up to symmetries of lower order, in the same way as the recursion operator R does. Thus, this operator of commutation serves the role of the recursion operator for the Korteweg–de Vries equation.

Note that a similar situation takes place for the equations considered in the next subsection.

5.6. Examples. In this subsection, we consider examples of equations with recursion operators, which formally do not satisfy the assumptions of Proposition 5.2. Nevertheless, additional series of nonlocal symmetries arise in the covering τ^* for these equations.

Consider an equation in two independent variables $x = x_1$, $t = x_2$ possessing a recursion operator of the form

$$R = \sum_{i=0}^{M} f_i D_x^i + a D_x^{-1} \circ b, \qquad f_i, a, b \in \mathcal{F}(\mathcal{E}^\infty).$$

To prove existence of the function $R\varphi \in \mathcal{F}(\mathcal{E}^*)$ for any $\varphi \in \ker \widetilde{\ell}_F$, it suffices to show that the function $b\varphi$ lies in the total derivative image, i.e., $b\varphi = \bar{D}_x \psi$ for some function $\psi \in \mathcal{F}(\mathcal{E}^*)$. This fact will be established, if we prove existence of a function $T \in \mathcal{F}(\mathcal{E}^*)$ such that $\bar{D}_t(b\varphi) = \bar{D}_x(T)$. It is not difficult to check this condition in each particular case.

EXAMPLE 5.1. Consider the modified Korteweg–de Vries equation

$$\mathcal{E} = \{u_t = u_{xxx} + u^2 u_x\}. \tag{5.6}$$

The recursion operator for this equation is of the form [**91**]:

$$R = \bar{D}_x^2 + \frac{2}{3} p_0^2 + \frac{2}{3} p_1 \bar{D}_x^{-1} \circ p_0.$$

Let us show that the operator $\widetilde{R}$ may be applied to any function $\varphi \in \ker \widetilde{\ell}_F$. In fact,

$$\widetilde{D}_t(p_0\varphi) = \widetilde{D}_x \left(\left(p_2 + \frac{1}{3} p_0^3 \right) \varphi + p_0 (\widetilde{D}_x^2(\varphi) + p_0^2 \varphi) - p_1 \widetilde{D}_x(\varphi) - \frac{1}{3} p_0^3 \varphi \right),$$

$$\widetilde{D}_t(\varphi) = \widetilde{D}_x^3(\varphi) + p_0^2 \widetilde{D}_x(\varphi) + 2 p_0 p_1 \varphi.$$

Therefore, equation (5.6) possesses a series of nonlocal symmetries $\eth_{\psi_n, A}$, where

$$\psi_n = \widetilde{R}^n \left(t(p_3 + p_0^2 p_1) + \frac{1}{3} x p_1 + \frac{1}{3} p_0 \right).$$

EXAMPLE 5.2. Consider the sine-Gordon equation

$$\mathcal{E} = \{u_{xt} = \sin u\}. \tag{5.7}$$

If φ satisfies the equation $\widetilde{D}_x \widetilde{D}_t(\varphi) = \varphi \cos p_{(0,0)}$, then, since $\widetilde{D}_t(p_{(2,0)}\varphi) = \widetilde{D}_x(p_{(1,0)} \widetilde{D}_t(\varphi))$, the recursion operator $\widetilde{R} = \widetilde{D}_x^2 + p_{(1,0)}^2 - p_{(1,0)} \widetilde{D}_x^{-1} \circ p_{(2,0)}$ (see [**91**]) is defined on $\ker \widetilde{\ell}_F$. Therefore, equation (5.7) possesses the series of nonlocal symmetries $\eth_{\psi_n, A}$, $\psi_n = \widetilde{R}^n (x p_{(1,0)} - t p_{(0,1)})$.

EXAMPLE 5.3. Consider the nonlinear Schrödinger equation in the form

$$\begin{cases} u_t = i u_{xx} + 2 u u_x v, \\ v_t = -i v_{xx} - 2 u v v_x. \end{cases}$$

This system possesses the following recursion operator [**14**]

$$
R = \begin{pmatrix} \bar{D}_x + i(p_1^1 \bar{D}_x^{-1} \circ p_0^2 & i(p_0^1 \bar{D}_x^{-1} \circ p_1^1 + p_1^1 \bar{D}_x^{-1} \circ p_0^1) \\ -p_0^1 \bar{D}_x^{-1} \circ p_1^2 + p_0^1 p_0^2) & \\ i(p_0^2 \bar{D}_x^{-1} \circ p_1^2 + p_1^2 \bar{D}_x^{-1} \circ p_0^2) & -\bar{D}_x + i(p_1^2 \bar{D}_x^{-1} \circ p_0^1 \\ & -p_0^2 \bar{D}_x^{-1} \circ p_1^1 + p_0^1 p_0^2) \end{pmatrix}
$$

Here $p_k^i = p_{k0}^i$, $x_1 = x$, $x_2 = t$.

To check that the operator $\widetilde{R}$ generates an infinite series of τ^*-symmetries, it suffices to show that $\widetilde{D}_x^{-1}(p_1^2 \varphi - p_1^1 \psi)$, $\widetilde{D}_x^{-1}(p_0^2 \varphi + p_0^1 \psi) \in \mathcal{F}(\widetilde{\mathcal{E}}^*)$ for any vector function $(\varphi, \psi) \in \ker \widetilde{\ell}_F$. But this fact is implied by the following equalities:

$$
\widetilde{D}_x^{-1}(p_1^2 \varphi - p_1^1 \psi) = \widetilde{D}_x(i(p_1^2 \widetilde{D}_x \varphi - p_1^2 \varphi + p_1^1 \widetilde{D}_x \psi - p_1^1 \psi) + 2p_0^1 p_0^2 (p_1^1 \psi - p_1^2 \varphi)),
$$
$$
\widetilde{D}_x^{-1}(p_0^2 \varphi + p_0^1 \psi) = \widetilde{D}_x(i(p_0^2 \widetilde{D}_x \varphi - p_1^2 \varphi + p_1^1 \widetilde{D}_x \psi - p_0^1 \psi) - 2p_0^1 p_0^2 (p_0^1 \psi - p_0^2 \varphi)).
$$

Thus, the operator $\widetilde{R}$ is determined on $\operatorname{sym}_{\tau^*} \mathcal{E}$ and the symmetry

$$
\Phi = (t p_2^1 + \frac{1}{2} x p_1^1, t p_2^2 + \frac{1}{2} x p_1^2 + \frac{1}{2} p_0^2)
$$

generates a series of symmetries for the system under consideration.

5.7. General problem of nonlocal symmetry reconstruction. Consider the general situation now. Namely, let $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be an arbitrary covering of the equation $\mathcal{E}^\infty$ and $\varphi \in \mathcal{F}(\widetilde{\mathcal{E}}, \pi)$ be an arbitrary solution of the equation $\widetilde{\ell}_F(\varphi) = 0$, where $\mathcal{E} = \{F = 0\}$. To find a τ-symmetry of the form $\widetilde{\Im}_{\varphi, A} = \widetilde{\Im}_\varphi + \sum_{i=1}^N a_i \partial/\partial w_i$, it needs to find functions $a_1, \dots, a_N$, $a_j \in \mathcal{F}(\widetilde{\mathcal{E}})$, satisfying the equations

$$
\widetilde{D}_i(a_j) = \widetilde{\Im}_{\varphi, A}(X_{ij}). \tag{5.8}
$$

As we already saw, this system may have no solution for a given φ. Note that formal conditions of compatibility for system (5.8) are of the form

$$
\widetilde{D}_i(Y_{jk}) = \widetilde{D}_j(Y_{ik}), \tag{5.9}
$$

where Y_{ij} denotes the right-hand side of (5.8). Note further that the integrability condition for the distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}}$, i.e., $[\widetilde{D}_i, \widetilde{D}_j] = 0$, $i, j = 1, \dots, n$, in local coordinates are of the form

$$
\widetilde{D}_i(X_{jk} = \widetilde{D}_j(X_{ik}) \tag{5.10}
$$

(see. (1.3)), where $\widetilde{D}_i = \bar{D}_i + \sum_{j=1}^N X_{ij} \partial/\partial w_j$, $i = 1, \dots, n$. Conditions (5.9) and (5.10) are of a similar structure. Using this fact, let us construct a new covering $\tau_1 \colon \widetilde{\mathcal{E}}_1 \to \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ by introducing new nonlocal variables $a_1, \dots, a_N$ and defining new total derivatives $\widetilde{D}_i^{(1)}$, $i = 1, \dots, n$, on $\widetilde{\mathcal{E}}_1$ by the formula $\widetilde{D}_i^{(1)} = \widetilde{D}_i + \sum_j Y_{ij} \partial/\partial a_j$. Introducing the covering

$$
\tau_1 \colon \widetilde{\mathcal{E}}_1 \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \mathcal{E}^\infty,
$$

we can now solve system (5.8), but it does not give us a solution of the initial problem, since the field $\widetilde{\Im}_{\varphi, A} = \widetilde{\Im}_\varphi + \sum_{i=1}^N a_j \partial/\partial w_j$ is not necessary a symmetry in the covering τ_1. This field is to be extended up to a field of the form

$$
\widetilde{\Im}_{\varphi, A, B} = \widetilde{\Im}_\varphi + \sum_i a_i \frac{\partial}{\partial w_i} + \sum_j b_j \frac{\partial}{\partial a_j}
$$

on the manifold $\widetilde{\mathcal{E}}_1$, where the functions b_j must satisfy a system of equations similar to (5.8). Again, this system may have no solution and we shall have to repeat the construction procedure for a new covering. As a result, we shall obtain a chain of coverings similar to that aroused when we constructed the universal Abelian covering. In more formal details, this procedure is described in the next two subsections.

5.8. Kiso's construction. Let $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be an arbitrary covering over the equation $\mathcal{E} = \{F = 0\}$ described by the fields (1.2) and let $\varphi \in \ker \widetilde{\ell}_F$, $\varphi = (\varphi_1, \ldots, \varphi_m)$, $\varphi_i \in \mathcal{F}(\widetilde{\mathcal{E}})$. In the paper [**55**], for evolution equations in one space variable the covering $\tau_\varphi\colon \widetilde{\mathcal{E}}_\varphi \to \mathcal{E}^\infty$ was constructed, where the function φ generates a symmetry. Below we describe a generalization of this construction to arbitrary equations.

Let us set $\widetilde{\mathcal{E}}_\varphi = \widetilde{\mathcal{E}} \times \mathbb{R}^\infty$ and let w_j^l, $j = 1, \ldots, N$, $l = 0, 1, 2, \ldots$, be coordinates in $\mathbb{R}^\infty$ (new nonlocal variables). The mapping $\tau_\varphi\colon \widetilde{\mathcal{E}}_\varphi \to \mathcal{E}^\infty$ is the composition of the projection to the first factor with τ.

Let us define vector fields $\widetilde{D}_i^\varphi$ on $\widetilde{\mathcal{E}}_\varphi$ by the formula

$$\widetilde{D}_i^\varphi = \bar{D}_i + \sum_{j,l>0} \left(\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} + S_w \right)^l (X_{ij}) \frac{\partial}{\partial w_j^l}, \qquad i = 1, \ldots, n, \qquad (5.11)$$

where[5] $\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} = \sum'_{\sigma,k} \widetilde{D}_\sigma^\varphi(\varphi_k) \partial/\partial p_\sigma^k$, $S_w = \sum_{j,l} w_j^{l+1} \partial/\partial w_j^l$, $w_j^0 = w_j$.

PROPOSITION 5.4. *Let $\mathcal{E}$ be an equation and $\widetilde{\mathcal{E}}_\varphi = \widetilde{\mathcal{E}} \times \mathbb{R}^\infty$. Then*

1. $[\widetilde{D}_i^\varphi, \widetilde{D}_k^\varphi] = 0$, i, $k = 1, \ldots, n$, *i.e., the set of fields (5.11) determines a covering structure in $\widetilde{\mathcal{E}}_\varphi$.*
2. *If $\varphi \in \ker \widetilde{\ell}_F$, $\varphi = (\varphi_1, \ldots, \varphi_m)$, $\varphi_i \in \mathcal{F}(\widetilde{\mathcal{E}})$, then $[\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} + S_w, \widetilde{D}_i^\varphi] = 0$, $i = 1, \ldots, n$, i.e., $\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} + S_w$ is a symmetry of type τ_φ for the equation $\mathcal{E}^\infty$.*

PROOF. Let us first show that $\widetilde{\ell}_F(\varphi) = 0$ implies $[\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} + S_w, \widetilde{D}_i^\varphi] = 0$, $i = 1, \ldots, n$. One has

$$[\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} + S_w, \widetilde{D}_i^\varphi] = \sum'_{\sigma,k} \widetilde{\mathfrak{I}}_\varphi(\bar{p}_{\sigma+1_i}^k) \frac{\partial}{\partial p_\sigma^k} + \sum_{j,l>0} (\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} + S_w)^{l+1} (X_{ij}) \frac{\partial}{\partial w_j^l}$$

$$- \sum'_{\sigma,k} \widetilde{D}_i^\varphi(\widetilde{D}_\sigma(\varphi_k)) \frac{\partial}{\partial p_\sigma^k} - \sum_{j,l>0} (\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} + S_w)^{l+1} (X_{ij}) \frac{\partial}{\partial w_j^l}$$

$$= \sum'_{\sigma,k} (\widetilde{\mathfrak{I}}_\varphi(\bar{p}_{\sigma+1_i}^k) - \widetilde{D}_{\sigma+1_i}(\varphi_k)) \frac{\partial}{\partial p_\sigma^k} = 0.$$

Consider now the commutator $[\widetilde{D}_i^\varphi, \widetilde{D}_k^\varphi]$. Since $[\bar{D}_i, \bar{D}_k] = 0$ and the coefficients of the vector fields $\bar{D}_j$ are independent of w_j^l, we have

$$[\widetilde{D}_i^\varphi, \widetilde{D}_k^\varphi] = \sum_{j,l>0} \{ \widetilde{D}_i^\varphi (\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} + S_w)^l (X_{kj}) - \widetilde{D}_k^\varphi (\widetilde{\mathfrak{I}}_\varphi^{(\varphi)} + S_w)^l (X_{ij}) \} \frac{\partial}{\partial w_j^l}.$$

---

[5]As above, "prime" denotes summation over internal coordinates.

By the first statement of our proposition, one has

$$[\widetilde{D}_i^\varphi, \widetilde{D}_k^\varphi] = \sum_{j, l > 0} (\widetilde{\vartheta}_\varphi^{(\varphi)} + S_w)^l (\widetilde{D}_i^\varphi(X_{kj}) - \widetilde{D}_k^\varphi(X_{ij})) \frac{\partial}{\partial w_j^l} = 0,$$

since $\widetilde{D}_i^\varphi(X_{kj}) - \widetilde{D}_k^\varphi(X_{ij}) = \widetilde{D}_i(X_{kj}) - \widetilde{D}_k(X_{ij}) = 0$. The proposition is proved. $\square$

5.9. Construction of the covering τ_S. Below we follow the paper [**53**].

Let us set $\widetilde{\mathcal{E}}_\tau = \widetilde{\mathcal{E}} \times \mathbb{R}^\infty$, where coordinates in $\mathbb{R}^\infty$ (nonlocal variables) are the variables[6] v_j^l, $j = 1, \ldots, N$, $l = 1, 2, \ldots p_\sigma^{(k)}$, $k > 0$. The mapping $\tau_S \colon \widetilde{\mathcal{E}}_\tau \to \mathcal{E}^\infty$ is the composition of the projection to the first factor with τ.

Let us define the system of fields $\widetilde{D}_1^\tau, \ldots, \widetilde{D}_n^\tau$ on $\widetilde{\mathcal{E}}_\tau$ as follows:

$$\widetilde{D}_i^\tau = \bar{D}_i^S + \sum_{l \geq 0, j} (S_p + S_v)^l (X_{ij}) \frac{\partial}{\partial v_j^l}, \qquad i = 1, \ldots, n, \tag{5.12}$$

where

$$\bar{D}_i^S = \frac{\partial}{\partial x_i} + \sum_{l \geq 0, \sigma}' S_p^l(\bar{p}_{\sigma+1_i}) \frac{\partial}{\partial p_\sigma^{(l)}}, \qquad p_\sigma^{(0)} = p_\sigma, \tag{5.13}$$

$$S_p = \sum_{l \geq 0, \sigma}' p_\sigma^{(l+1)} \frac{\partial}{\partial p_\sigma^{(l)}}, \quad S_v = \sum_{l \geq 0, j} v_j^{l+1} \frac{\partial}{\partial v_j^l}, \qquad v_j^0 = w_j. \tag{5.14}$$

PROPOSITION 5.5.   1. $[\widetilde{D}_i^\tau, \widetilde{D}_k^\tau] = 0$, $i, k = 1, \ldots, n$, *i.e., the set of fields* (5.12) *determines a covering structure in* $\widetilde{\mathcal{E}}_\tau$.
2. *The vector field* $S_\tau = S_p + S_v$ *is a nonlocal symmetry of type* τ_S *for the equation* $\mathcal{E}^\infty$.

PROOF. From formulas (5.12)–(5.14) it easily follows that $[S_\tau, \widetilde{D}_i^\tau] = 0$, $i = 1, \ldots, n$, i.e., S_τ is a nonlocal symmetry. Let us now compute the commutator $[\widetilde{D}_i^\tau, \widetilde{D}_j^\tau]$:

$$[\widetilde{D}_i^\tau, \widetilde{D}_k^\tau] = \sum' \left\{ \widetilde{D}_i^\tau \left(S_p^l(\bar{p}_{\sigma+1_j}) \right) - \widetilde{D}_j^\tau \left(S_p^l(\bar{p}_{\sigma+1_i}) \right) \right\} \frac{\partial}{\partial p_\sigma^{(l)}}$$

$$+ \sum \left\{ \widetilde{D}_i^\tau \left(S_\tau^l (X_{jk}) \right) - \widetilde{D}_j^\tau \left(S_\tau^l (X_{ik}) \right) \right\} \frac{\partial}{\partial v_k^l}$$

$$= \sum' S_P^l \left\{ \widetilde{D}_i^\tau \left(\bar{p}_{\sigma+1_j} \right) - \widetilde{D}_j^\tau \left(\bar{p}_{\sigma+1_i} \right) \right\} \frac{\partial}{\partial p_\sigma^{(l)}}$$

$$+ \sum S_\tau^l \left\{ \widetilde{D}_i^\tau (X_{jk}) - \widetilde{D}_j^\tau (X_{ik}) \right\} \frac{\partial}{\partial v_k^l} = 0,$$

since $[\bar{D}_i, \bar{D}_k] = [\widetilde{D}_i, \widetilde{D}_k] = 0$. The proposition is proved. $\square$

The following theorem is a corollary of Propositions 5.4 and 5.5.

THEOREM 5.6. *For any covering* $\tau \colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ *over the equation* $\mathcal{E} = \{F = 0\}$ *and for any set* Φ *of vector functions* $\varphi^1, \ldots, \varphi^K$, $\varphi^j = (\varphi_1^j, \ldots, \varphi_m^j)$, $\varphi_i^j \in \mathcal{F}(\widetilde{\mathcal{E}}_\tau)$, *satisfying the equation* $\widetilde{\ell}_F^\tau(\varphi) = 0$, *where* $\widetilde{\ell}_F^\tau$ *is the lifting of the operator* ℓ_F *to* $\widetilde{\mathcal{E}}_\tau$,

---

[6]To simplify notation, we omit the superscript corresponding to the independent variable number at p_σ. The superscript in parenthesis enumerates new nonlocal variables.

there exists a covering $\tau_\Phi : \widetilde{\mathcal{E}}_\Phi \to \widetilde{\mathcal{E}}^\infty$ *and* τ_Φ*-symmetries* $S_{\varphi^1}, \dots, S_{\varphi^K}$, *such that* $S_{\varphi^j}\big|_{\mathcal{E}^\infty} = \widetilde{\eth}_{\varphi^j}\big|_{\mathcal{E}^\infty}$, $j = 1, \dots, K$.

5.10. The universal property of the symmetry S_τ. In conclusion, we shall discuss relations between the constructions of §§5.8 and 5.9. Namely, the following statement holds:

PROPOSITION 5.7. *Let* $\tau : \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ *be a covering over the equation* $\mathcal{E} = \{F = 0\}$. *Then for any vector function* $\varphi \in \ker \widetilde{\ell}_F$, $\varphi = (\varphi_1, \dots, \varphi_m)$, $\varphi_i \in \mathcal{F}(\widetilde{\mathcal{E}})$, *there exists an embedding* $i_\varphi : \widetilde{\mathcal{E}}_\varphi \to \widetilde{\mathcal{E}}_\tau$ *such that the diagram*

$$\widetilde{\mathcal{E}}_\varphi \xrightarrow{\quad i_\varphi \quad} \widetilde{\mathcal{E}}_\tau$$
$$\tau_\varphi \searrow \quad \swarrow \tau_S$$
$$\widetilde{\mathcal{E}}$$

is commutative. Moreover, $\Pi_\varphi = i_\varphi(\widetilde{\mathcal{E}}_\varphi)$ *is an integral submanifold of the distribution on* $\widetilde{\mathcal{E}}_\tau$ *determined by the fields* (5.11) *and*

$$S_\tau\big|_{\Pi_\varphi} = (i_\varphi)_* \left(\widetilde{\eth}_\varphi^{(\varphi)} + S_w \right), \qquad \widetilde{D}_i^\tau\big|_{\Pi_\varphi} = (i_\varphi)_* \widetilde{D}_i^\varphi.$$

PROOF. Let us define the embedding i_φ by the formula $i_\varphi(p_\sigma) = (p_\sigma^{(i)}, v_j^l)$, where $p_\sigma^{(i)} = \left(\widetilde{\eth}_\varphi + S_w \right)^i (p_\sigma)$, $v_j^l = w_j^l$. Then the manifold Π_φ is determined by the equations $p_\sigma^{(i)} = \left(\widetilde{\eth}_\varphi + S_v \right)^i (p_\sigma)$. Let us show that the field $S_\tau = S_p + S_v$ is tangent to Π_φ. In fact,

$$\left\{ S_\tau \left(p_\sigma^{(i)} - \left(\widetilde{\eth}_\varphi + S_v \right)^i (p_\sigma) \right) \right\}\Big|_{\Pi_\varphi} = \left\{ p_\sigma^{(i+1)} - S_\tau \left(\widetilde{\eth}_\varphi + S_v \right)^i (p_\sigma) \right\}\Big|_{\Pi_\varphi}$$

$$= \left\{ \left(\left(\widetilde{\eth}_\varphi + S_v \right)^{i+1} - S_\tau \left(\widetilde{\eth}_\varphi + S_v \right)^i \right) (p_\sigma) \right\}\Big|_{\Pi_\varphi}$$

$$= \left\{ \left(\widetilde{\eth}_\varphi - S_p \right) \left(\widetilde{\eth}_\varphi + S_v \right)^i (p_\sigma) \right\}\Big|_{\Pi_\varphi}$$

$$= \left\{ \left({\sum_{\sigma'}}' \widetilde{D}_{\sigma'}(\varphi) \frac{\partial}{\partial p_{\sigma'}} - {\sum_{\sigma',i}}' p_{\sigma'}^{(i+1)}\Big|_{\Pi_\varphi} \frac{\partial}{\partial p_{\sigma'}^{(i)}} \right) \left(\widetilde{\eth}_\varphi + S_v \right)^i (p_\sigma) \right\}\Big|_{\Pi_\varphi}$$

$$= \left\{ \left({\sum_{\sigma'}}' \widetilde{D}_{\sigma'}(\varphi) \frac{\partial}{\partial p_{\sigma'}} - {\sum_{\sigma'}}' p_{\sigma'}^{(1)}\Big|_{\Pi_\varphi} \frac{\partial}{\partial p_{\sigma'}} \right) \left(\widetilde{\eth}_\varphi + S_v \right)^i (p_\sigma) \right\}\Big|_{\Pi_\varphi} =$$

$$= \left\{ {\sum_{\sigma'}}' \left(\widetilde{D}_{\sigma'}(\varphi) - \left(\widetilde{\eth}_\varphi + S_v \right)^i (p_{\sigma'}) \right) \frac{\partial}{\partial p_{\sigma'}} \left(\widetilde{\eth}_\varphi + S_v \right)^i (p_\sigma) \right\}\Big|_{\Pi_\varphi} = 0,$$

since $\left(\widetilde{\eth}_\varphi + S_v \right)^i (p_\sigma) = \widetilde{\eth}_\varphi(p_\sigma) = \widetilde{D}_\sigma(\varphi)$. Let us prove now that for any function $f \in \mathcal{F}(\widetilde{\mathcal{E}}_\varphi)$ the equality

$$S_\tau\big|_{\Pi_\varphi} \left((i_\varphi^*)^{-1}(f) \right) = (i_\varphi^*)^{-1} \left(\widetilde{\eth}_\varphi^{(\varphi)} + S_w \right)(f).$$

holds. In fact,

$$
S_\tau|_{\Pi_\varphi}\left((i_\varphi^*)^{-1}(f)\right) = \left(\sideset{}{'}\sum_{\sigma,i} p_\sigma^{(i+1)}\Big|_{\Pi_\varphi}\frac{\partial}{\partial p_\sigma^i} + S_v\right)\Bigg|_{\Pi_\varphi}\left((i_\varphi^*)^{-1}(f)\right)
$$

$$
= \left(\sideset{}{'}\sum_{\sigma} p_\sigma^{(1)}\Big|_{\Pi_\varphi}\frac{\partial}{\partial p_\sigma} + S_v\right)\Bigg|_{\Pi_\varphi}\left((i_\varphi^*)^{-1}(f)\right)
$$

$$
= \left(\sideset{}{'}\sum_{\sigma} \widetilde{D}_\sigma(\varphi)\frac{\partial}{\partial p_\sigma} + S_v\right)\Bigg|_{\Pi_\varphi}\left((i_\varphi^*)^{-1}(f)\right) = (i_\varphi^*)^{-1}\left(\widetilde{\mathfrak{H}}_\varphi^{(\varphi)} + S_w\right)(f).
$$

From these relations it follows that $S_\tau|_{\Pi_\varphi} = (i_\varphi)_*\left(\widetilde{\mathfrak{H}}_\varphi^{(\varphi)} + S_w\right)$.

Let us show that the fields $\widetilde{D}_i^\tau$ are tangent to Π_φ, $\varphi \in \ker\widetilde{\ell}_F$, i.e., that Π_φ is an invariant submanifold. Indeed,

$$
\left\{\widetilde{D}_i^\tau\left(p_\sigma^{(l)} - \left(\widetilde{\mathfrak{H}}_\varphi + S_v\right)^l(p_\sigma)\right)\right\}\Bigg|_{\Pi_\varphi} = S_p^l(\bar{p}_{\sigma+1_i})|_{\Pi_\varphi} - \widetilde{D}_i^\tau(\widetilde{\mathfrak{H}}_\varphi + S_v)^l(p_\sigma)\Big|_{\Pi_\varphi}
$$

$$
= S_\tau^l(\bar{p}_{\sigma+1_i})|_{\Pi_\varphi} - (i_\varphi^*)^{-1}\left(\widetilde{D}_i^\varphi(\widetilde{\mathfrak{H}}_\varphi + S_v)^l(p_\sigma)\right) = S_\tau^l(\bar{p}_{\sigma+1_i})|_{\Pi_\varphi}
$$

$$
- (i_\varphi^*)^{-1}\left((\widetilde{\mathfrak{H}}_\varphi + S_v)^l\widetilde{D}_i^\varphi(p_\sigma)\right) = S_\tau^l(\bar{p}_{\sigma+1_i})|_{\Pi_\varphi} - (i_\varphi^*)^{-1}\left((\widetilde{\mathfrak{H}}_\varphi + S_v)^l(p_{\sigma+1_i})\right) = 0.
$$

Let us prove finally that $\widetilde{D}_i^\tau\Big|_{\Pi_\varphi} = (i_\varphi)_*\widetilde{D}_i^\varphi$. In fact, for an arbitrary function $f \in \mathcal{F}(\widetilde{\mathcal{E}}_\varphi)$ we have

$$
\widetilde{D}_i^\tau\Big|_{\Pi_\varphi}((i_\varphi^*)^{-1}(f)) = \left(\bar{D}_i + \sum_{l\geq 0,j} S_\tau^l(X_{ij})\big|_{\Pi_\varphi}\frac{\partial}{\partial v_j^l}\right)((i_\varphi^*)^{-1}(f))
$$

$$
= \left(\bar{D}_i + \sum_{l\geq 0,j}(\widetilde{\mathfrak{H}}_\varphi + S_v)^l X_{ij}\big|_{\Pi_\varphi}\frac{\partial}{\partial v_j^l}\right)((i_\varphi^*)^{-1}(f)) = (i_\varphi^*)^{-1}\widetilde{D}_i^\varphi(f)
$$

The proposition is proved. $\qquad\square$

6. Symmetries of integro-differential equations

The above constructed theory of nonlocal symmetries is based on the concept of a covering, i.e., on introduction of nonlocal variables to the space of infinitely prolonged equation. Recall that such nonlocalities are of indefinite integral nature. Now we shall consider the situation, when the initial object is nonlocal and the corresponding nonlocalities are definite integrals. Such objects are *integro-differential equations* and below we construct a symmetry theory for such equations.

6.1. Transformation of integro-differential equations to boundary differential form. Let us represent a system of integro-differential equation in the form

$$
G_j(x, u, p_\sigma, I) = 0, \qquad j = 1, \ldots, m_1, \tag{6.1}
$$

where $x = (x_1, \ldots, x_n)$ are independent variables, $u = (u^1, \ldots, u^m)$ are dependent ones, $p_\sigma = (p_\sigma^1, \ldots, p_\sigma^m)$ are the derivatives of latter over the former, $I = (I_1, \ldots, I_{m_2})$ are integrals. Values of integrals may depend both on choice of a

point x and on choice of a section u. The most popular integro-differential equations (see [**41**]) contain integrals of three types:

$$\int_a^b f(x, s, u(s + x))\, ds, \tag{6.2}$$

$$\int_0^x f(t, x, s, u(t, s), u(t, x - s), p_\sigma(t, s), p_\sigma(t, x - s))\, ds, \tag{6.3}$$

$$\int_a^b f(t, x, s, u(t, s))\, ds, \tag{6.4}$$

where t, x are the independent variables, u is the dependent one, p_σ is its derivatives, and s is the variable of integration. These, as well as many others, types of integrals are unified in the following way.

Let M be the manifold of independent variables $x = (x_1, \ldots, x_n)$, N be the manifold of independent and integration variables $(x, s) = (x_1, \ldots, x_n, s_1, \ldots, s_{n_1})$, $\rho \colon N \to M$ be the corresponding fiber bundle (possibly, with singularities), N_x be the fiber of the bundle ρ over the point $x \in M$. Note that the integrand depends on x, s, and $u(x)$ and there exist smooth mappings $h_i \colon N \to M$, $i = 1, \ldots, m_2$, allowing one to rewrite (6.2)–(6.4) in the form

$$\int_{N_x} f(x, s, h_1{}^*(u), h_1{}^*(p_\sigma), \ldots, h_l{}^*(u), h_l{}^*(p_\sigma))\, ds. \tag{6.5}$$

EXAMPLE 6.1. For the integral (6.2) we have: $M = \mathbb{R}$, $N = \{\, (x, s) \in \mathbb{R}^2 \mid a \leq s \leq b \,\}$, N_x is the interval $[a, b]$ for any x, $l = 1$, $h \colon (x, s) \mapsto x + s$.

EXAMPLE 6.2. For the integral (6.3) we have: $M = \{\, (t, x) \in \mathbb{R}^2 \mid x \geq 0 \,\}$, $N = \{\, (t, x, s) \in \mathbb{R}^3 \mid x \geq 0,\ 0 \leq s \leq x \,\}$, $N_{(t,x)}$ is the interval $[0, x]$ (at singular points $\{x = 0\}$, the interval shrinks to a point), $l = 2$, $h_1 \colon (t, x, s) \mapsto (t, s)$, $h_2 \colon (t, x, s) \mapsto (t, x - s)$.

EXAMPLE 6.3. For the integral (6.4) in the case $b = \infty$ we have: $M = \{\, (t, x) \in \mathbb{R}^2 \mid x \geq a \,\}$, $N = \{\, (t, x, s) \in \mathbb{R}^3 \mid x \geq a, s \geq a \,\}$, $N_{(t,x)}$ is the ray $\{s \geq a\}$, $l = 1$, $h \colon (t, x, s) \mapsto (t, s)$. The case of the finite b is considered in a similar way.

To simplify the formulas below, we shall assume that all the integrals under consideration are one-dimensional. The case of multiple integrals can be considered in a similar way: it suffices to represent the integral as iterated one.

If the integral (6.5) is one-dimensional, then N_x is either an interval or an infinite semi-interval or a straight line. In the case $N_x = [a_x, b_x]$, we shall introduce a nonlocal variable v depending on x, s, such that

$$\begin{aligned} \frac{\partial v}{\partial s} &= f(x, s, h_1{}^*(u), h_1^*(p_\sigma), \ldots, h_l^*(u), h_l^*(p_\sigma)), \\ v|_{s=a_x} &= 0. \end{aligned} \tag{6.6}$$

The value of the integral (6.5) at a point $x \in M$ coincides with $v|_{s=b_x}$ in this case. Let us introduce the mappings

$$a \colon M \longrightarrow N, \quad x \longmapsto (x, a_x); \qquad b \colon M \longrightarrow N, \quad x \longmapsto (x, b_x).$$

Then

$$v|_{s=a_x} = a^*(v), \qquad v|_{s=b_x} = b^*(v).$$

The case $N_x = [a_x, \infty)$ (an in a similar manner, the cases $(-\infty, b_x]$ and $(-\infty, \infty)$) can be reduced to the case of the interval by passing to a new integration variable. Such a pass is realized by constructing a diffeomorphism $\mu \colon N \to N'$ taking infinite integration intervals $N_x = [a_x, \infty)$ to finite semi-intervals $\mu(N_x) = [\bar{a}_x, b_x)$. Adding to the manifold N' the limit point $\{\, b_x \mid x \in M \,\}$ of the semi-intervals $\mu(N_x)$, we shall obtain the manifold $\bar{N}$ and the fiber bundle $\bar{\rho} \colon \bar{N} \to M$, whose restriction to the subset $N' \subset \bar{N}$ coincides with the mapping $\rho \circ \mu^{-1}$, while the fibers of the bundle $\bar{\rho}$ are the intervals $(\bar{\rho})^{-1}(x) = [\bar{a}_x, b_x]$, $x \in M$. Thus, we can restrict ourselves with the case, when the sets N_x, $x \in M$ are intervals (see also Remark 6.1 below).

Thus, system (6.1) can be written in the form

$$
\begin{aligned}
&G_j(x, u, p_\sigma, \ldots, b_k{}^*(v^k), \ldots) = 0, \qquad j = 1, \ldots, m_1, \\
&a_k{}^*(v^k) = 0, \qquad k = 1, \ldots, m_2, \\
&\frac{\partial v^k}{\partial s_k} = f_k(x, s_k, \ldots, (h_i^k){}^*(u), (h_i^k){}^*(p_\sigma), \ldots), \qquad k = 1, \ldots, m_2,
\end{aligned}
\tag{6.7}
$$

where $x = (x_1, \ldots, x_n)$ are coordinates on the manifold M, $u = (u^1, \ldots, u^m)$ is a section of the bundle π over M, $p_\sigma = (p_\sigma^1, \ldots, p_\sigma^m)$ is the vector of its derivatives, v^k is a function on the manifold N_k, b_k, a_k are the embeddings of M into N_k, h_i^k, $i = 1, \ldots, l_k$ are the projections of N_k to M, $k = 1, \ldots, m_2$. Note that the variables v^k, $k = 1, \ldots, m_2$, are uniquely determined by the variables u^j, $j = 1, \ldots, m$, by means of the second and the third equations in system (6.7). This means that systems (6.1) and (6.7) are equivalent: solutions of one of them are reconstructed by solutions of the other.

System (6.7) describes sections over different manifolds $M, N_1, \ldots, N_{m_2}$. When the manifolds $N_1, \ldots, N_{m_2}$ are diffeomorphic to each other, all functions and sections in system (6.7) can be represented as functions and sections over the same manifold. In fact, let N be a manifold, $F_k \colon N \to N_k$, $k = 1, \ldots, m_2$, be diffeomorphisms, $a_0 \colon M \to N$ be an embedding (e.g., $a_0 = F_1^{-1} \circ a_1$), $h_0 \colon N \to M$ be a smooth mapping left inverse to a_0, i.e., $h_0 \circ a_0 = \mathrm{id}_M$. Let us set

$$
\bar{u} = h_0{}^*(u), \quad \bar{v}^k = F_k^*(v^k), \qquad k = 1, \ldots, m_2.
$$

Then

$$
a_0{}^*(\bar{u}) = a_0{}^*(h_0{}^*(u)) = (h_0 \circ a_0){}^*(u) = u,
$$
$$
v^k = (F_k^{-1}){}^*(\bar{v}^k), \quad (h_i^k){}^*(u) = (h_i^k){}^*(a_0{}^*(\bar{u})) = (a_0 \circ h_i^k){}^*(\bar{u}),
$$
$$
b_k^*(v^k) = b_k^*((F_k^{-1}){}^*(\bar{v}^k)) = (F_k^{-1} \circ b_k){}^*(\bar{v}^k),
$$

etc. Using these equalities and acting by the homomorphism $h_0{}^*$ on the first and second equations (6.7), we shall rewrite these equations in the form

$$
\begin{aligned}
&G_j(h_0{}^*(x), \bar{u}, h_0^*(p_\sigma), \ldots, \bar{b}_k^*(\bar{v}^k), \ldots) = 0, \\
&\bar{a}_k^*(\bar{v}^k) = 0,
\end{aligned}
\tag{6.8}
$$

where $\bar{b}_k = F_k^{-1} \circ b_k \circ h_0$, $\bar{a}_k = F_k^{-1} \circ a_k \circ h_0$ are the mappings from N to N. Let us now act to the third equation in (6.7) by the diffeomorphism $F_k{}^*$. Using the equality

$$
F_k^*\left(\frac{\partial v^k}{\partial s_k}\right) = (F_k^{-1})_*\left(\frac{\partial}{\partial s_k}\right)(F_k{}^*(v^k)) = X_k(\bar{v}^k),
$$

where $X_k = (F_k^{-1})_*(\partial/\partial s_k)$ is a vector field on N, we obtain the equation

$$X_k(\bar{v}^k) = f_k(F_k^*(x), F_k^*(s_k), (\bar{h}_i^k)^*(\bar{u}), (\bar{h}_i^k)^*(h_0^*(p_\sigma))), \qquad (6.9)$$

where $\bar{h}_i^k = a_0 \circ h_i^k \circ F_k$ is a mapping from N to N.

Note that functions of the form $h_0^*(p_\sigma^r)$ can be represented as derivatives of the functions $\bar{u}^r$, if

$$h_{0,*}(Y_i) = \frac{\partial}{\partial x_i}, \quad Y_i \in D(N), \qquad i = 1, \ldots, n.$$

In this case

$$h_0^*\left(\frac{\partial u^r}{\partial x_i}\right) = Y_i(\bar{u}^r), \qquad h_0^*\left(\frac{\partial^2 u^r}{\partial x_i \partial x_l}\right) = Y_i(Y_l(\bar{u}^r)),$$

etc. Therefore, $h_0^*(p_\sigma^r)$ is a function in x, s, $\bar{u}$, and $\bar{p}_\sigma$. Besides, $\bar{u}$ is a section of the form $h_0^*(u)$, $u \in \Gamma(\pi)$. These and only these sections satisfy the equation

$$(a_0 \circ h_0)^*(\bar{u}) = \bar{u}. \qquad (6.10)$$

Indeed,

$$(a_0 \circ h_0)^*(\bar{u}) = h_0^*(a_0^*(\bar{u}))$$

and one can set $u = a_0^*(\bar{u})$. If $\bar{u} = h_0^*(u)$, then

$$(a_0 \circ h_0)^*(\bar{u}) = (h_0 \circ a_0 \circ h_0)^*(u) = h_0^*(u) = \bar{u},$$

since $h_0 \circ a_0 = \mathrm{id}_M$.

Thus, system (6.7) (and consequently, system (6.1)) is equivalent to system of equations (6.8)–(6.10), $j = 1, \ldots, m_1$, $k = 1, \ldots, m_2$. The functions $\bar{u}^j$, $\bar{v}^k$, entering these equations, are functions on N, while $\bar{b}_k$, $\bar{a}_k$, $\bar{h}_i^k$, $\bar{h}_0 = a_0 \circ h_0$ are mappings from N to N. Coming back to the previous notation, i.e., denoting by u the section $(\bar{u}^1, \ldots, \bar{u}^m, \bar{v}^1, \ldots, \bar{v}^{m_2})$ over N and by $x = (x_1, \ldots, x_{n+m_2})$ coordinates on N, this system can be written as a system of equations of the form

$$G(x, u, p_\sigma, g^*(u), g^*(p_\sigma)) = 0, \qquad (6.11)$$

where $g = g_1, \ldots, g_l$ are mappings from N to N. We shall call equations of the form (6.11) *boundary differential equations*. The reason for this name is that in examples, as a rule, the image of g_k, $k = 1, \ldots, m_2$, lies in the boundary of N, while the value of $g_k^*(u)$ is determined by the values of u on the boundary of N. The other term for these equations is *functional differential equations* (see, for example, [**41**]).

For prolongation of equations of the form (6.11), we can use both derivations and the action of the homomorphism g^*, where g is a mapping from N to N. It is easy to see that the new (prolonged) equation will contain sections of the form $g^*(g_i^*(u)) = (g_i \circ g)^*(u)$, where $g_1, \ldots, g_l$ are the mappings used in (6.11). Therefore, besides g, we must consider all compositions $g_i \circ g$, $i = 1, \ldots, l$. Taking for g the mappings participating in equations (6.11) and their prolongations only, we shall obtain the semigroup containing $g_1, \ldots, g_l$, the identical mapping $g_0 = \mathrm{id}_N$ and various compositions of the mappings $g_1, \ldots, g_l$. We shall call this semigroup the *semigroup of boundary differential system* (6.11) and denote it by $\mathcal{G}$.

EXAMPLE 6.4. Let a system $\mathcal{E}_0$ of integro-differential equations contain one integral of type (6.3) and one integral of type (6.4) with $a = 0$, $b = \infty$. Let us find the semigroup of mappings for the corresponding system $\mathcal{E}_1$ of boundary

$$h_1 : (t, x, s) \mapsto (t, s)$$
$$h_2 : (t, x, s) \mapsto (t, x - s)$$
$$a_1 : (t, x) \mapsto (t, x, 0)$$
$$a_2 : (t, x) \mapsto (t, x, 0)$$
$$b_1 : (t, x) \mapsto (t, x, x)$$
$$b_2 : (t, x) \mapsto (t, x, \infty)$$
$$h : (t, x, s) \mapsto (t, s)$$
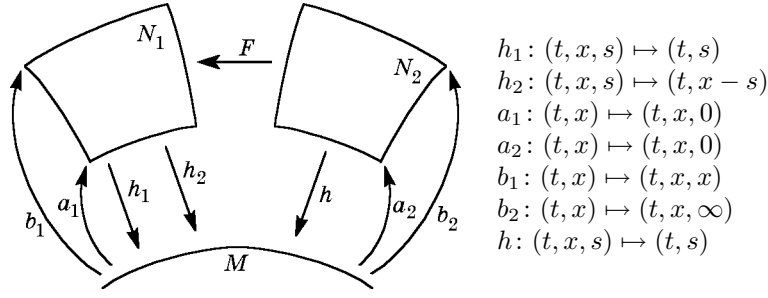
FIGURE 6.2

differential equations. The manifold of independent variables for the system $\mathcal{E}_0$ is $M = \{ (t, x) \mid x \geq 0 \}$ (see Examples 6.2 and 6.3). The manifold N_1 for the integral (6.3) is $N_1 = \{ (t, x, s) \mid x \geq 0, 0 \leq s \leq x \}$, while for the integral (6.4) it is $N_2 = \{ (t, x, s) \mid x \geq 0, s \geq 0 \}$ ($a = 0$, $b = \infty$). One can see the mappings defined on these manifolds in Figure 6.2.

REMARK 6.1. The above described scheme of transformation of an integro-differential equation to a system of boundary differential equations needs a reduction of an improper integral of the form (6.4) with $b = \infty$ to a definite integral with finite limits of integration. It is being done by construction of the diffeomorphism $\mu \colon N_2 \to N_2'$ and of the manifold $\bar{N}$ obtained from N_2' by adding some limit points (see p. 271). Such an approach makes solution essentially more difficult. Instead, we can assume that the sets (t, x, ∞), (t, ∞, s), (t, ∞, ∞), $x \geq 0$, $s \geq 0$ are added to the manifold N_2 and that a smooth structure on the set obtained is introduced by defining the ring of smooth functions on this set. Namely, a function f in the variables x, t, s is called *smooth*, if there exist the limits

$$f(t, x, \infty) = \lim_{s \to \infty} f(t, x, s),$$
$$f(t, \infty, s) = \lim_{x \to \infty} f(t, x, s),$$
$$f(t, \infty, \infty) = \lim_{\substack{x \to \infty \\ s \to \infty}} f(t, x, s),$$

and they are equal to the values of the function f at the added point. Thus obtained manifold is diffeomorphic to $\bar{N}_2$ and will be also denoted by N_2. We introduce similar smooth structures on M and N_1. Such an approach allows to define uniquely the mappings b_2, g_3 (see below) and various compositions containing g_3.

The manifolds N_1 and N_2 are diffeomorphic and $F \colon N_2 \to N_1$, $F(t, x, s) = (t, x + s, s)$ is their diffeomorphism. Let us choose for the manifold of independent variables of the desired system of boundary differential equations the manifold $N = N_2$ and take for the embedding the mapping $a_0 = F^{-1} \circ a_1 \colon M \to N$. Then the mapping $h_0 = h_2 \circ F \colon N \to M$ can be taken for the left inverse mapping. Consequently, the mappings

$$g_1 = \bar{a}_1 = \bar{h}_2 = F^{-1} \circ a_1 \circ h_2 \circ F = \bar{a}_2 = a_2 \circ h_2 \circ F, \quad g_1 \colon (t, x, s) \longmapsto (t, x, 0);$$
$$g_2 = \bar{b}_1 = F^{-1} \circ b_1 \circ h_2 \circ F, \qquad\qquad g_2 \colon (t, x, s) \longmapsto (t, 0, x);$$
$$g_3 = \bar{b}_2 = b_2 \circ h_2 \circ F, \qquad\qquad g_3 \colon (t, x, s) \longmapsto (t, x, \infty);$$
$$g_4 = \bar{h}_1 = F^{-1} \circ a_1 \circ h_1 \circ F = \bar{h} = F^{-1} \circ a_1 \circ h, \quad g_4 \colon (t, x, s) \longmapsto (t, s, 0)$$

generate a semigroup $\mathcal{G}$ consisting of eleven elements $g_0 = \mathrm{id}_M$, g_1, g_2, g_3, g_4, g_2^2, $g_3 \circ g_2$, $g_2 \circ g_4$, $g_3 \circ g_4$, $g_4 \circ g_3$, $g_3 \circ g_4 \circ g_3$.

EXAMPLE 6.5. Let us find the system of boundary differential equations for the *Smoluchowski coagulation equation* [**111**, **144**]:

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2} \int_0^x K(x-s,s)u(t,x-s)u(t,s)\,ds - u(t,x)\int_0^\infty K(x,s)u(t,s)\,ds,$$
(6.12)

where $K(x,s)$ is a fixed function satisfying the condition $K(s,x) = K(x,s)$ for all $x \geq 0$ and $s \geq 0$. Let us apply the above general procedure to this particular equation. Equation (6.12) contains one integral of type (6.3) and one integral of type (6.4). The manifold N and the semigroup are described in Example 6.4. For any solution $u(t,x)$ of equation (6.12), let us introduce the function $v^1(t,x,s)$ on N_1 and the function $v^2(t,x,s)$ on N_2:

$$\frac{\partial v^1}{\partial s} = K(x-s,s)h_1{}^*(u)h_2{}^*(u),$$
(6.13)

$$a_1{}^*(v^1) = 0,$$
(6.14)

$$\frac{\partial v^2}{\partial s} = K(x,s)h^*(u),$$

$$a_2{}^*(v^2) = 0$$
(6.15)

(the mappings h, h_1, h_2, a_1, b_1, a_2, b_2, F are defined in Example 6.4). Then the coagulation equation is written down in the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}b_1{}^*(v^1) - ub_2{}^*(v^2).$$
(6.16)

Let us introduce the notation $u^1 = (h_2 \circ F)^*(u)$, $u^2 = F^*(v^1)$, $u^3 = v^2$. Then, acting to equations (6.14), (6.15) and (6.16) by the homomorphism $(h_2 \circ F)^*$ and to equation (6.13) by isomorphism F^* and using the equality $(F^{-1})_*(\partial/\partial s) = \partial/\partial s - \partial/\partial x$, we obtain the system of boundary differential equations

$$\frac{\partial u^1}{\partial t} = \frac{1}{2}g_2{}^*(u^2) - u^1 g_3{}^*(u^3),$$

$$\frac{\partial u^2}{\partial s} - \frac{\partial u^2}{\partial x} = K g_4^*(u^1)u^1,$$
(6.17)

$$\frac{\partial u^3}{\partial s} = K g_4{}^*(u^1),$$

$$g_1{}^*(u^1) = u^1, \quad g_1^*(u^2) = 0, \quad g_1^*(u^3) = 0,$$

equivalent to equation (6.12).

Let us give geometric interpretation to boundary differential equations using a parallel with differential equations.

6.2. Spaces of $(k,\mathcal{G})$-jets. Let us generalize the concept of jet spaces in such a way that boundary differential equations could be interpreted as submanifolds of these spaces. Let $\pi\colon E \to M$ be a smooth locally trivial bundle over the manifold M with boundary, $k = 1, 2, \ldots, \infty$, $\pi_k\colon J^k(\pi) \to M$ be the corresponding bundle of k-jets, and $\mathcal{G}$ be a finite set of smooth mappings from M to M containing the identity mapping id_M. Let us denote by $\pi_k^{\mathcal{G}}$ the Whitney product of the pullbacks
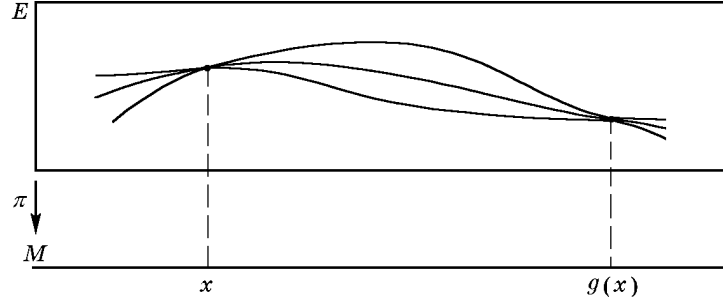
FIGURE 6.3

$g^*(\pi_k)$: $\pi_k^{\mathcal{G}} = \bigoplus_{g \in \mathcal{G}} g^*(\pi_k)$. We denote by $J^k(\pi; \mathcal{G})$ the total space of the bundle $\pi_k^{\mathcal{G}}$. The set $J^k(\pi; \mathcal{G})$ is a finite-dimensional smooth manifold, if k is finite, and an infinite-dimensional one, if k is infinite.

Every point of $J^k(\pi; \mathcal{G})$ over $x \in M$ is a set of k-jets $\theta_k^g \in J^k(\pi)$, $g \in \mathcal{G}$, satisfying the condition $\pi_k(\theta_k^g) = g(x)$. For any $g \in \mathcal{G}$ there exists a smooth section s_g of the bundle π such that $[s_g]_{g(x)}^k = \theta_k^g$. The set of k-jets $\{[s_g]_{g(x)}^k\}_{g \in \mathcal{G}}$ will be called the $(k, \mathcal{G})$-*jet of the family of sections* $\{s_g\}$ at the point x. Any family of sections $\{s_g\}_{g \in \mathcal{G}}$ determines the section $j_k(\{s_g\})$ of the bundle $\pi_k^{\mathcal{G}}$:

$$j_k(\{s_g\})(x) = \{[s_g]_{g(x)}^k\}_{g \in \mathcal{G}}. \qquad (6.18)$$

If all sections s_g, $g \in \mathcal{G}$, coincide with the section s, then the corresponding set of jets will be called $(k, \mathcal{G})$-*jet of the section* s at the point x and will be denoted by $[s]_x^{(k, \mathcal{G})}$. The section $x \mapsto [s]_x^{(k, \mathcal{G})}$ will be denoted by $j_k(s)$ and the subset of $(k, \mathcal{G})$-jets of sections of the bundle π will be denoted by $J^k(\pi; \mathcal{G})_0$. Geometrically, the $(k, \mathcal{G})$-jet of a section s is interpreted as the class of sections of the bundle π tangent to the section s with order $\geq k$ at all points $g(x)$, $g \in \mathcal{G}$ (see Figure 6.3). We shall call $J^k(\pi; \mathcal{G})$ the *manifold (space) of* $(k, \mathcal{G})$-*jets*; the bundle $\pi_k^{\mathcal{G}}$ will be called the *bundle of* $(k, \mathcal{G})$-*jets*.

Let us describe the set $J^k(\pi; \mathcal{G})_0$. If a point $x \in M$ is such that all the points $g(x)$, $g \in \mathcal{G}$, are different, then for any set of sections $\{s_g\}_{g \in \mathcal{G}}$ there exists a smooth section s tangent to the section s_g at the point $g(x)$ with order $\geq k$ for any $g \in \mathcal{G}$. Consequently, the fiber $(\pi_k^{\mathcal{G}})^{-1}(x)$ lies in $J^k(\pi; \mathcal{G})_0$. If for a point $x \in M$ two mappings $g_1, g_2 \in \mathcal{G}$, $g_1 \neq g_2$, exist, such that $g_1(x) = g_2(x)$, then $(k, \mathcal{G})$-jet of the family of sections $\{s_g\}_{g \in \mathcal{G}}$ is the $(k, \mathcal{G})$-jet of a section if and only if

$$[s_{g_1}]_{g_1(x)}^k = [s_{g_2}]_{g_2(x)}^k.$$

A submanifold $\mathcal{E}$ of the manifold $J^k(\pi; \mathcal{G})$ will be called a *boundary differential equation* of order k in the bundle π with respect to the set of mappings $\mathcal{G}$ (or simply, an *equation*). A *solution* of the equation $\mathcal{E} \subset J^k(\pi; \mathcal{G})$ is a section $s \in \Gamma(\pi)$ such that $j_k(s)(M) \subset \mathcal{E}$.

REMARK 6.2. Submanifolds of the form $j_k(s)(M)$ are contained in $J^k(\pi; \mathcal{G})_0$. It might seem that in the boundary differential case the set $J^k(\pi; \mathcal{G})_0$ must be taken for an analog of the space $J^k(\pi)$. But since in the sequel we shall use the language of differential geometry and the set $J^k(\pi; \mathcal{G})_0$ is not a manifold, we extended $J^k(\pi; \mathcal{G})_0$ to the manifold $J^k(\pi; \mathcal{G})$ and take the latter for an analog of $J^k(\pi)$.

Let us introduce the following canonical coordinate system on the manifold $J^k(\pi; \mathcal{G})$. Let $x_1, \ldots, x_n$ be coordinates in a neighborhood $\mathcal{U} \subset M$, $u^1, \ldots, u^m$ be coordinates in the fiber of the bundle $\pi|_{\mathcal{U}}$, x_i, p_σ^j, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $|\sigma| \leq k$, be the corresponding coordinates in $J^k(\pi)$. Then, denoting by $p_{\sigma g}^j$ the corresponding coordinates in the fiber of the bundle $g^*(\pi_k)$, we obtain the coordinate system

$$x_i, \ p_{\sigma g}^j, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad |\sigma| \leq k, \quad g \in \mathcal{G}, \tag{6.19}$$

on $J^k(\pi; \mathcal{G})$. The p-coordinates of the $(k, \mathcal{G})$-jet of the family of sections $\{s_g\}_{g \in \mathcal{G}}$ at the point x are equal to

$$p_{\sigma g}^j = \frac{\partial^{|\sigma|} s_g^j}{\partial x^\sigma}(g(x)),$$

where $s_g^1, \ldots, s_g^m$ are the components of the section s_g. We have the same coordinate system (6.19) on the manifold $J^\infty(\pi; \mathcal{G})$, but with no restriction imposed on $|\sigma|$.

EXERCISE 6.1. Describe the set $J^\infty(\pi; \mathcal{G})_0$ in canonical coordinates.

As in the case of "usual" jets, for $k > l$ and for $k = \infty$ the bundle

$$\pi_{k,l}^{\mathcal{G}} \colon J^k(\pi; \mathcal{G}) \longrightarrow J^l(\pi; \mathcal{G}), \tag{6.20}$$

is defined. Namely,

$$\pi_{k,l}^{\mathcal{G}}(\{\theta_k^g\}_{g \in \mathcal{G}}) = \{\pi_{k,l}(\theta_k^g)\}_{g \in \mathcal{G}}. \tag{6.21}$$

The following equality fulfills:

$$\pi_{k,l}^{\mathcal{G}} \circ j_k(\{s_g\}) = j_l(\{s_g\}), \tag{6.22}$$

where $\{s_g\}_{g \in \mathcal{G}}$ is an arbitrary family of sections of the bundle π.

EXERCISE 6.2. Show that the space on infinite jets $J^\infty(\pi; \mathcal{G})$ is the inverse limit of the tower of finite jets generated by the mapping $\pi_{l+1,l}^{\mathcal{G}}$, $l \geq 0$.

If $\mathcal{G}_1$ is a subset in $\mathcal{G}$, then the formula

$$\pi_k^{\mathcal{G}, \mathcal{G}_1}(\{\theta_k^g\}_{g \in \mathcal{G}}) = \{\theta_k^g\}_{g \in \mathcal{G}_1} \tag{6.23}$$

determines the bundle

$$\pi_k^{\mathcal{G}, \mathcal{G}_1} \colon J^k(\pi; \mathcal{G}) \longrightarrow J^k(\pi; \mathcal{G}_1), \tag{6.24}$$

and the following equalities hold:

$$\pi_k^{\mathcal{G}_1} \circ \pi_k^{\mathcal{G}, \mathcal{G}_1} = \pi_k^{\mathcal{G}}, \tag{6.25}$$

$$\pi_{k,l}^{\mathcal{G}_1} \circ \pi_k^{\mathcal{G}, \mathcal{G}_1} = \pi_l^{\mathcal{G}, \mathcal{G}_1} \circ \pi_{k,l}^{\mathcal{G}}, \tag{6.26}$$

$$\pi_k^{\mathcal{G}, \mathcal{G}_1} \circ j_k(\{s_g\}) = j_k(\{s_g\}_{g \in \mathcal{G}_1}), \tag{6.27}$$

where $\{s_g\}$ is an arbitrary family of sections of the bundle π, $g \in \mathcal{G}$, and $\{s_g\}_{g \in \mathcal{G}_1}$ denotes the sections s_g such that $g \in \mathcal{G}_1$.

REMARK 6.3. If the set $\mathcal{G}$ consists of the identical mapping only, then the space $J^k(\pi; \{\mathrm{id}_M\})$ coincides with the "usual" jet space $J^k(\pi)$ defined in Ch. 3 and 4. If the set $\mathcal{G}$ contains other mappings, besides the identical one, then $J^0(\pi; \mathcal{G}) \neq E = J^0(\pi)$. Note that for any k the bundle

$$\pi_0^{\mathcal{G}, \{\mathrm{id}_M\}} \circ \pi_{k,0}^{\mathcal{G}} \colon J^k(\pi; \mathcal{G}) \longrightarrow J^0(\pi; \mathcal{G}) \longrightarrow E.$$

is defined.

In the case when the set $\mathcal{G}$ is infinite, the manifold of $(k, \mathcal{G})$-jets is defined in the following way. Let $\{\mathcal{G}_\alpha\}$ be all finite subsets of the set $\mathcal{G}$ containing id_M. Then for any pair $\mathcal{G}_\alpha \subset \mathcal{G}_\beta$ the mappings $\pi_k^{\mathcal{G}_\beta, \mathcal{G}_\alpha} \colon J^k(\pi; \mathcal{G}_\beta) \to J^k(\pi; \mathcal{G}_\alpha)$ are defined and $\pi_k^{\mathcal{G}_\beta, \mathcal{G}_\alpha} \circ \pi_k^{\mathcal{G}_\gamma, \mathcal{G}_\beta} = \pi_k^{\mathcal{G}_\gamma, \mathcal{G}_\alpha}$, if $\mathcal{G}_\alpha \subset \mathcal{G}_\beta \subset \mathcal{G}_\gamma$. Let us define $J^k(\pi; \mathcal{G})$ as the inverse limit of the manifolds $J^k(\pi; \mathcal{G}_\alpha)$ over the system of mappings $\{\pi_k^{\mathcal{G}_\beta, \mathcal{G}_\alpha}\}_{\mathcal{G}_\alpha \subset \mathcal{G}_\beta}$. In other words, a point θ_k of the set $J^k(\pi; \mathcal{G})$ is a family of points $\theta_k^{\mathcal{G}_\alpha} \in J^k(\pi; \mathcal{G}_\alpha)$ such that (a) $\mathcal{G}_\alpha$ is an arbitrary finite subset of $\mathcal{G}$ and (b) if $\mathcal{G}_\alpha \subset \mathcal{G}_\beta$, then

$$\pi_k^{\mathcal{G}_\beta, \mathcal{G}_\alpha}(\theta_k^{\mathcal{G}_\beta}) = \theta_k^{\mathcal{G}_\alpha}. \tag{6.28}$$

In this case, the mapping $\pi_k^{\mathcal{G}} \colon J^k(\pi; \mathcal{G}) \to M$ is defined by the formula $\pi_k^{\mathcal{G}}(\theta_k) = \pi_k^{\mathcal{G}_\gamma}(\theta_k^{\mathcal{G}_\gamma})$, where $\theta_k^{\mathcal{G}_\gamma}$ is an arbitrary element of the set $\theta_k = \{\theta_k^{\mathcal{G}_\alpha}\}_{\mathcal{G}_\alpha \subset \mathcal{G}}$. From formulas (6.25) and (6.28) it follows that the point $\pi_k^{\mathcal{G}_\gamma}(\theta_k^{\mathcal{G}_\gamma}) \in M$ is independent of the choice of elements $\theta_k^{\mathcal{G}_\gamma}$ from the set θ_k.

The points $\theta_k^{\mathcal{G}_\alpha} \in J^k(\pi; \mathcal{G}_\alpha)$, in the case of finite set $\mathcal{G}_\alpha$, are themselves sets of k-jets $\{\theta_k^g\}_{g \in \mathcal{G}_\alpha}$ while condition (6.28) means only that the set $\{\theta_k^g\}_{g \in \mathcal{G}_\alpha}$ is a part of the set $\{\theta_k^g\}_{g \in \mathcal{G}_\beta}$ provided $\mathcal{G}_\alpha \subset \mathcal{G}_\beta$. Therefore a point of $J^k(\pi; \mathcal{G})$, in the case of infinite set $\mathcal{G}$, may be understood as an infinite set of k-jets $\{\theta_k^g\}_{g \in \mathcal{G}}$, or more exactly, as $\{[s_g]_{g(x)}^k\}_{g \in \mathcal{G}}$. Formulas (6.21), (6.23), and (6.18) define the maps (6.20), (6.24), $j_k(\{s_g\}) \colon M \to J^k(\pi; \mathcal{G})$ in the case of infinite $\mathcal{G}$ as well, and as in the finite case, equalities (6.22), (6.25), (6.26), (6.27) hold, as well as the equality $\pi_k^{\mathcal{G}} \circ j_k(\{s_g\}) = \mathrm{id}_M$, etc. (see Ch. 4).

Similar to the case $J^\infty(\pi)$, the set $J^k(\pi; \mathcal{G})$, when $\mathcal{G}$ is infinite and k is either finite or infinite, is endowed with a structure of infinite-dimensional manifold. Coordinates in $J^k(\pi; \mathcal{G})$ are given by the set of functions (6.19) with infinite set $\mathcal{G}$. The concepts of smooth functions, smooth mappings, tangent vectors, vector fields, differential forms, etc., are introduced on $J^k(\pi; \mathcal{G})$ in the same way as in the case of $J^\infty(\pi)$ (see Ch. 4).

Let us denote by $\mathcal{F}(\pi; \mathcal{G})$ and by $\mathcal{F}_k(\pi; \mathcal{G})$ the algebras of smooth functions on $J^\infty(\pi; \mathcal{G})$ and on $J^k(\pi; \mathcal{G})$ respectively and denote by $\Lambda^*(\pi; \mathcal{G})$ the module of differential forms on $J^\infty(\pi; \mathcal{G})$. The monomorphisms $(\pi_{\infty,k}^{\mathcal{G}})^*$ and $(\pi_\infty^{\mathcal{G},\mathcal{G}'})^*$, where $\mathcal{G}' \subset \mathcal{G}$, embed the algebras $\mathcal{F}_k(\pi; \mathcal{G})$ and $\mathcal{F}(\pi; \mathcal{G}')$ to the algebra $\mathcal{F}(\pi; \mathcal{G})$. The images of these monomorphisms will be also denoted by $\mathcal{F}_k(\pi; \mathcal{G})$ and $\mathcal{F}(\pi; \mathcal{G}')$. In particular, $\mathcal{F}(\pi) = \mathcal{F}(\pi; \{\mathrm{id}_M\})$ is a subalgebra of the algebra $\mathcal{F}(\pi; \mathcal{G})$. The above introduced embeddings allow us to define two filtrations in the algebra $\mathcal{F}(\pi; \mathcal{G})$ (as well as in the module $\Lambda^*(\pi; \mathcal{G})$): filtration with respect to k, similar to filtration in $\mathcal{F}(\pi)$ (see Ch. 4) and filtration corresponding to the embeddings of finite subsets $\mathcal{G}_\alpha$ of the set $\mathcal{G}$. The second filtration puts into correspondence to a function φ from $\mathcal{F}(\pi; \mathcal{G})$ the finite subset $\mathcal{G}_\alpha$ of the set $\mathcal{G}$, if φ is a function on $J^k(\pi; \mathcal{G}_\alpha)$. The first filtration possesses the same properties as filtration in the algebra $\mathcal{F}(\pi)$ (see Ch. 4). For the second filtration the properties 1.3 should be rewritten as follows: $\deg \varphi$ is to be understood as the corresponding subset $\mathcal{G}_\alpha \subset \mathcal{G}$ while the inequality $\leq$ must be changed by the embedding $\subseteq$.

To simplify the subsequent exposition, we shall not use the notion of filtration, though this exposition is completely in parallel with that of Ch. 4. For example, a smooth mapping G from $J^\infty(\pi; \mathcal{G})$ to $J^\infty(\xi; \mathcal{G}')$, where $\xi \colon Q \to M$ is a vector bundle and $\mathcal{G}'$ is another set of mappings from M to M, is a mapping of sets such that for any function $\varphi \in \mathcal{F}(\xi; \mathcal{G}')$ the element $G^*(\varphi) = \varphi \circ G$ is a smooth function

on the space $J^\infty(\pi; \mathcal{G})$ an for any number k and any finite subset of the set $\mathcal{G}_\alpha \subset \mathcal{G}'$ there exist a number l and a finite subset $\mathcal{G}_\beta \subset \mathcal{G}$ such that

$$G^*(\mathcal{F}_k(\xi; \mathcal{G}_\alpha)) \subset \mathcal{F}_l(\pi; \mathcal{G}_\beta).$$

A tangent vector X_θ to the manifold $J^\infty(\pi; \mathcal{G})$ at a point θ is defined as the set $\{X_x, X_{\theta_k^{\mathcal{G}_\alpha}}\}$ of tangent vectors to finite-dimensional manifolds M and $J^k(\pi; \mathcal{G}_\alpha)$ at points $x = \pi_\infty^{\mathcal{G}}(\theta)$ and $\theta_k^{\mathcal{G}_\alpha} = \pi_k^{\mathcal{G}, \mathcal{G}_\alpha}(\pi_{\infty, k}^{\mathcal{G}}(\theta))$ respectively, such that

$$(\pi_{k+1, k}^{\mathcal{G}_\alpha})_*(X_{\theta_{k+1}^{\mathcal{G}_\alpha}}) = X_{\theta_k^{\mathcal{G}_\alpha}}, \quad (\pi_k^{\mathcal{G}_\beta, \mathcal{G}_\alpha})_*(X_{\theta_k^{\mathcal{G}_\beta}}) = X_{\theta_k^{\mathcal{G}_\alpha}}, \quad (\pi_k^{\mathcal{G}_\alpha})_*(X_{\theta_k^{\mathcal{G}_\alpha}}) = X_x.$$

A vector field on the manifold $J^\infty(\pi; \mathcal{G})$ is, by definition, a derivation X of the algebra $\mathcal{F}(\pi; \mathcal{G})$ such that for any number k and any finite subset $\mathcal{G}_\alpha \subset \mathcal{G}$ there exist a number l and a finite subset $\mathcal{G}_\beta \subset \mathcal{G}$ such that

$$X(\mathcal{F}_k(\pi; \mathcal{G}_\alpha)) \subset \mathcal{F}_l(\pi; \mathcal{G}_\beta).$$

The set of vector fields on $J^\infty(\pi; \mathcal{G})$ will be denoted by $\mathrm{D}(\pi; \mathcal{G})$. The *inner product* of a vector field X on $J^\infty(\pi; \mathcal{G})$ with a form $\omega \in \Lambda^i(\pi; \mathcal{G})$ gives us the form $X \lrcorner \omega \in \Lambda^{i-1}(\pi; \mathcal{G})$, whose value at a point $\theta = \{x, \theta_k^{\mathcal{G}_\alpha}\}$ equals $X_{\theta_l^{\mathcal{G}_\beta}} \lrcorner \omega_{\theta_l^{\mathcal{G}_\beta}}$, if ω is a form on $J^l(\pi; \mathcal{G}_\beta)$.

EXERCISE 6.3. Define the de Rham differential $d\omega$ and the Lie derivative $X(\omega)$ in the case $\omega \in \Lambda^*(\pi; \mathcal{G}_1)$, $X \in D(\pi; \mathcal{G}_2)$, $\mathcal{G}_1 \neq \mathcal{G}_2$.

6.3. Boundary differential operators. Let π and π' be smooth vector locally trivial bundles over the manifold M, $\mathcal{G}$ be a finite set of smooth mappings from M to M containing the identical mapping id_M. A mapping $\Delta \colon \Gamma(\pi) \to \Gamma(\pi')$ is called a *boundary differential operator* of order k with respect to the set $\mathcal{G}$, if for any section $s \in \Gamma(\pi)$ the value of the section $\Delta(s)$ at an arbitrary point $x \in M$ is determined by the k-jets of the section s at the points $g(x)$, $g \in \mathcal{G}$. Similar to the purely differential case (see Ch. 4), any section φ of the bundle $(\pi_k^{\mathcal{G}})^*(\pi')$ determines a boundary differential operator $\Delta_\varphi \colon \Gamma(\pi) \to \Gamma(\pi')$ by the formula

$$\Delta_\varphi(s) = \varphi \circ j_k(s). \tag{6.29}$$

In coordinates $x_1, \ldots, x_n, \ldots, p_{\sigma g}^j, \ldots$ on $J^k(\pi; \mathcal{G})$ this correspondence looks as follows. If $\varphi = \{\varphi^i(x_1, \ldots, x_n, \ldots, p_{\sigma g}^j, \ldots)\}$ and $s = \{s^j(x_1, \ldots, x_n)\}$, then

$$\Delta_\varphi(s) = (\varphi^1(\ldots, x_i, \ldots, g^*(\frac{\partial^{|\sigma|} s^j}{\partial x^\sigma}), \ldots), \ldots, \varphi^{m'}(\ldots, x_i, \ldots, g^*(\frac{\partial^{|\sigma|} s^j}{\partial x^\sigma}), \ldots)),$$

where $m' = \dim \pi'$.

A particular feature of the boundary differential case is that there exist boundary differential operators of order k such that no section of the bundle $(\pi_k^{\mathcal{G}})^*(\pi')$ correspond to these operators. The reason is that using formula (6.29) and having the function Δ_φ, we can reconstruct the values of the section φ at the $(k, \mathcal{G})$-jets of sections of the bundle π only, i.e., at points of the subset $J^k(\pi; \mathcal{G})_0$. Therefore, formula (6.29) determines the boundary differential operator Δ_φ not only in the case when φ is a section the bundle $(\pi_k^{\mathcal{G}})^*(\pi')$, but also in the case when φ is a section of the restriction of the bundle $(\pi_k^{\mathcal{G}})^*(\pi')$ to the subset $J^k(\pi; \mathcal{G})_0$.

EXAMPLE 6.6. Let $M = \mathbb{R}$ and $\pi = \pi' \colon \mathbb{R}^2 \to \mathbb{R}$ be a trivial bundle. Let the set $\mathcal{G}$ consist of two mappings: the identical mapping $g_0 = \mathrm{id}_\mathbb{R}$ and the projection g of the manifold $\mathbb{R}$ to the point $0 \in \mathbb{R}$. We have the coordinate system x, p_{0g_0}, p_{0g}

on $J^0(\pi; \mathcal{G})$. In this coordinate system, points of the set $J^0(\pi; \mathcal{G}) \smallsetminus J^0(\pi; \mathcal{G})_0$ form the plane $\{x = 0\}$ without the line $\{x = 0, p_{0g_0} = p_{0g}\}$. The boundary differential operator

$$\Delta \colon s \longmapsto \begin{cases} \exp\left(-\dfrac{1}{x^2 + (s(x) - s(0))^2}\right), & x > 0 \\ 0, & x \le 0, \end{cases}$$

where $s \in \Gamma(\pi)$, determines a smooth function on $J^0(\pi; \mathcal{G})_0$, which cannot be extended to entire $J^0(\pi; \mathcal{G})$.

EXERCISE 6.4. Prove this fact.

In what follows, only operators of the form Δ_φ, where φ is a section of the bundle $(\pi_k^\mathcal{G})^*(\pi')$, will be called *boundary differential operators*. Not only boundary differential operators satisfy this condition, but rather "exotic" operators also (see Example 6.6) and thus this restriction is not too essential. We shall denote by $\mathcal{F}_k(\pi, \pi'; \mathcal{G})$ the set of sections of the bundle $(\pi_k^\mathcal{G})^*(\pi')$. Similar to the differential case, for any k there exist the embeddings

$$\mathcal{F}_k(\pi, \pi'; \mathcal{G}) \subset \mathcal{F}_{k+1}(\pi, \pi'; \mathcal{G}).$$

Therefore, we can set

$$\mathcal{F}(\pi, \pi'; \mathcal{G}) = \bigcup_{k=0}^\infty \mathcal{F}_k(\pi, \pi'; \mathcal{G}).$$

Let us show now that the composition of two boundary differential operators of the form Δ_φ is again a boundary differential operator of the same form. First, we prove this fact for the composition $j_{k'}^{\mathcal{G}'} \circ j_k^\mathcal{G}$, where $j_k^\mathcal{G}$ denotes the operator taking an arbitrary section s of the bundle π to the section $j_k(s)$ of the bundle $\pi_k^\mathcal{G}$. In the composition $j_{k'}^{\mathcal{G}'} \circ j_k^\mathcal{G}$, the operator $j_{k'}^{\mathcal{G}'}$ takes sections of the bundle $\pi_k^\mathcal{G}$ to sections of the bundle $(\pi_k^\mathcal{G})_{k'}^{\mathcal{G}'}$. Denote by $\mathcal{G} \circ \mathcal{G}'$ the set of mappings of the manifold M of the form $g \circ g'$, $g \in \mathcal{G}$, $g' \in \mathcal{G}'$. Let us construct the mapping

$$\Phi_{k,k'}^{\mathcal{G},\mathcal{G}'} \colon J^{k+k'}(\pi; \mathcal{G} \circ \mathcal{G}') \longrightarrow J^{k'}(\pi_k^\mathcal{G}; \mathcal{G}')$$

by setting

$$\Phi_{k,k'}^{\mathcal{G},\mathcal{G}'}(\{[s_g]_{g(x)}^{k+k'}\}_{g \in \mathcal{G} \circ \mathcal{G}'}) = \{[j_k(\{s_g\}_{g \in \mathcal{G}})]_{g'(x)}^{k'}\}_{g' \in \mathcal{G}'}.$$

Then for any section $s \in \Gamma(\pi)$ one has

$$\Phi_{k,k'}^{\mathcal{G},\mathcal{G}'} \circ j_{k+k'}^{\mathcal{G} \circ \mathcal{G}'}(s) = (j_{k'}^{\mathcal{G}'} \circ j_k^\mathcal{G})(s), \tag{6.30}$$

i.e., $j_{k'}^{\mathcal{G}'} \circ j_k^\mathcal{G} = \Delta_\varphi$, where φ is a section of the bundle $(\pi_{k+k'}^{\mathcal{G} \circ \mathcal{G}'})^*((\pi_k^\mathcal{G})_{k'}^{\mathcal{G}'})$ determined by the mapping $\Phi_{k,k'}^{\mathcal{G},\mathcal{G}'}$. Therefore, the composition $j_{k'}^{\mathcal{G}'} \circ j_k^\mathcal{G}$ is a boundary differential operator of order $k + k'$ with respect to the set $\mathcal{G} \circ \mathcal{G}'$.

For any morphism Φ of the bundles π and π' let us define its lifting

$$\Phi^{(k)} \colon J^k(\pi; \mathcal{G}) \to J^k(\pi'; \mathcal{G})$$

by the formula

$$\Phi^{(k)}(\{[s_g]_{g(x)}^k\}_{g \in \mathcal{G}}) = \{[\Phi \circ s_g]_{g(x)}^k\}_{g \in \mathcal{G}}.$$

Then, obviously, for any section $s \in \Gamma(\pi)$ the equality

$$\Phi^{(k)} \circ j_k(s) = j_k(\Phi \circ s) \tag{6.31}$$

holds.

Let now $\Delta_\varphi \colon \Gamma(\pi) \to \Gamma(\pi')$ and $\Delta_{\varphi'} \colon \Gamma(\pi') \to \Gamma(\pi'')$ be two boundary differential operators of orders k and k' with respect to the sets of mappings $\mathcal{G}$ and $\mathcal{G}'$ respectively. Then using consecutively equalities (6.29), (6.31), and (6.30), we obtain

$$\Delta_{\varphi'}(\Delta_\varphi(s)) = \Delta_{\varphi'}(\varphi \circ j_k^{\mathcal{G}}(s)) = \varphi' \circ j_{k'}^{\mathcal{G}'}(\varphi \circ j_k^{\mathcal{G}}(s))$$
$$= \varphi' \circ \varphi^{(k)} \circ j_{k'}^{\mathcal{G}'}(j_k^{\mathcal{G}}(s)) = \varphi' \circ \varphi^{(k)} \circ \Phi_{k,k'}^{\mathcal{G},\mathcal{G}'} \circ j_{k+k'}^{\mathcal{G}\circ\mathcal{G}'}(s).$$

Therefore, the mapping

$$\Phi = \varphi' \circ \varphi^{(k)} \circ \Phi_{k,k'}^{\mathcal{G},\mathcal{G}'} \colon J^{k+k'}(\pi; \mathcal{G} \circ \mathcal{G}') \longrightarrow E'',$$

where E'' is the space of the bundle π'', is the section of the bundle $(\pi_{k+k'}^{\mathcal{G}\circ\mathcal{G}'})^*(\pi'')$ determining the operator $\Delta_{\varphi'} \circ \Delta_\varphi$. In other words, the composition $\Delta_{\varphi'} \circ \Delta_\varphi = \Delta_\Phi$ is a boundary differential operator of order $k+k'$ with respect to the set of mappings $\mathcal{G} \circ \mathcal{G}'$.

If $\Delta \colon \Gamma(\pi) \to \Gamma(\pi')$ is a boundary differential operator of order k with respect to the set of mappings $\mathcal{G}$ and $\mathcal{G}'$ is another set of mappings of the manifold M, then the composition

$$\Delta_l^{\mathcal{G}'} = j_l^{\mathcal{G}'} \circ \Delta \colon \Gamma(\pi) \to \Gamma(\pi'^{\mathcal{G}'}_l)$$

is a again boundary differential operator of order $k + l$ with respect to the set of mappings $\mathcal{G} \circ \mathcal{G}'$. The operator $\Delta_l^{\mathcal{G}'}$ will be called the $(l, \mathcal{G}')$-*prolongation* of the boundary differential operator Δ.

EXERCISE 6.5. Let $\mathcal{G}$ and $\mathcal{G}'$ be sets of mappings of the manifold M containing the identical mapping and $\mathcal{G}$ be finite. Let us show that the set of mappings

$$\{\varphi_{\Delta_l^{\mathcal{G}'}} \colon J^{k+l}(\pi; \mathcal{G} \circ \mathcal{G}') \longrightarrow J^l(\pi'; \mathcal{G}')\}_{l \geq 0}$$

defines a smooth mapping of $J^\infty(\pi; \mathcal{G} \circ \mathcal{G}')$ to $J^\infty(\pi'; \mathcal{G}')$.

Let now $\nabla \colon \Gamma(\xi) \to \Gamma(\xi')$ be a linear boundary differential operator with respect to the set of mappings $\mathcal{G}'$, π be a bundle over the same base M, $\mathcal{G}$ be a set of mappings of M. Similar to the differential case, let us define the *lifting*

$$\widehat{\nabla} \colon \mathcal{F}(\pi, \xi; \mathcal{G}) \to \mathcal{F}(\pi, \xi'; \mathcal{G} \circ \mathcal{G}')$$

of the operator ∇. Let θ be a point of the space $J^\infty(\pi; \mathcal{G})$, $\{s_g\}_{g \in \mathcal{G}}$ be a family of sections whose infinite jet at the point x is θ, φ be a section of the bundle $(\pi_k^{\mathcal{G}})^*(\xi)$. Let us set by definition

$$\widehat{\nabla}(\varphi)(\theta) = \nabla(j_k(\{s_g\})^*(\varphi))(x).$$

EXERCISE 6.6. Prove the identity $\widehat{\nabla}(\varphi) = \varphi_{\nabla \circ \Delta}$ at points of $J^\infty(\pi; \mathcal{G})_0$, where $\Delta = \Delta_\varphi$. Construct an example, when this equality does not hold at other points of the space $J^\infty(\pi; \mathcal{G})$. Thus, we cannot use the "global" definition of the lifting given in Ch. 4.

A vector field X on the manifold M and the homomorphism $g_1{}^*$, where g_1 is a mapping from M to M, are linear boundary differential operators acting from $C^\infty(M)$ to $C^\infty(M)$ with respect to the sets of mappings $\{\mathrm{id}_M\}$ and $\{\mathrm{id}_M, g_1\}$. Therefore, the liftings $\widehat{X}$ and $\widehat{g_1{}^*}$ are defined.

EXERCISE 6.7. Prove that $\widehat{X}$ is a vector field on $J^\infty(\pi; \mathcal{G})$ satisfying equalities (1.12) of Ch. 4 for $\varphi \in \mathcal{F}(\pi; \mathcal{G})$.

A smooth mapping $g_1 \colon M \to M$ lifts to the smooth mapping

$$\widehat{g_1} \colon J^k(\pi; \mathcal{G} \circ \mathcal{G}_{g_1}) \to J^k(\pi; \mathcal{G}),$$

where $\mathcal{G}_{g_1} = \{\mathrm{id}_M, g_1\}$, $k = 0, 1, \ldots, \infty$. Namely, the mapping $\widehat{g_1}$ takes the point $\theta_k = \{\theta_k^g\}_{g \in \mathcal{G} \circ \mathcal{G}_{g_1}}$ to the point $\widetilde{\theta}_k = \{\widetilde{\theta}_k^g\}_{g \in \mathcal{G}}$, where $\widetilde{\theta}_k^g = \theta_k^{g \circ g_1}$. If $\pi_k^{\mathcal{G}}(\theta) = x$, then θ_k^g is a k-jet at the point $g(x)$ and $\widetilde{\theta}_k^g$ is a k-jet at the point $g(x_1)$, where $x_1 = g_1(x)$. Therefore, the diagram

$$
\begin{array}{ccc}
J^k(\pi; \mathcal{G} \circ \mathcal{G}_{g_1}) & \xrightarrow{\widehat{g_1}} & J^k(\pi; \mathcal{G}) \\
{\scriptstyle \pi_k^{\mathcal{G} \circ \mathcal{G}_{g_1}}} \Big\downarrow & & \Big\downarrow {\scriptstyle \pi_k^{\mathcal{G}}} \\
M & \xrightarrow{\quad g_1 \quad} & M
\end{array}
\tag{6.32}
$$

is commutative, as well as the diagram

$$
\begin{array}{ccc}
J^k(\pi; \mathcal{G} \circ \mathcal{G}_{g_1}) & \xrightarrow{\widehat{g_1}} & J^k(\pi; \mathcal{G}) \\
{\scriptstyle \pi_k^{\mathcal{G} \circ \mathcal{G}_{g_1}, \mathcal{G}' \circ \mathcal{G}_{g_1}}} \Big\downarrow & & \Big\downarrow {\scriptstyle \pi_k^{\mathcal{G}, \mathcal{G}'}} \\
J^k(\pi; \mathcal{G}' \circ \mathcal{G}_{g_1}) & \xrightarrow{\widehat{g_1}} & J^k(\pi; \mathcal{G}')
\end{array}
\tag{6.33}
$$

where $\mathcal{G}' \subset \mathcal{G}$. The following formula is also valid:

$$\widehat{g_1} \circ j_k(\{s_{\widetilde{g}}\}) = j_k(\{s_{\widetilde{g}}\}) \circ g_1, \tag{6.34}$$

where $\widetilde{g} \in \mathcal{G} \circ \mathcal{G}_{g_1}$, $g \in \mathcal{G}$, $\widetilde{s}_g = s_{g \circ g_1}$, since

$$
\begin{aligned}
\widehat{g_1}(j_k(\{s_{\widetilde{g}}\})(x)) &= \widehat{g_1}(\{[s_{\widetilde{g}}]_{\widetilde{g}(x)}^k\}_{\widetilde{g} \in \mathcal{G} \circ \mathcal{G}_{g_1}}) \\
&= \{[s_{g \circ g_1}]_{g(g_1(x))}^k\}_{g \in \mathcal{G}} = j_k(\{s_{g \circ g_1}\})(g_1(x)).
\end{aligned}
$$

The mapping $\widehat{g_1}$ is smooth for $k = 0, 1, \ldots, \infty$, since for any number $l \geq 0$ and any finite subset $\mathcal{G}_\alpha \subset \mathcal{G}$ one has

$$(\widehat{g_1})^*(\mathcal{F}_l(\pi; \mathcal{G}_\alpha)) \subset \mathcal{F}_l(\pi; \mathcal{G}_\alpha \circ \mathcal{G}_{g_1}).$$

EXERCISE 6.8. Prove that in the case $k = \infty$ the equality $\widehat{g}^* = \widehat{g}^*$ holds.

Note that the equality $\mathcal{G} \circ \mathcal{G}_{g_1} = \mathcal{G}$ means invariance of the set of mappings $\mathcal{G}$ with respect to the right action of g_1, i.e., $g \circ g_1 \in \mathcal{G}$ for any $g \in \mathcal{G}$. If the set $\mathcal{G}$ is invariant with respect to the right action of g_1 and g_2, then the equality

$$\widehat{g_1 \circ g_2} = \widehat{g_1} \circ \widehat{g_2} \tag{6.35}$$

holds. In fact, $\widehat{g_2}(\{\theta_k^g\}_{g \in \mathcal{G}}) = \{\widetilde{\theta}_k^g\}_{g \in \mathcal{G}}$, where $\widetilde{\theta}_k^g = \theta_k^{g \circ g_2}$, and

$$\widehat{g_1}(\{\widetilde{\theta}_k^g\}_{g \in \mathcal{G}}) = \{\widetilde{\theta}_k^{g \circ g_1}\}_{g \in \mathcal{G}} = \{\theta_k^{g \circ g_1 \circ g_2}\}_{g \in \mathcal{G}} = \widehat{g_1 \circ g_2}(\{\theta_k^g\}_{g \in \mathcal{G}}).$$

Equality (6.35) is valid, for example, if $\mathcal{G}$ is a semigroup and g_1, g_2 are its elements.

Let us describe coordinate representation of liftings of vector fields and mappings and coordinate representation of $(l, \mathcal{G}')$-prolongations of boundary differential operators. Let $x_1, \ldots, x_n, \ldots, p_{\sigma g}^j, \ldots$ be coordinates in $J^\infty(\pi; \mathcal{G})$, $X = \partial/\partial x_i$, $\varphi = \varphi(x_1, \ldots, x_n, \ldots, p_{\sigma g}^j, \ldots) \in \mathcal{F}(\pi; \mathcal{G})$, and $s = (s^1, \ldots, s^m) \in \Gamma(\pi)$. Then

$$\left(\frac{\partial}{\partial x_i} \circ \Delta_\varphi \right)(s) = \frac{\partial}{\partial x_i} \left(\varphi \left(x_1, \ldots, x_n, \ldots, g^* \left(\frac{\partial^{|\sigma|} s^j}{\partial x^\sigma} \right), \ldots \right) \right)$$

$$= \frac{\partial \varphi}{\partial x_i} + \sum_{j, \sigma, g} \frac{\partial}{\partial x_i} \left(g^* \left(\frac{\partial^{|\sigma|} s^j}{\partial x^\sigma} \right) \right) \frac{\partial \varphi}{\partial p_{\sigma g}^j}.$$

Therefore, by the equality $\partial/\partial x_i \circ g^* = \sum_l \partial g^*(x_l)/\partial x_i g^* \circ \partial/\partial x_l$, we have

$$\widehat{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial x_i} + \sum_{|\sigma|, j, g, l} \frac{\partial g^*(x_l)}{\partial x_i} p_{\sigma+1_l, g}^j \frac{\partial}{\partial p_{\sigma g}^j}. \tag{6.36}$$

Similar to the differential case, we shall denote the operator $\widehat{\partial/\partial x_i}$ by D_i and call it the *total derivative* along the variable x_i.

The mapping $\widehat{g_1}$, where g_1 is a mapping from M to M, obviously acts on the coordinate functions in the following way: $\widehat{g_1}^*(x_i) = g_1^*(x_i)$, $\widehat{g_1}^*(p_{\sigma g}^j) = p_{\sigma g_2}^j$, where $g_2 = g \circ g_1$.

Finally, let $\Delta \colon \Gamma(\pi) \to \Gamma(\pi')$ be a boundary differential operator, $x_1, \ldots, x_n$, $v^1, \ldots, v^{m'}, \ldots, q_{\sigma g'}^j, \ldots$ be coordinates in $J^\infty(\pi'; \mathcal{G}')$, and the operator Δ be determined by the relations

$$v^{j'} = \varphi^{j'}(x_1, \ldots, x_n, \ldots, p_{\sigma g}^j, \ldots), \qquad j' = 1, \ldots, m'.$$

Then the $(l, \mathcal{G}')$-prolongation of the operator Δ is described by the relations

$$v^{j'} = \varphi^{j'},$$
$$q_{\tau g'}^{j'} = g'^*(D_\tau \varphi^{j'}), \qquad j' = 1, \ldots, m', \quad |\tau| = 0, \ldots, l, \quad g' \in \mathcal{G}'. \tag{6.37}$$

where $D_\tau = D_1^{l_1} \circ \cdots \circ D_n^{l_n}$ for $\tau = (l_1, \ldots, l_n)$.

6.4. The Cartan distribution on $J^\infty(\pi; \mathcal{G})$. If $\mathcal{G}$ is a finite set, then, in parallel with the differential case, we denote by $L_{\theta_{k+1}}$ the tangent plane to the graph of the section $j_k(\{s_g\})$ at a point $\theta_k \in J^k(\pi; \mathcal{G})$, where $\{s_g\}_{g \in \mathcal{G}}$ is a family of sections of the bundle π such that its $(k+1, \mathcal{G})$-jet is $\theta_{k+1} \in J^{k+1}(\pi; \mathcal{G})$ and $\theta_k = \pi_{k+1,k}^{\mathcal{G}}(\theta_{k+1})$. In the case of finite $\mathcal{G}$ and k, the *Cartan distribution* $\mathcal{C}^k$ on $J^k(\pi; \mathcal{G})$ at the point θ_k is defined as the span of the spaces $L_{\theta_{k+1}}$, where $\theta_{k+1} \in (\pi_{k+1,k}^{\mathcal{G}})^{-1}(\theta_k)$.

From formulas (6.22) and (6.27) it follows that the projections $\pi_{k,k-1}^{\mathcal{G}}$ and $\pi_k^{\mathcal{G}, \mathcal{G}_1}$, where $\mathcal{G}_1$ is a subset in $\mathcal{G}$, take the graph of the section $j_k(\{s_g\})$ to a graph of a similar type. For tangent maps, this fact implies the equalities

$$(\pi_{k,k-1}^{\mathcal{G}})_*(L_{\theta_{k+1}}) = L_{\theta_k}, \qquad (\pi_k^{\mathcal{G}, \mathcal{G}_1})_*(L_{\theta_{k+1}}) = L_{\widetilde{\theta}_{k+1}}, \tag{6.38}$$

where $\theta_k = \pi_{k+1,k}^{\mathcal{G}}(\theta_{k+1})$, $\widetilde{\theta}_{k+1} = \pi_{k+1}^{\mathcal{G}, \mathcal{G}_1}(\theta_{k+1})$. Consequently,

$$(\pi_{k,k-1}^{\mathcal{G}})_*(\mathcal{C}_{\theta_k}^k) = L_{\theta_k} \subset \mathcal{C}_{\theta_{k-1}}^{k-1}, \tag{6.39}$$

$$(\pi_k^{\mathcal{G}, \mathcal{G}_1})_*(\mathcal{C}_{\theta_k}^k) = \mathcal{C}_{\widetilde{\theta}_k}^k, \tag{6.40}$$

where $\theta_{k-1} = \pi^{\mathcal{G}}_{k,k-1}(\theta_k)$, $\widetilde{\theta}_k = \pi^{\mathcal{G},\mathcal{G}_1}_k(\theta_k)$.

Formula (6.39) means that the distribution $\mathcal{C}^k$ on $J^k(\pi;\mathcal{G})$ determines the distribution $\mathcal{C} = \mathcal{C}(\pi;\mathcal{G})$ on $J^\infty(\pi;\mathcal{G})$ in the case of finite set $\mathcal{G}$. If $\mathcal{G}$ is an infinite set, then formula (6.40) allows to define the distributions $\mathcal{C}^k$ on $J^k(\pi;\mathcal{G})$ and $\mathcal{C} = \mathcal{C}(\pi;\mathcal{G})$ on $J^\infty(\pi;\mathcal{G})$. Namely, a tangent vector $X_\theta = \{X_x, X_{\theta^{\mathcal{G}_\alpha}_k}\}$ to the manifold $J^\infty(\pi;\mathcal{G})$ at a point $\theta = \{x, \theta^{\mathcal{G}_\alpha}_k\}$ lies, by definition, in the plane $\mathcal{C}_\theta$, if for any number $k \geq 0$ and any finite subset $\mathcal{G}_\alpha \subset \mathcal{G}$ the vector $X_{\theta^{\mathcal{G}_\alpha}_k}$ lies in the Cartan distribution $\mathcal{C}^k$ on $J^k(\pi;\mathcal{G}_\alpha)$ at the point $\theta^{\mathcal{G}_\alpha}_k$. We shall call the distribution $\mathcal{C}_\theta$, $\theta \in J^\infty(\pi;\mathcal{G})$, the *Cartan distribution* on $J^\infty(\pi;\mathcal{G})$. Let us describe the Cartan distribution on $J^\infty(\pi;\mathcal{G})$ from various points of view.

PROPOSITION 6.1. *At an arbitrary point* $\theta = \{[s_g]^\infty_{g(x)}\}_{g\in\mathcal{G}} \in J^\infty(\pi;\mathcal{G})$, *the Cartan plane* $\mathcal{C}_\theta$ *is the tangent plane to the graph of the section* $j_\infty(\{s_g\})$ *at the point* θ. *Namely, a vector* $X_\theta = \{X_x, X_{\theta^{\mathcal{G}_\alpha}_k}\}$ *lies in the plane* $\mathcal{C}_\theta$ *if and only if for any number* $k \geq 0$ *and any finite subset* $\mathcal{G}\alpha \subset \mathcal{G}$ *the vector* $X_{\theta^{\mathcal{G}_\alpha}_k}$ *is tangent to the submanifold* $j_k(\{s_g\})(M) \subset J^k(\pi;\mathcal{G}_\alpha)$.

PROOF. If the vector $X_\theta = \{X_x, X_{\theta^{\mathcal{G}_\alpha}_k}\}$ at the point $\theta = \{x, \theta^{\mathcal{G}_\alpha}_k\}$ belongs to the plane $\mathcal{C}_\theta$, then, by the definition of the Cartan distribution, for any number $k \geq 0$ and a finite subset $\mathcal{G}_\alpha \subset \mathcal{G}$ the vector $X_{\theta^{\mathcal{G}_\alpha}_{k+1}}$ lies in the plane $\mathcal{C}^{k+1}_{\theta^{\mathcal{G}_\alpha}_{k+1}}$. Therefore, from (6.39) we obtain

$$X_{\theta^{\mathcal{G}_\alpha}_k} = (\pi^{\mathcal{G}_\alpha}_{k+1,k})_*(X_{\theta^{\mathcal{G}_\alpha}_{k+1}}) \in L_{\theta^{\mathcal{G}_\alpha}_{k+1}} = T_{\theta^{\mathcal{G}_\alpha}_k}(j_k(\{s_g\})(M)),$$

where $\theta = \{[s_g]^\infty_{g(x)}\}_{g\in\mathcal{G}}$ and $\theta^{\mathcal{G}_\alpha}_{k+1} = \{[s_g]^{k+1}_{g(x)}\}_{g\in\mathcal{G}_\alpha}$.

Conversely, if the vector $X_{\theta^{\mathcal{G}_\alpha}_k}$ is tangent to the submanifold $j_k(\{s_g\})(M)$, then it lies in the Cartan distribution on $J^k(\pi;\mathcal{G}_\alpha)$. Since this is true for all $k \geq 0$ and $\mathcal{G}_\alpha \subset \mathcal{G}$, we have $X_\theta = \{X_x, X_{\theta^{\mathcal{G}_\alpha}_k}\} \in \mathcal{C}_\theta$. $\square$

PROPOSITION 6.2. *The Cartan distribution on* $J^\infty(\pi;\mathcal{G})$ *determines a connection in the bundle* $\pi^{\mathcal{G}}_\infty$: *at any point* $\theta \in J^\infty(\pi;\mathcal{G})$ *the tangent mapping* $(\pi^{\mathcal{G}}_\infty)_{*,\theta}$ *isomorphically takes the Cartan plane* $\mathcal{C}_\theta$ *to the tangent space* $T_x(M)$, *where* $x = \pi^{\mathcal{G}}_\infty(\theta)$.

PROOF. At any point $\theta = \{x, \theta^{\mathcal{G}_\alpha}_k\} \in J^\infty(\pi;\mathcal{G})$ we shall construct the inverse to the mapping $(\pi^{\mathcal{G}}_\infty)_{*,\theta}\colon \mathcal{C}_\theta \to T_x(M)$. For any number $k \geq 0$ and a finite subset $\mathcal{G}_\alpha \subset \mathcal{G}$, the tangent mapping $(\pi^{\mathcal{G}_\alpha}_k)_{*,\theta^{\mathcal{G}_\alpha}_k}$ isomorphically maps the plane $L_{\theta^{\mathcal{G}_\alpha}_{k+1}}$ to the space $T_x(M)$. Therefore, any vector $X_x \in T_x(M)$ uniquely determines a set of vectors $X_{\theta^{\mathcal{G}_\alpha}_k} \in L_{\theta^{\mathcal{G}_\alpha}_{k+1}}$, $k \geq 0$, $\mathcal{G}_\alpha \subset \mathcal{G}$, such that $(\pi^{\mathcal{G}_\alpha}_k)_*(X_{\theta^{\mathcal{G}_\alpha}_k}) = X_x$. From uniqueness of this set and from formulas (6.38) it follows that for any number $k \geq 0$ and a finite subset $\mathcal{G}_\beta \subset \mathcal{G}_\alpha$ of the set $\mathcal{G}$ the equalities

$$(\pi^{\mathcal{G}_\alpha}_{k,k-1})_*(X_{\theta^{\mathcal{G}_\alpha}_k}) = X_{\theta^{\mathcal{G}_\alpha}_{k-1}}, \qquad (\pi^{\mathcal{G}_\alpha,\mathcal{G}_\beta}_k)_*(X_{\theta^{\mathcal{G}_\alpha}_k}) = X_{\theta^{\mathcal{G}_\beta}_k}$$

hold. Therefore, the set of vectors $\{X_x, X_{\theta^{\mathcal{G}_\alpha}_k}\}$ determines a tangent vector X_θ lying in the Cartan plane $\mathcal{C}_\theta$. From uniqueness of vectors $\{X_{\theta^{\mathcal{G}_\alpha}_k}\}$ it also follows that the above constructed mapping $X_x \mapsto X_\theta$ is inverse to the mapping $(\pi^{\mathcal{G}}_\infty)_*|_{\mathcal{C}_\theta}$. $\square$

Let us now prove generalizations of Propositions 2.1–2.3 of Ch. 4.

PROPOSITION 6.3. *The set $\mathcal{CD}(\pi;\mathcal{G})$ of vector fields lying in the Cartan distribution is spanned by the liftings to $J^\infty(\pi;\mathcal{G})$ of fields on the manifold M:*

$$\mathcal{CD}(\pi;\mathcal{G}) = \{\, \varphi_1\widehat{X_1} + \cdots + \varphi_l\widehat{X_l} \mid \varphi_i \in \mathcal{F}(\pi;\mathcal{G}),\ X_i \in \mathrm{D}(M),\ l = 1,2,\dots \,\}.$$

PROOF. Let us prove that any vector of the form $\widehat{X}_\theta$, where $\theta = \{x, \theta_k^{\mathcal{G}_\alpha}\} \in J^\infty(\pi;\mathcal{G})$, $X \in \mathrm{D}(M)$, lies in the plane $\mathcal{C}_\theta$. This is equivalent to the fact that any projection $\widehat{X}_{\theta_k^{\mathcal{G}_\alpha}}$ of the vector $\widehat{X}_\theta$ to the finite-dimensional space of jets $J^k(\pi;\mathcal{G}_\alpha)$ lies in the corresponding plane $L_{\theta_{k+1}^{\mathcal{G}_\alpha}}$. Consider an arbitrary function φ on $J^k(\pi;\mathcal{G}_\alpha)$ constant on the graph of the section $j_k(\{s_g\})$ whose $(k+1,\mathcal{G}_\alpha)$-jet equals $\theta_{k+1}^{\mathcal{G}_\alpha}$, where $\{s_g\}_{g\in\mathcal{G}_\alpha}$ is a family of sections of the bundle π. Then

$$\widehat{X}_\theta(\varphi) = \widehat{X}_{\theta_k^{\mathcal{G}_\alpha}}(\varphi) = X(j_k(\{s_g\})^*(\varphi))(x) = X(\mathrm{const})(x) = 0.$$

Since this is true for any function φ of the chosen type, the vector $\widehat{X}_{\theta_k^{\mathcal{G}_\alpha}}$ is tangent to the graph of the section $j_k(\{s_g\})$ and consequently lies in $L_{\theta_{k+1}^{\mathcal{G}_\alpha}}$. Hence, $\widehat{X}_\theta \in \mathcal{C}_\theta$ and $\widehat{X} \in \mathcal{CD}(\pi;\mathcal{G})$.

Consider now an arbitrary field X from $\mathcal{CD}(\pi;\mathcal{G})$. By the definition of vector fields, there exists a number k such that $X(C^\infty(M)) \subset \mathcal{F}_k(\pi;\mathcal{G})$. Similar to the differential case (see Ch. 4), the restriction X_M of the field X to $C^\infty(M)$ can be locally represented in the form $X_M = \varphi_1 X_1 + \cdots + \varphi_l X_l$, where $X_1,\dots,X_l \in \mathrm{D}(M)$, $\varphi_1,\dots,\varphi_l \in \mathcal{F}_k(\pi;\mathcal{G})$. Let us show that the field X locally coincides with the field

$$\widehat{X}_M = \varphi_1\widehat{X}_1 + \cdots + \varphi_l\widehat{X}_l \in \mathcal{CD}(\pi;\mathcal{G}).$$

To this end, consider an arbitrary point $\theta = \{x, \theta_k^{\mathcal{G}_\alpha}\} \in J^\infty(\pi;\mathcal{G})$, where the field $\widehat{X}_M$ is defined. The vectors X_θ and $\widehat{X}_{M,\theta}$ lie in $\mathcal{C}_\theta$ and, by the construction of the field $\widehat{X}_M$, their projections to M by the mapping $\pi_\infty^{\mathcal{G}}$ coincide. From here and from Proposition 6.2 it follows that $X_\theta = \widehat{X}_{M,\theta}$. Standard arguments using partition of unity [**146, 115**] gives global representation of the field $\varphi_1\widehat{X}_1 + \cdots + \varphi_l\widehat{X}_l$. $\square$

PROPOSITION 6.4. *A field $X \in \mathrm{D}(\pi;\mathcal{G})$ lies in the Cartan distribution on $J^\infty(\pi;\mathcal{G})$ if and only if for any function $\varphi \in \mathcal{F}(\pi)$ and any mapping $g \in \mathcal{G}$ the identity*

$$X \lrcorner (\widehat{g} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_g})^*(U_1(\varphi)) = 0,$$

holds, where $U_1(\varphi)$ is the Cartan form on $J^\infty(\pi)$ and $\mathcal{G}_g$ is the set consisting of two mappings, id_M and g.

The forms $(\widehat{g} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_g})^*(U_1(\varphi))$ will be called the *Cartan forms* on $J^\infty(\pi;\mathcal{G})$. The module generated by these forms is called the *Cartan module* and is denoted by $\mathcal{C}\Lambda^1(\pi;\mathcal{G})$.

PROOF. From formulas (6.34) and (6.27) it follows that the mappings $\widehat{g}$ and $\pi_\infty^{\mathcal{G},\mathcal{G}_g}$ preserve the Cartan distribution. Therefore, if $X_\theta \in \mathcal{C}_\theta$, $\theta \in J^\infty(\pi;\mathcal{G})$, then

$$X_\theta \lrcorner (\widehat{g} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_g})^*(U_1(\varphi)) = (\widehat{g} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_g})_*(X_\theta) \lrcorner U_1(\varphi) = 0.$$

Since the bundle $\pi_\infty^{\mathcal{G}}$ is the Whitney product of the bundles $g^*(\pi_\infty)$, $g \in \mathcal{G}$, the fiber $(\pi_\infty^{\mathcal{G}})^{-1}(x)$ is the direct product of the fibers $(g^*(\pi_\infty))^{-1}(x)$, $g \in \mathcal{G}$, for any

point $x \in M$. On the other hand, by the definition of the bundle $g^*(\pi_\infty)$, the fiber $(g^*(\pi_\infty))^{-1}(x)$ coincides with the fiber $\pi_\infty^{-1}(g(x))$. Thus we obtain the equality

$$(\pi_\infty^{\mathcal{G}})^{-1}(x) = \prod_{g \in \mathcal{G}} \pi_\infty^{-1}(g(x)).$$

The mapping $(\widehat{g} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_g})\Big|_{(\pi_\infty^{\mathcal{G}})^{-1}(x)}$ is the projection to the component $\pi_\infty^{-1}(g(x))$ of this direct product. In fact, any point θ of the fiber $(\pi_\infty^{\mathcal{G}})^{-1}(x)$ is the set $\{\theta^g\}_{g \in \mathcal{G}}$ of infinite jets at the corresponding points $g(x)$, $g \in \mathcal{G}$, i.e., $\theta^g \in \pi_\infty^{-1}(g(x))$. The mapping $\pi_\infty^{\mathcal{G},\mathcal{G}_g}$ takes the point θ to the set consisting of two jets $\{\theta^{g_0}, \theta^g\}$, where $g_0 = \mathrm{id}_M$, and the mapping $\widehat{g}$ takes this set to the jet θ^g.

Thus, linear space of vertical vectors of the bundle $\pi_\infty^{\mathcal{G}}$ at the point $\theta \in J^\infty(\pi; \mathcal{G})$ is the direct sum of linear spaces of vertical vectors of the bundle π_∞ at the points $\theta^g \in J^\infty(\pi)$ over all $g \in \mathcal{G}$:

$$V_\theta(J^\infty(\pi; \mathcal{G})) \simeq \bigoplus_{g \in \mathcal{G}} V_{\theta^g}(J^\infty(\pi)),$$

where by V_θ the space of vertical fields at the point θ of the corresponding jet bundle is denoted.

From Proposition 6.2 it follows that the tangent space to the manifold $J^\infty(\pi; \mathcal{G})$ at the point θ is the direct sum of the space of vertical vectors and the Cartan plane $\mathcal{C}_\theta$. We obtain the splitting

$$T_\theta\big(J^\infty(\pi; \mathcal{G})\big) \simeq \left(\bigoplus_{g \in \mathcal{G}} V_{\theta^g}\big(J^\infty(\pi)\big)\right) \bigoplus \mathcal{C}_\theta, \qquad \theta = \{\theta^g\}_{g \in \mathcal{G}}. \tag{6.41}$$

The tangent mapping $(\widehat{g} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_g})_*$ isomorphically maps the component $\mathcal{C}_\theta$ to the Cartan plane $\mathcal{C}_{\theta^g} \subset T_{\theta^g}(J^\infty(\pi))$, the component $V_{\theta^g}(J^\infty(\pi))$ is mapped identically to itself, and all other components are taken to zero.

Consider now an arbitrary tangent vector X_θ to the manifold $J^\infty(\pi; \mathcal{G})$ at the point θ and its splitting with respect to (6.41): $X_\theta = \sum_{g \in \mathcal{G}} X_{\theta^g} + Y_\theta$, where $X_{\theta^g} \in V_{\theta^g}(J^\infty(\pi))$, $Y_\theta \in \mathcal{C}_\theta$. If the vector X_θ does not lie in the Cartan distribution, then there exists at least one mapping $g_1 \in \mathcal{G}$ such that the vertical vector $X_{\theta^{g_1}}$ does not vanish. Then from Proposition 2.1 of Ch 4 it follows that there exists a function $\varphi \in \mathcal{F}(\pi)$ such that $X_{\theta^{g_1}} \lrcorner U_1(\varphi) \neq 0$. From here we have

$$X_\theta \lrcorner (\widehat{g_1} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_{g_1}})^*(U_1(\varphi))$$

$$= \left(\sum_{g \in \mathcal{G}} X_{\theta^g}\right) \lrcorner (\widehat{g_1} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_{g_1}})^*(U_1(\varphi)) + Y_\theta \lrcorner (\widehat{g_1} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_{g_1}})^*(U_1(\varphi))$$

$$= (\widehat{g_1} \circ \pi_\infty^{\mathcal{G},\mathcal{G}_{g_1}})_* \left(\sum_{g \in \mathcal{G}} X_{\theta^g}\right) \lrcorner U_1(\varphi) + 0 = X_{\theta^{g_1}} \lrcorner U_1(\varphi) \neq 0,$$

i.e., X_θ does not satisfy our identity in this case. $\qquad \square$

PROPOSITION 6.5. *Any maximal integral manifold of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$ is locally the graph of infinite jet for a family $\{s_g\}_{g \in \mathcal{G}}$ of sections of the bundle π.*

PROOF. The proof is similar to that of Proposition 2.3 of Ch. 4. Namely, let $\mathcal{R}^\infty$ be a maximal integral manifold of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$. The projections

$$\pi^{\mathcal{G}}_{\infty,k}\Big|_{\mathcal{R}^\infty} : \mathcal{R}^\infty \longrightarrow \mathcal{R}^k \overset{\text{def}}{=} \pi^{\mathcal{G}}_{\infty,k}(\mathcal{R}^\infty) \subset J^k(\pi; \mathcal{G}), \qquad k = 0, 1, \ldots,$$

and

$$\pi^{\mathcal{G}}_{\infty}\Big|_{\mathcal{R}^\infty} : \mathcal{R}^\infty \longrightarrow \mathcal{R} \overset{\text{def}}{=} \pi^{\mathcal{G}}_{\infty}(\mathcal{R}^\infty) \subset M$$

are local diffeomorphisms, since the Cartan planes $\mathcal{C}_\theta$, $\theta \in J^\infty(\pi; \mathcal{G})$ (and consequently, the tangent planes to $\mathcal{R}^\infty$) do not contain nontrivial vertical vectors (see Proposition 6.2). Therefore, the projection $\pi^{\mathcal{G}}_0\big|_{\mathcal{R}^0} : \mathcal{R}^0 \to \mathcal{R}$ is also a local diffeomorphism. Hence, there exists a mapping $s' : \mathcal{R} \to \mathcal{R}^0$ such that $\pi^{\mathcal{G}}_0\big|_{\mathcal{R}^0} \circ s'$ is the identity. Let us extend the mapping s', similar to the differential case (see Ch. 4), to a section $\widetilde{s}$ of the bundle $\pi^{\mathcal{G}}_0$. Since $\pi^{\mathcal{G}}_0$ is the Whitney product of the bundles $g^*(\pi_0) = g^*(\pi)$, $g \in \mathcal{G}$, the section $\widetilde{s}$ of the bundle $\pi^{\mathcal{G}}_0$ is the set $\{s_g\}_{g \in \mathcal{G}}$ of sections of the bundle π. The manifolds $j_k(\{s_g\})(M) \subset J^k(\pi; \mathcal{G})$ for $k = 0, 1, \ldots, \infty$ are integral manifolds of the Cartan distribution and contain the manifolds $\mathcal{R}^k$ for the corresponding k. But since $\mathcal{R}^\infty$ is a maximal integral manifold, we obtain that at least locally the manifolds $\mathcal{R}^\infty$ and $j_\infty(\{s_g\})(M)$ coincide. $\qquad \square$

From this proposition it follows that besides the manifolds $j_\infty(s)(M)$, which correspond to solutions of equations $\mathcal{E} \subset J^k(\pi; \mathcal{G})$, other maximal integral manifolds of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$ may exist. The following proposition allows to distinguish the needed ones between all maximal integral manifolds.

PROPOSITION 6.6. *Let the set $\mathcal{G}$ be a semigroup of mappings of the manifold M and N be the graph of a section of the bundle $\pi^{\mathcal{G}}_\infty$. Then N is a graph of the form $j_\infty(s)$, $s \in \Gamma(\pi)$, if and only if N is a maximal integral manifold of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$ invariant with respect to any mapping $\widehat{g}$, $g \in \mathcal{G}$, i.e., such that $\widehat{g}(N) \subset N$.*

PROOF. From the definition of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$ it follows that the manifold $N = j_\infty(s)(M)$ is a maximal integral manifold of this distribution. Besides, if $\theta = [s]_x^{(\infty,\mathcal{G})} \in N$, then $\widehat{g}(\theta) = [s]_{g(x)}^{(\infty,\mathcal{G})} \in N$.

Let now N be a maximal integral manifold invariant with respect to $\mathcal{G}$. Since N is the graph of a section of the bundle $\pi^{\mathcal{G}}_\infty$, we have that $\mathcal{R} = \pi^{\mathcal{G}}_\infty(N) = M$ and s' is a section of the bundle $\pi^{\mathcal{G}}_0$ defined at all points of M (see the proof of the previous proposition). Therefore, N is the graph of a section of the form $j_\infty(\{s_g\})$, where $\{s_g\}_{g \in \mathcal{G}}$ is a family of sections of the bundle π determined by the section s'.

Consider an arbitrary mapping $g_1 \in \mathcal{G}$, an arbitrary point x of the manifold M, and the corresponding point $\theta = j_\infty(\{s_g\})(x) = \{[s_g]_{g(x)}^\infty\}_{g \in \mathcal{G}}$ of the manifold N. By the assumption, the point $\widehat{g_1}(\theta)$, which, by the definition of $\widehat{g_1}$, is the infinite jet at the point $x_1 = g_1(x)$, lies in the manifold $N = j_\infty(\{s_g\})(M)$. Therefore, $\widehat{g_1}(\theta) = j_\infty(\{s_g\})(x_1)$. Hence, $\{[s_{g \circ g_1}]_{g(x_1)}^\infty\}_{g \in \mathcal{G}} = \{[s_g]_{g(x_1)}^\infty\}_{g \in \mathcal{G}}$ and consequently

$$[s_{g \circ g_1}]_{(g \circ g_1)(x)}^\infty = [s_g]_{(g \circ g_1)(x)}^\infty$$

for any $g \in \mathcal{G}$. Substituting $g = \text{id}_M$ to this equality and denoting s_{id_M} by s, we obtain the equality $[s_{g_1}]_{g_1(x)}^\infty = [s]_{g_1(x)}^\infty$ for any $g_1 \in \mathcal{G}$. Consequently,

$$\theta = \{[s]_{g(x)}^\infty\}_{g \in \mathcal{G}} = j_\infty(s)(x).$$

Since this is valid for any $x \in M$, we have $N = j_\infty(s)(M)$. $\qquad\square$

Everywhere below we assume that $\mathcal{G}$ is a semigroup of mappings of the manifold M.

6.5. $\mathcal{G}$-invariant symmetries of the Cartan distribution on $J^\infty(\pi;\mathcal{G})$. Let us describe infinitesimal transformations of the space $J^\infty(\pi;\mathcal{G})$ preserving maximal integral manifolds of the form $j_\infty(s)(M)$. Note that any manifold $j_\infty(s)(M)$ lies in $J^\infty(\pi;\mathcal{G})_0$. Therefore, in general, such a transformation is a transformation of the set $J^\infty(\pi;\mathcal{G})_0$. But by the reasons exposed in Remark 6.2 we shall assume that our transformation X is extended to the entire space $J^\infty(\pi;\mathcal{G})$. From Proposition 6.1 it follows that tangent planes to manifolds of the form $j_\infty(s)(M)$ are the Cartan planes at points of the set $J^\infty(\pi;\mathcal{G})_0$. Consequently (see Ch. 4), at points of the set $J^\infty(\pi;\mathcal{G})_0$ we have the embedding

$$X(\mathcal{C}\Lambda^1(\pi;\mathcal{G})) \subset \mathcal{C}\Lambda^1(\pi;\mathcal{G}). \qquad (6.42)$$

On the other hand, a field lying in $\mathcal{C}D(\pi;\mathcal{G})$ is tangent to all integral manifolds of the distribution $\mathcal{C}(\pi;\mathcal{G})$. Therefore, it is tangent to manifolds of the form $j_\infty(s)(M)$ and hence is the desired but "trivial" transformation (cf. Ch. 4). By this reason, it suffices to find vertical infinitesimal transformations only, such that they preserve the class of manifolds of the form $j_\infty(s)(M)$.

Let X be such a vertical field, θ be an arbitrary point of the manifold $N = j_\infty(s)(M)$, $\mathcal{G}$ be a semigroup of mappings of the manifold M, and g be its element. Assume also that the field X possesses a one-parameter group of diffeomorphisms $\{A_t\}$. Any manifold $N_t = A_t(N)$ is of the form $j_\infty(s_t)(M)$, $s_t \in \Gamma(\pi)$, and hence $\widehat{g}(N_t) \subset N_t$ (see Proposition 6.6). Hence, $\widehat{g}(A_t(\theta)) \in N_t$. Since $\widehat{g}(\theta) \in \widehat{g}(N) \subset N$, the point $A_t(\widehat{g}(\theta))$ also lies in N_t. Moreover, since X is a vertical field, the mapping A_t preserves every fiber of the bundle $\pi_\infty^{\mathcal{G}}$. From here and from the commutative diagram (6.32) we have

$$\pi_\infty^{\mathcal{G}}(A_t(\widehat{g}(\theta))) = \pi_\infty^{\mathcal{G}}(\widehat{g}(\theta)) = g(\pi_\infty^{\mathcal{G}}(\theta)),$$

as well as

$$\pi_\infty^{\mathcal{G}}(\widehat{g}(A_t(\theta))) = g(\pi_\infty^{\mathcal{G}}(A_t(\theta))) = g(\pi_\infty^{\mathcal{G}}(\theta)).$$

Therefore, the points $A_t(\widehat{g}(\theta))$ and $\widehat{g}(A_t(\theta))$ are projected to the same point under the mapping $\pi_\infty^{\mathcal{G}}$ and lie on the graph N_t of a section of the bundle $\pi_\infty^{\mathcal{G}}$. Hence, these points coincide. But since N is an arbitrary manifold of the form $j_\infty(s)(M)$, we obtain that θ is an arbitrary point of the set $J^\infty(\pi;\mathcal{G})_0$. Consequently, at points of the set $J^\infty(\pi;\mathcal{G})_0$ we have the equality $A_t \circ \widehat{g} = \widehat{g} \circ A_t$ for sufficiently small t, while for any function $\varphi \in \mathcal{F}(\pi;\mathcal{G})$ the equality $\widehat{g}^*(A_t^*(\varphi)) = A_t^*(\widehat{g}^*(\varphi))$ holds. Differentiating the last equality with respect to t at $t = 0$, we obtain the formula $\widehat{g}^*(X(\varphi)) = X(\widehat{g}^*(\varphi))$, which is valid at points of the set $J^\infty(\pi;\mathcal{G})_0$ for any function $\varphi \in \mathcal{F}(\pi;\mathcal{G})$.

Thus, a vertical vector field X preserving the class of submanifolds of the form $j_\infty(s)(M)$, $s \in \Gamma(\pi)$, and possessing the one-parameter group of diffeomorphisms is to satisfy condition (6.42) and to commute with any homomorphism $\widehat{g}^*$, $g \in \mathcal{G}$. These conditions are to be satisfied at points of the set $J^\infty(\pi;\mathcal{G})_0$ only. But if the closure of the set $J^\infty(\pi;\mathcal{G})_0$ coincides with the entire space $J^\infty(\pi;\mathcal{G})$ (and in all examples under consideration the situation is exactly of this kind), then the field X satisfies these conditions at all point of the space $J^\infty(\pi;\mathcal{G})$. This fact explains

why we must consider vertical fields X on $J^\infty(\pi; \mathcal{G})$ satisfying conditions (6.42) and such that $\widehat{g}^* \circ X = X \circ \widehat{g}^*$ for any $g \in \mathcal{G}$. We shall call such fields $\mathcal{G}$-*invariant* (*infinitesimal*) *symmetries* of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$. Denote by $\mathrm{sym}^{\mathcal{G}} \mathcal{C}(\pi)$ the set of such fields.

PROPOSITION 6.7. *Any vector field $X \in \mathrm{sym}^{\mathcal{G}} \mathcal{C}(\pi)$ is uniquely determined by its restriction to the subalgebra $\mathcal{F}_0(\pi) \subset \mathcal{F}(\pi; \mathcal{G})$.*

PROOF. An automorphism X of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$ vanishing on the subalgebra $\mathcal{F}_0(\pi) \subset \mathcal{F}(\pi; \mathcal{G})$ equals zero on the subalgebra $\mathcal{F}(\pi) \subset \mathcal{F}(\pi; \mathcal{G})$ as well. This fact is proved exactly in the same way as Proposition 2.4 of Ch. 4 was proved. The fact that X is a derivation of the algebra $\mathcal{F}(\pi; \mathcal{G})$, but not of $\mathcal{F}(\pi)$, is inessential.

Let $X \in \mathrm{sym}^{\mathcal{G}} \mathcal{C}(\pi)$ and $X|_{\mathcal{F}(\pi)} = 0$. Consider in M an arbitrary coordinate neighborhood and the corresponding local coordinates $x_1, \ldots, x_n, \ldots, p^j_{\sigma g}, \ldots$ in $J^\infty(\pi; \mathcal{G})$. If $g_0 = \mathrm{id}_M$, then $p^j_{\sigma g_0} \in \mathcal{F}(\pi)$ and $X(p^j_{\sigma g_0}) = 0$. If $g \neq g_0$, then

$$X(p^j_{\sigma g}) = X(\widehat{g}^*(p^j_{\sigma g_0})) = \widehat{g}^*(X(p^j_{\sigma g_0})) = 0$$

and consequently $X = 0$. $\qquad\square$

Let $X \in \mathrm{sym}^{\mathcal{G}} \mathcal{C}(\pi)$. Then, similar to the differential case (see Ch. 4), the restriction $X_0 = X|_{\mathcal{F}_0(\pi)}$ is a derivation of the algebra $\mathcal{F}_0(\pi)$ with values in the algebra $\mathcal{F}_k(\pi; \mathcal{G}_\alpha)$ for some number $k \geq 0$ and a finite subset $\mathcal{G}_\alpha \subset \mathcal{G}$. Therefore, at any point $\theta_k \in J_k(\pi; \mathcal{G}_\alpha)$ the vector X_{0, θ_k} is identified with a tangent vector to the fiber $\pi^{-1}(x)$ of the bundle π, where $x = \pi_k^{\mathcal{G}_\alpha}(\theta_k)$. Since π is a linear bundle, tangent vectors to its fibers are identified with points of these fibers. Hence, the vector X_{0, θ_k} identifies with a point of the fiber $\pi^{-1}(x)$ while the derivation X_0 identifies with a section of the bundle $(\pi_k^{\mathcal{G}_\alpha})^*(\pi)$. Thus we obtain the mapping

$$\mathrm{sym}^{\mathcal{G}} \mathcal{C}(\pi) \longrightarrow \mathcal{F}(\pi, \pi; \mathcal{G}). \tag{6.43}$$

Injectivity of this mapping is implied by Proposition 6.7. Let us prove surjectivity of the mapping (6.43).

Let φ be a section of the bundle $(\pi_k^{\mathcal{G}_\alpha})^*(\pi)$. Consider a coordinate neighborhood $\mathcal{U} \subset M$ and the corresponding neighborhood $\mathcal{U}^\infty = (\pi_\infty^{\mathcal{G}})^{-1}(\mathcal{U}) \subset J^\infty(\pi; \mathcal{G})$. Let us define a vector field in the neighborhood $\mathcal{U}^\infty$ by setting

$$\partial_{\varphi, \mathcal{U}} = \sum_{j, \sigma, g} \widehat{g}^*(D_\sigma(\varphi^j)) \frac{\partial}{\partial p^j_{\sigma g}},$$

where φ^j is the j-th component of the restriction of the section φ to the neighborhood $\mathcal{U}^\infty$, D_σ is the composition of total derivatives corresponding to the multi-index σ. Let us show that the derivation $\partial_{\varphi, \mathcal{U}}$ in the neighborhood $\mathcal{U}^\infty$ is a $\mathcal{G}$-symmetry of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$.

The field $\partial_{\varphi, \mathcal{U}}$ is vertical. Let us prove that it commutes with any homomorphism $\widehat{g}_1^*$, $g_1 \in \mathcal{G}$. The function $\widehat{g}_1^*(x_i) = g_1^*(x_i)$ is a function on M for any $i = 1, \ldots, n$ and consequently

$$\partial_{\varphi, \mathcal{U}}(\widehat{g}_1^*(x_i)) = 0 = \widehat{g}_1^*(\partial_{\varphi, \mathcal{U}}(x_i)).$$

For the function $p^j_{\sigma g}$, using formula (6.35), we obtain

$$\eth_{\varphi,\mathcal{U}}(\widehat{g_1}^*(p^j_{\sigma g})) = \eth_{\varphi,\mathcal{U}}(p^j_{\sigma,g\circ g_1}) = (\widehat{g\circ g_1})^*(D_\sigma(\varphi^j))$$
$$= (\widehat{g_1}^* \circ \widehat{g}^*)(D_\sigma(\varphi^j)) = \widehat{g_1}^*(\eth_{\varphi,\mathcal{U}}(p^j_{\sigma g})),$$

i.e., the equality $\eth_{\varphi,\mathcal{U}}(\widehat{g_1}^*(\psi)) = \widehat{g_1}^*(\eth_{\varphi,\mathcal{U}}(\psi))$ holds for any coordinate function ψ on $\mathcal{U}^\infty$ and consequently for any function on $\mathcal{U}^\infty$.

Consider the Cartan form

$$\omega = (\widehat{g_1} \circ \pi^{\mathcal{G},\mathcal{G}_{g_1}}_\infty)^*(U_1(\psi)),$$

where $g_1 \in \mathcal{G}$, $\psi \in \mathcal{F}(\pi)$. From the commutative diagram (6.33), where $k = \infty$, $\mathcal{G}' = \{\mathrm{id}_M\}$, and $\mathcal{G} \circ \mathcal{G}_{g_1} = \mathcal{G}$, due to the fact that $\mathcal{G}$ is a semigroup and $g_1 \in \mathcal{G}$, we obtain

$$\omega = (\pi^{\mathcal{G},\{\mathrm{id}_M\}}_\infty \circ \widehat{g_1})^*(U_1(\psi)).$$

Since $(\pi^{\mathcal{G},\{\mathrm{id}_M\}}_\infty)^*$ is an embedding of the module $\Lambda^1(\pi)$ to $\Lambda^1(\pi;\mathcal{G})$, we have $\omega = \widehat{g_1}^*(U_1(\psi))$. Thus, we obtain

$$\eth_{\varphi,\mathcal{U}}(\omega) = \eth_{\varphi,\mathcal{U}}(\widehat{g_1}^*(U_1(\psi))) = \widehat{g_1}^*(\eth_{\varphi,\mathcal{U}}(U_1(\psi)))$$
$$= \widehat{g_1}^*\left(d\eth_{\varphi,\mathcal{U}}(\psi) - \sum_i \eth_{\varphi,\mathcal{U}}(D_i(\psi))\,dx_i\right).$$

Using calculations similar to those in the differential case (see Ch. 4) and the fact that $\psi \in \mathcal{F}(\pi)$, it is easy to prove the equalities

$$\eth_{\varphi,\mathcal{U}}(D_i(\psi)) = D_i(\eth_{\varphi,\mathcal{U}}(\psi)), \qquad i = 1,\dots,n.$$

Therefore,

$$\eth_{\varphi,\mathcal{U}}(\omega) = \widehat{g_1}^*\left(d\eth_{\varphi,\mathcal{U}}(\psi) - \sum_i D_i(\eth_{\varphi,\mathcal{U}}(\psi))\,dx_i\right).$$

The form $\Omega = d\eth_{\varphi,\mathcal{U}}(\psi) - \sum_i D_i(\eth_{\varphi,\mathcal{U}}(\psi))\,dx_i$ is an element of the Cartan module, since elements of this module and they only vanish on the Cartan distribution (this is easily derived from Proposition 6.4) and $D_j \lrcorner \Omega = 0$, $j = 1,\dots,n$. The mapping $\widehat{g_1}$ preserves the Cartan distribution (see formula (6.34)) and therefore the induced mapping $\widehat{g_1}^*$ preserves the Cartan module. This means that

$$\eth_{\varphi,\mathcal{U}}(\omega) = \widehat{g_1}^*(\Omega) \in \mathcal{C}\Lambda^1(\pi;\mathcal{G}) \quad \text{and} \quad \eth_{\varphi,\mathcal{U}}(\mathcal{C}\Lambda^1(\pi;\mathcal{G})) \subset \mathcal{C}\Lambda^1(\pi;\mathcal{G}).$$

Since the restriction of the field $\eth_{\varphi,\mathcal{U}}$ to the subalgebra $\mathcal{F}_0(\pi|_{\mathcal{U}})$ is the derivation $\varphi^1\partial/\partial p^1_0 + \cdots + \varphi^m\partial/\partial p^m_0$, the field $\eth_{\varphi,\mathcal{U}}$ is taken to the section $\varphi|_{\mathcal{U}}$ by the mapping (6.43). Let now $\mathcal{U},\mathcal{U}' \subset M$ be two coordinate neighborhoods in M. Then the restrictions of the fields $\eth_{\varphi,\mathcal{U}}$ and $\eth_{\varphi,\mathcal{U}'}$ to the algebra $\mathcal{F}_0(\pi|_{\mathcal{U}\cap\mathcal{U}'})$ coincide. Thus, by Proposition 6.7, we obtain that these field coincide in the neighborhood $(\pi^{\mathcal{G}}_\infty)^{-1}(\mathcal{U}\cap\mathcal{U}')$. Thus, similar to the differential case, any section φ of the bundle $(\pi^{\mathcal{G}}_k)^*(\pi)$ determines the field $\eth_\varphi \in \mathrm{sym}^{\mathcal{G}}\mathcal{C}(\pi)$. Therefore, we proved the following result:

THEOREM 6.8. *Any $\mathcal{G}$-symmetry of the Cartan distribution on $J^\infty(\pi;\mathcal{G})$ is of the form $\eth_\varphi$, where $\varphi \in \mathcal{F}(\pi,\pi;\mathcal{G})$. The algebra $\mathrm{sym}^{\mathcal{G}}\mathcal{C}(\pi)$ is identified with the module $\mathcal{F}(\pi,\pi;\mathcal{G})$.*

Similar to the differential case, the derivations $\eth_\varphi$, $\varphi \in \mathcal{F}(\pi, \pi; \mathcal{G})$, are called *evolutionary derivations*. They determine evolution of sections of the bundle π governed by evolution equations

$$\frac{\partial u^j}{\partial t} = \varphi^j\left(x, \ldots, g^*\left(\frac{\partial^{|\sigma|} u^l}{\partial x^\sigma}\right), \ldots\right), \qquad j = 1, \ldots, m.$$

The field $\eth_\varphi$, $\varphi \in \mathcal{F}(\pi, \pi; \mathcal{G}')$, may be understood both as a field on $J^\infty(\pi; \mathcal{G}')$, and as a field on any jet space $J^\infty(\pi; \mathcal{G})$, where $\mathcal{G}$ is a semigroup of mappings such that $\mathcal{G}' \subset \mathcal{G}$. If it is necessary to stress that the field $\eth_\varphi$ acts on $J^\infty(\pi; \mathcal{G})$, we shall use the notation $\eth_\varphi^{\mathcal{G}}$.

PROPOSITION 6.9. *Assume that $\mathcal{G}'$ is a subsemigroup in the semigroup $\mathcal{G}$ and let $\varphi \in \mathcal{F}(\pi, \pi; \mathcal{G}')$. Then the field $\eth_\varphi^{\mathcal{G}}$ projects to the filed $\eth_\varphi^{\mathcal{G}'}$ under the mapping $\pi_\infty^{\mathcal{G}, \mathcal{G}'}$.*

PROOF. The induced mapping $(\pi_\infty^{\mathcal{G}, \mathcal{G}'})^*$ embeds the algebra $\mathcal{F}(\pi; \mathcal{G}')$ to $\mathcal{F}(\pi; \mathcal{G})$. From the definition of the fields $\eth_\varphi^{\mathcal{G}}$ and $\eth_\varphi^{\mathcal{G}'}$ it follows that they coincide on the algebra $\mathcal{F}(\pi; \mathcal{G}')$. Hence, at any point $\theta \in J^\infty(\pi; \mathcal{G})$, for an arbitrary function $f \in \mathcal{F}(\pi; \mathcal{G}')$ we have

$$((\pi_\infty^{\mathcal{G}, \mathcal{G}'})_*(\eth_\varphi^{\mathcal{G}}\big|_\theta))(f) = \eth_\varphi^{\mathcal{G}}\big|_\theta((\pi_\infty^{\mathcal{G}, \mathcal{G}'})^*(f)) = \eth_\varphi^{\mathcal{G}'}\big|_{\theta'}(f),$$

where $\theta' = \pi_\infty^{\mathcal{G}, \mathcal{G}'}(\theta) \in J^\infty(\pi; \mathcal{G}')$. Consequently, $(\pi_\infty^{\mathcal{G}, \mathcal{G}'})_*(\eth_\varphi^{\mathcal{G}}\big|_\theta) = \eth_\varphi^{\mathcal{G}'}\big|_{\theta'}$. $\qquad\square$

From Proposition 6.9 it follows in particular, that $\mathcal{G}$-invariant symmetries of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$ correspond to classical symmetries of the Cartan distribution on $J^\infty(\pi)$ for any semigroup $\mathcal{G}$. These symmetries will be called *classical symmetries* of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$.

The *Jacobi bracket* $\{\varphi, \psi\}$ of two sections $\varphi, \psi \in \mathcal{F}(\pi, \pi; \mathcal{G})$ is defined by the formula

$$\eth_{\{\varphi, \psi\}} = [\eth_\varphi, \eth_\psi].$$

The equality $\{\varphi, \psi\} = \eth_\varphi(\psi) - \eth_\psi(\varphi)$ holds for the Jacobi bracket, which in local coordinates rewrites as

$$\{\varphi, \psi\}^j = \sum_{\alpha, \sigma, g}(\widehat{g}^*(D_\sigma(\varphi^\alpha))\frac{\partial \psi^j}{\partial p_{\sigma g}^\alpha} - \widehat{g}^*(D_\sigma(\psi^\alpha))\frac{\partial \varphi^j}{\partial p_{\sigma g}^\alpha}), \quad j = 1, \ldots, m.$$

EXERCISE 6.9. Prove the last two equalities.

The formula

$$l_\psi(\varphi) = \eth_\varphi(\psi)$$

defines the *universal linearization operator* l_ψ of the nonlinear boundary differential operator Δ_ψ, $\psi \in \mathcal{F}(\pi; \mathcal{G})$. The operator l_ψ acts from $\mathcal{F}(\pi, \pi; \mathcal{G})$ to $\mathcal{F}(\pi, \pi; \mathcal{G})$ and its properties are similar to those in the differential case (see Ch. 4). In addition, the formula

$$l_{\widehat{g}^*(\psi)} = \widehat{g}^* \circ l_\psi$$

is easily proved.

EXERCISE 6.10. State and prove the above mentioned properties of the operator l_ψ.

6.6. Higher symmetries of boundary differential equations. Let us first introduce the notion of the infinite prolongation for boundary differential equations. Let $\mathcal{E} \subset J^k(\pi; \mathcal{G})$ be an equation of order k, $\mathcal{G}$ be a semigroup. The l-*prolongation* of the equation $\mathcal{E}$ is called the set $\mathcal{E}^{(l)} \subset J^{k+l}(\pi; \mathcal{G})$ consisting of points $\theta_{k+l} = \{[s_g]_{g(x)}^{k+l}\}_{g \in \mathcal{G}}$ such that all points $\widehat{g}(\theta_k)$, $g \in \mathcal{G}$, where $\theta_k = \pi_{k+l,k}^{\mathcal{G}}(\theta_{k+l})$, lie in $\mathcal{E}$ while the graph of the section $j_k(\{s_g\})$ is tangent to the equation $\mathcal{E}$ with order $\geq l$ at all points $\widehat{g}(\theta_k)$, $g \in \mathcal{G}$.

REMARK 6.4. The set $\mathcal{E}^{(0)}$ always contains in $\mathcal{E}$, though, contrary to the differential case, may not coincide with $\mathcal{E}$. Besides, the equation $\mathcal{E}$ may not be invariant with respect to the mappings $\widehat{g}$, $g \in \mathcal{G}$, but any prolongation $\mathcal{E}^{(l)}$ of this equation, including $\mathcal{E}^{(0)}$, is invariants with respect to these mappings.

If the equation $\mathcal{E}$ is determined by the system

$$G_j\left(x, \ldots, g^*\left(\frac{\partial^{|\sigma|} s^l}{\partial x^\sigma}\right), \ldots\right) = 0, \qquad j = 1, \ldots, r,$$

where s^l, $l = 1, \ldots, m$, are the components of the section $s \in \Gamma(\pi)$, $|\sigma| \leq k$, $g \in \mathcal{G}$, then its zero prolongation $\mathcal{E}^{(0)}$ is described by the system

$$G_j\left(g_1^*(x), \ldots, (g \circ g_1)^*\left(\frac{\partial^{|\sigma|} s^l}{\partial x^\sigma}\right), \ldots\right) = 0, \qquad j = 1, \ldots, r, \quad g_1 \in \mathcal{G},$$

while the l-prolongation $\mathcal{E}^{(l)}$ is given by the equations

$$\widehat{g_1}^*(D_\sigma(G_j)) = 0, \qquad j = 1, \ldots, r, \quad g_1 \in \mathcal{G}, \quad |\sigma| \leq l,$$

where D_σ is the composition of total derivative corresponding to the multi-index σ. Thus, to obtain all consequences of the equation $\mathcal{E} \subset J^k(\pi; \mathcal{G})$ we must use both the total derivative operators D_i, $i = 1, \ldots, n$, and the action of homomorphisms $\widehat{g_1}^*$, $g_1 \in \mathcal{G}$.

EXERCISE 6.11. Show that if the equation $\mathcal{E} \subset J^k(\pi; \mathcal{G})$ is given by a boundary differential operator $\Delta \colon \Gamma(\pi) \to \Gamma(\xi)$, then the l-prolongation $\mathcal{E}^{(l)} \subset J^{k+l}(\pi; \mathcal{G})$ corresponds to the $(l, \mathcal{G})$-prolongation $\Delta_l^{\mathcal{G}} = j_l^{\mathcal{G}} \circ \Delta \colon \Gamma(\pi) \to \Gamma(\xi_l^{\mathcal{G}})$ of this operator.

Similar to the differential case (see Ch. 4), the set $\mathcal{E}^{(l+1)}$ projects to the set $\mathcal{E}^{(l)}$ by the mapping $\pi_{k+l+1,k+l}^{\mathcal{G}}$ for any l. The inverse limit of the tower of equations $\mathcal{E}^{(l)}$, $l \geq 0$, together with the projections $\pi_{k+l+1,k+l}^{\mathcal{G}}$, is called the *infinite prolongation* of the equation $\mathcal{E}$ and is denoted by $\mathcal{E}^\infty$. A point $\theta = [s]_x^{(\infty, \mathcal{G})} \in J^\infty(\pi; \mathcal{G})_0$ belongs to the set $\mathcal{E}^\infty$ if and only if the Taylor series of the section s at the points $g(x) \in M$, $g \in \mathcal{G}$, satisfy the equation $\mathcal{E}$. Therefore, points of the set $\mathcal{E}^\infty \cap J^\infty(\pi; \mathcal{G})_0$ are natural to be called *formal solutions* of the boundary differential equation $\mathcal{E}$.

The set of restrictions to $\mathcal{E}^{(l)}$ of smooth functions on the enveloping space $J^{k+l}(\pi; \mathcal{G})$ will be denoted by $\mathcal{F}_l(\mathcal{E})$. For any $l \geq 0$ the embedding $\mathcal{F}_l(\mathcal{E}) \subset \mathcal{F}_{l+1}(\mathcal{E})$ is defined. Elements of the set $\mathcal{F}(\mathcal{E}) = \bigcup_{l=0}^\infty \mathcal{F}_l(\mathcal{E})$ will be called smooth functions on the equation $\mathcal{E}^\infty$. The set $\mathcal{F}(\mathcal{E})$ is an algebra, because the sets $\mathcal{F}_l(\mathcal{E})$ are algebras.

If $\theta = \{[s_g]_{g(x)}^\infty\}_{g \in \mathcal{G}} \in \mathcal{E}^\infty$, then the graph of the section $j_\infty(\{s_g\})$ is tangent to the set $\mathcal{E}^\infty$ at the point θ. Hence, by Proposition 6.1, the Cartan plane $\mathcal{C}_\theta$ at the point $\theta \in \mathcal{E}^\infty$ lies in the tangent space $T_\theta(\mathcal{E}^\infty)$. The distribution $\mathcal{C}_\theta(\mathcal{E}) = \mathcal{C}_\theta$, $\theta \in \mathcal{E}^\infty$, is called the *Cartan distribution* on $\mathcal{E}^\infty$. From this definition it follows that integral manifolds of the distribution $\mathcal{C}(\mathcal{E})$ are integral manifolds of the Cartan

distribution on $J^\infty(\pi; \mathcal{G})$. Besides, since any prolongation of the equation $\mathcal{E}$ is invariant with respect to any mapping $\widehat{g}$, $g \in \mathcal{G}$, this condition is also satisfied by the infinite prolongation $\mathcal{E}^\infty$. From here and from Proposition 6.6 we obtain the following statement:

PROPOSITION 6.10. *The graph of a section* $\pi^\mathcal{G}_\infty\big|_{\mathcal{E}^\infty}$ *is a maximal integral manifold of the Cartan distribution on* $\mathcal{E}^\infty$ *invariant with respect to any mapping* $\widehat{g}$, $g \in \mathcal{G}$, *if and only if it is of the form* $j_\infty(s)(M)$, *where* $s \in \Gamma(\pi)$ *is a solution of the equation* $\mathcal{E}$.

The restriction of the mapping $\widehat{g}$, $g \in \mathcal{G}$, to $\mathcal{E}^\infty$ will be also denoted by $\widehat{g}$.

Since the planes of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$ at points of $\mathcal{E}^\infty$ coincide with the planes of this distribution on $\mathcal{E}^\infty$, any field lying in the Cartan distribution on $J^\infty(\pi; \mathcal{G})$ is tangent to the equation $\mathcal{E}^\infty$. The set of restrictions of such fields to $\mathcal{E}^\infty$ will be denoted by $\mathcal{CD}(\mathcal{E}^\infty)$.

Similar to the differential case, let us define the module of Cartan forms on $\mathcal{E}^\infty$ as the set $\mathcal{C}\Lambda^1(\mathcal{E}^\infty) \subset \Lambda^1(\mathcal{E}^\infty)$ of 1-forms annihilating by vectors of the distribution $\mathcal{C}(\mathcal{E})$ at any point $\theta \in \mathcal{E}^\infty$. A vector field X on $\mathcal{E}^\infty$ will be called a *higher (infinitesimal) symmetry* of the equation $\mathcal{E} \subset J^k(\pi; \mathcal{G})$, if it satisfies the conditions

$$X(\mathcal{C}\Lambda^1(\mathcal{E}^\infty)) \subset \mathcal{C}\Lambda^1(\mathcal{E}^\infty), \qquad \widehat{g}^* \circ X = X \circ \widehat{g}^*$$

for any $g \in \mathcal{G}$. The set of such fields forms a Lie algebra denoted by $\mathrm{sym}(\mathcal{E})$.

The restriction to $\mathcal{E}^\infty$ of a $\mathcal{G}$-invariant symmetry of the Cartan distribution on $J^\infty(\pi; \mathcal{G})$ tangent to $\mathcal{E}^\infty$ is a symmetry of this equation. Such symmetries will be called *exterior*.

THEOREM 6.11. *If the equation* $\mathcal{E} \subset J^k(\pi; \mathcal{G})$ *is such that*

$$(\pi^{\mathcal{G}, \{\mathrm{id}_M\}}_0 \circ \pi^\mathcal{G}_{\infty,0})(\mathcal{E}^\infty) = J^0(\pi), \tag{6.44}$$

then any higher symmetry of the equation $\mathcal{E}$ *is a restriction to* $\mathcal{E}^\infty$ *of some exterior symmetry of this equation.*

PROOF. The proof of this fact is similar to the proof of Theorem 3.7 of Ch. 4. Namely, if X is a higher symmetry, then the restriction of X to $\mathcal{F}_0(\pi)$ extends to a derivation $X'_0 \colon \mathcal{F}_0(\pi) \to \mathcal{F}(\pi; \mathcal{G})$. The derivation X'_0 determines an evolutionary derivation $\eth_\varphi$ such that $\eth_\varphi\big|_{\mathcal{F}_0(\pi)} = X'_0$ (cf. the arguments after the proof of Proposition 6.7). The restriction of the field $\eth_\varphi$ to $\mathcal{E}^\infty$ coincides with X. Indeed, any coordinate function on $J^\infty(\pi; \mathcal{G})$ is obtained from elements of the algebra $\mathcal{F}_0(\pi)$ by action of fields $\widehat{Y}$, $Y \in D(M)$, and of homomorphisms $\widehat{g}^*$, $g \in \mathcal{G}$. Coordinate functions on $\mathcal{E}^\infty$ are obtained by restriction to $\mathcal{E}^\infty$ of some coordinate functions from $J^\infty(\pi; \mathcal{G})$. Since the fields $\eth_\varphi$ and X commute both with $\widehat{Y}$ and with $\widehat{g}^*$ while $\eth_\varphi\big|_{\mathcal{F}_0(\pi)} = X\big|_{\mathcal{F}_0(\pi)}$, the fields $\eth_\varphi$ and X coincide at any function on $\mathcal{E}^\infty$. $\qquad\square$

Let us derive defining equations for higher symmetries. Let the equation $\mathcal{E}$ be given by the relations

$$G_j = 0, \qquad G_j \in \mathcal{F}_k(\pi; \mathcal{G}), \qquad j = 1, \dots, r. \tag{6.45}$$

From Theorems 6.8 and 6.11 it follows that any higher symmetry of the equation $\mathcal{E}$ can be obtained by restriction to $\mathcal{E}^\infty$ of some evolutionary derivation $\eth_\varphi$, $\varphi \in$

$\mathcal{F}(\pi, \pi; \mathcal{G})$ tangent to the equation $\mathcal{E}^\infty$. Similar to the differential case (see Ch. 4), the condition of tangency of the field ϑ_φ to the equation $\mathcal{E}^\infty$ is written in the form

$$\vartheta_\varphi(G_j) = \sum_{l,\sigma,g} \psi'_{l,\sigma,g} \widehat{g}^*(D_\sigma(G_l)), \qquad j = 1, \dots, r.$$

The right-hand sides of these equalities vanish on $\mathcal{E}^\infty$ while the left-hand ones, restricted to $\mathcal{E}^\infty$, are of the form

$$\vartheta_\varphi(G_j)\big|_{\mathcal{E}^\infty} = l_{G_j}(\varphi)\big|_{\mathcal{E}^\infty} = l^{\mathcal{E}}_{G_j}(\bar{\varphi}), \qquad j = 1, \dots, r,$$

where $\bar{\varphi} = \varphi|_{\mathcal{E}^\infty}$, $l^{\mathcal{E}}_G = l_G|_{\mathcal{E}^\infty}$. Denote by $\mathcal{F}(\mathcal{E}, \pi; \mathcal{G})$ the set of restrictions to $\mathcal{E}^\infty$ of elements of the module $\mathcal{F}(\pi, \pi; \mathcal{G})$ and define for elements $\bar{\varphi}, \bar{\psi} \in \mathcal{F}(\mathcal{E}, \pi; \mathcal{G})$ the bracket

$$\{\bar{\varphi}, \bar{\psi}\}_{\mathcal{E}} \stackrel{\text{def}}{=} l^{\mathcal{E}}_\psi(\varphi) - l^{\mathcal{E}}_\varphi(\psi) = \{\varphi, \psi\}|_{\mathcal{E}^\infty},$$

where $\bar{\varphi} = \varphi|_{\mathcal{E}^\infty}$, $\bar{\psi} = \psi|_{\mathcal{E}^\infty}$.

THEOREM 6.12. *If an equation $\mathcal{E} \subset J^k(\pi; \mathcal{G})$ is given by the relations (6.45) and satisfies (6.44), then the Lie algebra* $\operatorname{sym}(\mathcal{E})$ *is isomorphic to the algebra of solutions of the system*

$$l^{\mathcal{E}}_{G_j}(\varphi) = 0, \qquad j = 1, \dots, r, \quad \varphi \in \mathcal{F}(\mathcal{E}, \pi; \mathcal{G}), \tag{6.46}$$

where the Lie algebra structure is determined by the bracket $\{\cdot, \cdot\}_{\mathcal{E}}$.

EXERCISE 6.12. Write down equations (6.46) for the system of boundary differential equations (6.17).

6.7. Examples. The generating sections φ of symmetries for integro-differential equations contain much more variables than the corresponding sections in the differential case. By this reason, to solve equation (6.46) without aid of computer is rather difficult.

When one analyzes methods of solution of equations (6.46) (see Examples in Ch. 3 and Ch. 6), one sees that the main trick is differentiation of both sides of equations over some variable. Most often, such a variable contains in the equation, but either all components of the section φ are independent of this variable, or only few of them depend on it. After differentiating, we obtain a new equation on components of φ. If the new equation is sufficiently simple and contains a derivative of the component φ^i over a variable v, then, solving this equation, we derive dependence of φ^i on v. Substituting the obtained expression for φ to equation (6.46) and differentiating the result over another variable (w, for example), we obtain new equations and repeat the procedure until the final solution of equation (6.46).

The first of the examples below was computed using a program written by means of the MAPLE V software. This program produces and analyzes system of equation (6.46), seeks for suitable variables for differentiation, differentiates the corresponding equations over these variables, and chooses the simplest equations in the obtained ones. The latter were solved by hand and the corresponding procedures are described below. When solving the equations, we used the following lemma, which may be useful in solving other equations.

LEMMA 6.13. *Let M be a smooth manifold. Then*

1. *If a submanifold N of the manifold M is given by the equations*

$$f_i = 0, \qquad i = 1, \dots, k,$$

where $f_i \in C^\infty(M)$ and the covectors $df_i|_x$, $i = 1, \dots, k$, are linear independent at all points $x \in N$, then any function f on M vanishing on N is represented in the form

$$f = \sum_{i=1}^{k} f_i a_i, \tag{6.47}$$

where a_i, $i = 1, \dots, k$, are smooth functions on M.

2. *Let a second submanifold $N_1 \subset M$ be also determined by the equations*

$$g_j = 0, \qquad j = 1, \dots, l, \quad g_j \in C^\infty(M),$$

and the covectors $df_i|_x$, $dg_j|_x$, $i = 1, \dots, k$, $j = 1, \dots, l$, be linear independent at all points $x \in N_1$. Then any smooth function f on M, vanishing both on N and on N_1 represents in the form

$$f = \sum_{i,j} f_i g_j b_{ij}, \tag{6.48}$$

where $b_{ij} \in C^\infty(M)$, $i = 1, \dots, k$, $j = 1, \dots, l$.

PROOF. The result is a consequence of [**80**, Lemma 2.1, Ch.3]. $\qquad\square$

EXAMPLE 6.7. Let us find symmetries of equation (6.12). This equation is equivalent to the system of boundary differential equations (6.17). Let us use the following notation. Let $\mathcal{G}$ the semigroup of mappings considered Example 6.4. If v is a coordinate in the fiber of the jet space with the semigroup $\mathcal{G}$, then the coordinates corresponding to the derivatives of v over x, s, and t will be denoted by v_1, v_2, and v_3 respectively. For example, the coordinate u_{13}^2 corresponds to the derivative $\partial^2 u^2 / \partial x \partial t$. By $g_{[ab]}$, where the letters a, b are either of the symbols 0, x, s, or ∞, we denote the mapping belonging to the semigroup $\mathcal{G}$ and taking the point with coordinates t, x, s to the point with coordinates (t, a, b). For example, $g_1 = g_{[x0]}$, $g_2 = g_{[0x]}$, etc. (See Example 6.4.) By $v_{[ab]}$, where v is a function on the jet space, we denote the function $g_{[ab]}{}^*(v)$. For example, $g_2{}^*(u^2) = u_{[0,x]}^2$, $g_1{}^*(\partial u^1 / \partial x) = u_{1[x0]}^1$, etc. We shall write just v instead of $v_{[xs]}$. Note that if the restriction of a function v to the section $j_k(h)$ is $f(t, x, s)$, then the restriction of the function $v_{[ab]}$ to $j_k(h)$ is $f(t, a, b)$.

EXERCISE 6.13. Describe e, f via a, b, c, d, if $g_{[ef]} = g_{[ab]} \circ g_{[cd]}$.

Using the above introduced notation, let us rewrite system (6.17) in the form

$$
\begin{aligned}
u_3^1 &= \frac{1}{2} u_{[0x]}^2 - u^1 u_{[x\infty]}^3, \quad u_2^2 = u_1^2 + K u_{[s0]}^1 u^1, \\
u_2^3 &= K u_{[s0]}^1, \quad u_{[x0]}^1 = u^1, \quad u_{[x0]}^2 = 0, \quad u_{[x0]}^3 = 0.
\end{aligned}
\tag{6.49}
$$

Linearization of system (6.49) is of the form

$$D_3(\varphi^1) = \frac{1}{2}\varphi^2_{[0x]} - u^1\varphi^3_{[x\infty]} - u^3_{[x\infty]}\varphi^1, \tag{6.50}$$

$$D_2(\varphi^2) = D_1(\varphi^2) + Ku^1_{[s0]}\varphi^1 + Ku^1\varphi^1_{[s0]}, \tag{6.51}$$

$$D_2(\varphi^3) = K\varphi^1_{[s0]}, \tag{6.52}$$

$$\varphi^1_{[x0]} = \varphi^1, \tag{6.53}$$

$$\varphi^2_{[x0]} = 0, \tag{6.54}$$

$$\varphi^3_{[x0]} = 0, \tag{6.55}$$

where D_1, D_2, D_3 are the total derivatives with respect to x, s, t respectively, $\varphi^2_{[0x]} = g_{[0x]}{}^*(\varphi^2)$, etc.

We shall solve system (6.50)–(6.55) assuming that the desired functions φ^1, φ^2, φ^3 depend on coordinates in $\pi^{\mathcal{G}}_{\infty,1}(\mathcal{E}^\infty) \subset J^1(\pi;\mathcal{G})$, i.e., on the variables

$$
\begin{aligned}
&x,\ s,\ t,\ u^1,\ u^2,\ u^3,\ u^1_1,\ u^2_1,\ u^3_1,\ u^3_1,\ u^3_3,\ u^1_{[00]},\ u^1_{[\infty 0]},\ u^1_{[s0]},\ u^2_{[0\infty]}, \\
&u^2_{[0s]},\ u^2_{[0x]},\ u^2_{[\infty\infty]},\ u^2_{[s\infty]},\ u^2_{[x\infty]},\ u^3_{[0\infty]},\ u^3_{[0s]},\ u^3_{[0x]},\ u^3_{[\infty\infty]}, \\
&u^3_{[s\infty]},\ u^3_{[x\infty]},\ u^1_{1[00]},\ u^1_{1[\infty 0]},\ u^1_{1[s0]},\ u^2_{1[0\infty]},\ u^2_{1[0s]},\ u^2_{1[0x]}, \\
&u^2_{1[\infty\infty]},\ u^2_{1[s\infty]},\ u^2_{1[x\infty]},\ u^2_{3[0\infty]},\ u^2_{3[0s]},\ u^2_{3[0x]},\ u^2_{3[\infty\infty]}, \\
&u^2_{3[s\infty]},\ u^2_{3[x\infty]},\ u^3_{1[0\infty]},\ u^3_{1[0s]},\ u^3_{1[0x]},\ u^3_{1[\infty\infty]},\ u^3_{1[s\infty]}, \\
&u^3_{1[x\infty]},\ u^3_{3[0\infty]},\ u^3_{3[0s]},\ u^3_{3[0x]},\ u^3_{3[\infty\infty]},\ u^3_{3[s\infty]},\ u^3_{3[x\infty]}.
\end{aligned}
\tag{6.56}
$$

From equation (6.53) it follows that the function φ^1 depends only on the variables (6.56) by the action of the homomorphism $g_{[x,0]}{}^*$. These variables are

$$
\begin{aligned}
&x,\ t,\ u^1_{[00]},\ u^1_{[\infty 0]},\ u^1,\ u^2_{[0\infty]},\ u^2_{[0x]},\ u^2_{[\infty\infty]},\ u^2_{[x\infty]},\ u^3_{[0\infty]}, \\
&u^3_{[0x]},\ u^3_{[\infty\infty]},\ u^3_{[x\infty]},\ u^1_{1[00]},\ u^1_{1[\infty 0]},\ u^1_1,\ u^2_{1[0\infty]},\ u^2_{1[0x]}, \\
&u^2_{1[\infty\infty]},\ u^2_{1[x\infty]},\ u^2_{3[0\infty]},\ u^2_{3[0x]},\ u^2_{3[\infty\infty]},\ u^2_{3[x\infty]},\ u^3_{1[0\infty]}, \\
&u^3_{1[0x]},\ u^3_{1[\infty\infty]},\ u^3_{1[x\infty]},\ u^3_{3[0\infty]},\ u^3_{3[0x]},\ u^3_{3[\infty\infty]},\ u^3_{3[x\infty]}.
\end{aligned}
$$

The above-mentioned differentiating program was applied to equations (6.50)–(6.52). The result is a new set of equations, etc. The equations essential for subsequent exposition will be written down and analyzed.

Step 1. Choose the variables corresponding to second order derivatives. Differentiation over these variables shows that the function φ^1 is independent of

$$
\begin{aligned}
&u^2_{1[0\infty]},\ u^2_{1[0x]},\ u^2_{1[\infty\infty]},\ u^2_{1[x\infty]},\ u^2_{3[0\infty]},\ u^2_{3[0x]},\ u^2_{3[\infty\infty]},\ u^2_{3[x\infty]}, \\
&u^3_{1[0\infty]},\ u^3_{1[0x]},\ u^3_{1[\infty\infty]},\ u^3_{1[x\infty]},\ u^3_{3[0\infty]},\ u^3_{3[0x]},\ u^3_{3[\infty\infty]},\ u^3_{3[x\infty]}.
\end{aligned}
$$

The function φ^2 is independent of the variables

$$
\begin{aligned}
&u^3_1,\ u^3_3,\ u^1_{1[s0]},\ u^1_{1[x0]},\ u^2_{1[0s]},\ u^2_{1[0x]},\ u^2_{1[s\infty]},\ u^2_{1[x\infty]},\ u^2_{3[0s]},\ u^2_{3[0x]}, \\
&u^2_{3[s\infty]},\ u^2_{3[x\infty]},\ u^3_{1[s\infty]},\ u^3_{1[x\infty]},\ u^3_{3[s\infty]},\ u^3_{3[x\infty]}.
\end{aligned}
$$

The function φ^3 is independent of

$$
u^2_1,\ u^2_3,\ u^1_{1[s0]},\ u^2_{1[0s]},\ u^2_{1[s\infty]},\ u^2_{3[0s]},\ u^2_{3[s\infty]},\ u^3_{1[s\infty]},\ u^3_{3[s\infty]}.
$$

Step 2. Differentiation of equation (6.51) over the variables u_1^3, $u_{1[0s]}^2$, $u_{1[0x]}^2$, $u_{1[s\infty]}^2$, $u_{1[x\infty]}^2$, $u_{1[s\infty]}^3$, $u_{1[x\infty]}^3$ and equation (6.52) over u_1^2, $u_{1[0s]}^2$, $u_{1[s\infty]}^2$, $u_{1[s\infty]}^3$ proves independence of the function φ^2 of the variables

$$u^3, \ u_{[0s]}^2, \ u_{[0x]}^2, \ u_{[s\infty]}^2, \ u_{[x\infty]}^2, \ u_{[s\infty]}^3, \ u_{[x\infty]}^3$$

and the function φ^3 of

$$u^2, \ u_{[0s]}^2, \ u_{[s\infty]}^2, \ u_{[s\infty]}^3.$$

Step 3. Let us now differentiate equation (6.50) over $u_{3[x\infty]}^2$, equation (6.51) over $u_{[0x]}^2$, $u_{[x\infty]}^3$, u_1^1, $u_{[0s]}^2$, $u_{1[s0]}^1$, and equation (6.52) over $u_{[0s]}^2$, $u_{1[s0]}^1$. Then we obtain the following equations:

$$\frac{\partial \varphi^1}{\partial u_{[x\infty]}^2} = 0, \tag{6.57}$$

$$\frac{\partial \varphi^2}{\partial u_3^2} K u_{[s0]}^1 - \frac{\partial \varphi^2}{\partial u_{3[0x]}^3} K_{[0x]} = 2 K u_{[s0]}^1 \frac{\partial \varphi^1}{\partial u_{[0x]}^2}, \tag{6.58}$$

$$-\frac{\partial \varphi^2}{\partial u_3^2} K u_{[s0]}^1 u^1 + \frac{\partial \varphi^2}{\partial u_{3[0x]}^3} K_{[0x]} u^1 = K u_{[s0]}^1 \frac{\partial \varphi^1}{\partial u_{[x\infty]}^3}, \tag{6.59}$$

$$-\frac{\partial \varphi^2}{\partial u^1} + \frac{\partial \varphi^2}{\partial u_1^2} K u_{[s0]}^1 - K u_{[s0]}^1 \frac{\partial \varphi^1}{\partial u_1^1} = 0, \tag{6.60}$$

$$\frac{\partial \varphi^2}{\partial u_{3[0s]}^3} K_{[0s]} + \frac{\partial \varphi^2}{\partial u_3^2} K u^1 = 2 K u^1 \frac{\partial \varphi_{[s0]}^1}{\partial u_{[0s]}^2}, \tag{6.61}$$

$$\frac{\partial \varphi^2}{\partial u_{[s0]}^1} = K u^1 \frac{\partial \varphi_{[s0]}^1}{\partial u_{1[s0]}^1}, \tag{6.62}$$

$$\frac{\partial \varphi^3}{\partial u_{3[0s]}^3} K_{[0s]} + K \frac{\partial \varphi^3}{\partial u_3^3} = 2 K \frac{\partial \varphi_{[s0]}^1}{\partial u_{[0s]}^2}, \tag{6.63}$$

$$\frac{\partial \varphi^3}{\partial u_{[s0]}^1} = K \frac{\partial \varphi_{[s0]}^1}{\partial u_{1[s0]}^1}. \tag{6.64}$$

Let us set $\varphi^4 = \partial \varphi^1 / \partial u_1^1$, $\varphi^5 = 2 \partial \varphi^1 / \partial u_{[0x]}^2$. Then the functions φ^4, φ^5 depend on the same variables as φ^1, i.e., on

$$t, \ u_{[00]}^1, \ u_{[\infty 0]}^1, \ u_{[0\infty]}^2, \ u_{[\infty \infty]}^2, \ u_{[0\infty]}^3, \ u_{[\infty \infty]}^3, u_{1[00]}^1, u_{1[\infty 0]}^1 \tag{6.65}$$

and on x, u^1, u_1^1, $u_{[0x]}^2$, $u_{[0x]}^3$, $u_{[x\infty]}^3$ (we take into account equation (6.57) here).

Multiplying equation (6.58) by u^1, adding it to equation (6.59) and cancelling $K u_{[s0]}^1$, we obtain the equation

$$\frac{\partial \varphi^1}{\partial u_{[x\infty]}^3} = -u^1 \varphi^5. \tag{6.66}$$

The right-hand side of equation (6.63) is $K\varphi_{[s0]}^5$ and consequently it is independent of the variables listed in (6.65) and of x, s, $u_{[s0]}^1$, $u_{1[s0]}^1$, $u_{[0s]}^2$, $u_{[0s]}^3$, $u_{[s\infty]}^3$. The left-hand side of this equation is independent of $u_{1[s0]}^1$, $u_{[0s]}^2$, $u_{[s\infty]}^3$. Therefore,

φ^5 cannot depend on u_1^1, $u_{[0x]}^2$, $u_{[x\infty]}^3$. From equation (6.64), by similar reasons, it follows that the function φ^4 is also independent of u_1^1, $u_{[0x]}^2$, $u_{[x\infty]}^3$.

Due to the equalities

$$\frac{\partial \varphi_{[s0]}^1}{\partial u_{1[s0]}^1} = g_{[s0]}{}^* \Big(\frac{\partial \varphi^1}{\partial u_1^1}\Big) = \varphi_{[s0]}^4,$$

equation (6.62) is rewritten in the form

$$\frac{\partial \varphi^2}{\partial u_{[s0]}^1} = K u^1 \varphi_{[s0]}^4. \tag{6.67}$$

Let us differentiate equation (6.60) over $u_{[s0]}^1$. The function $\partial \varphi^1/\partial u_1^1 = \varphi^4$ is independent of $u_{[s0]}^1$ and hence from equation (6.67) we obtain

$$\frac{\partial^2 \varphi^2}{\partial u_{[s0]}^1 \partial u_1^2} = 0 \quad \text{and} \quad \frac{\partial^2 \varphi^2}{\partial u_{[s0]}^1 \partial u^1} = K \varphi_{[s0]}^4.$$

Therefore, after differentiating equation (6.60) we obtain the equality

$$\frac{\partial \varphi^2}{\partial u_1^2} = \varphi^4 + \varphi_{[s0]}^4, \tag{6.68}$$

and equation (6.60) rewrites in the form

$$\frac{\partial \varphi^2}{\partial u^1} = K u_{[s0]}^1 \varphi_{[s0]}^4. \tag{6.69}$$

Comparing the derivative of equation (6.68) with respect to u^1 with the derivative of equation (6.69) with respect u_1^2, we obtain independence of the function φ^4 of u^1.

From equality (6.67) it can be easily deduced that the derivatives of the function φ^2 involved in equation (6.58) do not depend on $u_{[s0]}^1$. Hence, equation (6.58) splits into two equations:

$$\frac{\partial \varphi^2}{\partial u_3^2} = \varphi^5, \tag{6.70}$$

$$\frac{\partial \varphi^2}{\partial u_{3[0x]}^3} = 0. \tag{6.71}$$

In a similar way, from (6.69) one obtains that equation (6.61) splits into

$$\frac{\partial \varphi^2}{\partial u_3^2} = \varphi_{[s0]}^5, \tag{6.72}$$

$$\frac{\partial \varphi^2}{\partial u_{3[0s]}^3} = 0. \tag{6.73}$$

From equations (6.70) and (6.72) we have the equality $\varphi^5 = \varphi_{[s0]}^5$. Its left-hand side may depend on the variables (6.65) and on x, u^1, $u_{[0x]}^3$ only, while the right-hand side depends on (6.65) and s, $u_{[s0]}^1$, $u_{[0s]}^3$. Therefore, the function φ^5 depends on the variables (6.65) only.

Let us represent the functions φ^1, φ^2, φ^3 in the form

$$\varphi^1 = \varphi^4 u_1^1 + \varphi^5 (1/2 u_{[0x]}^2 - u^1 u_{[x\infty]}^3) + \varphi^6, \tag{6.74}$$

$$\varphi^2 = \varphi_{[s0]}^4 K u^1 u_{[s0]}^1 + (\varphi^4 + \varphi_{[s0]}^4) u_1^2 + \varphi^5 u_3^2 + \varphi^7, \tag{6.75}$$

$$\varphi^3 = \varphi_{[s0]}^4 K u_{[s0]}^1 + \varphi^8, \tag{6.76}$$

where φ^6, φ^7, φ^8 are some new functions. Looking through the lists of arguments of the functions φ^4, φ^5, $\varphi_{[s0]}^4$, it can be easily seen that the functions φ^6, φ^7, φ^8 may depend only on the same variables as the functions φ^1, φ^2, and φ^3 respectively. Differentiating equation (6.74) over $u_{[0x]}^2$, $u_{[x\infty]}^3$, u_1^1 and using formula (6.66) and the definitions of the functions φ^4 and φ^5, we obtain that φ^6 is independent of $u_{[0x]}^2$, $u_{[x\infty]}^3$, and u_1^1. In a similar way it is proved that φ^7 is independent of $u_{[s0]}^1$, u_1^2, u^1, u_3^2, $u_{3[0x]}^3$, $u_{3[0s]}^3$, and φ^8 does not depend on $u_{[s0]}^1$. To this end, it is sufficient to differentiate equations (6.75) and (6.76) over the corresponding variables and to use equations (6.67)–(6.71), (6.73) and (6.64).

Step 4. Differentiating equations (6.51) over u_1^2, we obtain the equality

$$g_{[s0]}{}^* \left(\frac{\partial \varphi^4}{\partial x} \right) - \frac{\partial \varphi^4}{\partial x} + K_{[0s]} u_{[s0]}^1 g_{[s0]}{}^* \left(\frac{\partial \varphi^4}{\partial u_{[0x]}^3} \right) - K_{[0x]} u^1 \frac{\partial \varphi^4}{\partial u_{[0x]}^3} = 0.$$

Since the function φ^4 is independent of u^1 and $u_{[s0]}^1$, we have

$$\frac{\partial \varphi^4}{\partial u_{[0x]}^3} = 0, \qquad g_{[s0]}{}^* \left(\frac{\partial \varphi^4}{\partial x} \right) = \frac{\partial \varphi^4}{\partial x}.$$

From the last equality it follows that the derivative $\partial \varphi^4 / \partial x$ is independent of x. Therefore, $\varphi^4 = \varphi^9 x + \widetilde{\varphi}^9$, where φ^9 are $\widetilde{\varphi}^9$ are functions depending on the variables (6.65).

Substituting the obtained representations for the functions φ^2, φ^4 to equations (6.75) and (6.54) and taking into account the equalities $u_{1[x,0]}^2 = 0$, $u_{3[x,0]}^2 = 0$ (which are implied by the fifth equation of system (6.49)), we rewrite (6.54) in the form

$$\widetilde{\varphi}^9 K_{[x0]} u^1 u_{[00]}^1 + \varphi_{[x0]}^7 = 0.$$

Since the function φ^7 is independent of u^1, the function $\varphi_{[x0]}^7$ is also independent of u^1 and consequently, $\widetilde{\varphi}^9 = 0 = \varphi_{[x0]}^7$. Hence, $\varphi^4 = \varphi^9 x$.

Step 5. From now on, when calculating $\varphi_{[x,\infty]}^3$, it is necessary to take into account the equality $g_{[x,\infty]}{}^*(K u_{[s0]}^1 s) = 0$, which holds because the improper integral in equation (6.12) converges. Taking the second derivative of equation (6.51) over the variables u^1 and $u_{[s0]}^1$, we obtain the equation

$$-\frac{\partial \varphi_{[s0]}^6}{\partial u_{[s0]}^1} + \varphi^9 g + \frac{\partial \varphi^7}{\partial u^2} - \frac{\partial \varphi^6}{\partial u^1} = 0, \tag{6.77}$$

where

$$g = 1 + \frac{1}{K} \left(s \frac{\partial K}{\partial s} + x \frac{\partial K}{\partial x} \right) \tag{6.78}$$

is a function in x and s. The function φ^6 depends linearly on u^1, since the first three summands in (6.77) are independent of u^1. Substituting the expression $\varphi^6 =$

$u^1\varphi^{10} + \varphi^{11}$ to (6.77), where φ^{10}, φ^{11} are functions depending on the same variables as φ^6, except for u^1, we obtain

$$\varphi^7 = u^2(\varphi^{10} + \varphi^{10}_{[s0]} - \varphi^9 g) + \varphi^{12},$$

where the function φ^{12} depends on the same variables as φ^7, except for u^2.

Step 6. Differentiating equation (6.50) over u^1_1, $u^3_{[x\infty]}$, $u^3_{1[x\infty]}$, $u^3_{3[x\infty]}$ and twice differentiating (6.51) over u^2 and u^1, we obtain the equalities

$$xD_3(\varphi^9) + u^1\frac{\partial\varphi^8_{[x\infty]}}{\partial u^1_1} = 0, \tag{6.79}$$

$$u^1 D_3(\varphi^5) = u^1\frac{\partial\varphi^8_{[x\infty]}}{\partial u^3_{[x\infty]}} + \varphi^{11}, \tag{6.80}$$

$$\frac{\partial\varphi^8_{[x\infty]}}{\partial u^3_{1[x\infty]}} = \varphi^9 x, \qquad \frac{\partial\varphi^8_{[x\infty]}}{\partial u^3_{3[x\infty]}} = \varphi^5, \tag{6.81}$$

$$\frac{\partial\varphi^{10}}{\partial u^3_{[0x]}} = 0. \tag{6.82}$$

Taking into account all differential consequences of equation (6.51), the latter is transformed to

$$\frac{\partial\varphi^{12}}{\partial s} - \frac{\partial\varphi^{12}}{\partial x} = 0. \tag{6.83}$$

Substituting the obtained expression for φ^2 to equation (6.54), we have $\varphi^{12}_{[x0]} = 0$. Using equation (6.83), we see that φ^{12} is a function of the following form:

$$\varphi^{12} = \widetilde{\varphi}(x + s, u^3_{[0s]}, u^3_{1[0s]}, t, u^1_{[00]}, u^1_{[\infty0]}, u^2_{[0\infty]}, u^2_{[\infty\infty]}, u^3_{[0\infty]},$$
$$u^3_{[0x]}, u^3_{[\infty\infty]}, u^1_{1[00]}, u^1_{1[\infty0]}, u^2_{1[0\infty]}, u^2_{1[\infty\infty]}, u^3_{3[0\infty]},$$
$$u^2_{3[\infty\infty]}, u^3_{1[0\infty]}, u^3_{1[0x]}, u^3_{1[\infty\infty]}, u^3_{3[0\infty]}, u^3_{3[\infty\infty]}).$$

The homomorphism $g_{[x0]}{}^*$ changes only the first three arguments of the function φ^{12}:

$$\varphi^{12}_{[x0]} = \widetilde{\varphi}(x, u^3_{[00]}, u^3_{1[00]}, t, u^1_{[00]}, u^1_{[\infty0]}, u^2_{[0\infty]}, u^2_{[\infty\infty]}, u^3_{[0\infty]},$$
$$u^3_{[0x]}, u^3_{[\infty\infty]}, u^1_{1[00]}, u^1_{1[\infty0]}, u^2_{1[0\infty]}, u^2_{1[\infty\infty]}, u^3_{3[0\infty]}, \tag{6.84}$$
$$u^2_{3[\infty\infty]}, u^3_{1[0\infty]}, u^3_{1[0x]}, u^3_{1[\infty\infty]}, u^3_{3[0\infty]}, u^3_{3[\infty\infty]}).$$

Since the arguments of the function (6.84) are independent, from the equality $\varphi^{12}_{[x0]} = 0$ we obtain $\varphi^{12} = \widetilde{\varphi} = 0$.

The summands $xD_3(\varphi^9)$ and φ^{11} in equations (6.79) and (6.80) are independent of u^1. Therefore, setting $u^1 = 0$ in these equations, we obtain

$$D_3(\varphi^9) = 0, \tag{6.85}$$

$$\varphi^{11} = 0. \tag{6.86}$$

Equations (6.86) and (6.80) imply the equality

$$\frac{\partial\varphi^8_{[x\infty]}}{\partial u^3_{[x\infty]}} = D_3(\varphi^5). \tag{6.87}$$

Equations (6.87) and (6.81) show that equation (6.50) can be simplified by setting

$$\varphi^8 = u^3 D_3(\varphi^5) - u_1^3 x \varphi^9 + u_3^3 \varphi^5 + \varphi^{13},$$

where φ^{13} is a function of the same variables as φ^8.

Step 7. At this stage, we differentiate equation (6.85) over the variables, of which the function φ^9 is independent. As a result, we obtain the equations implying that the function φ^9 is independent of the variables

$$u^1_{[\infty 0]}, \ u^2_{[0\infty]}, \ u^2_{[\infty\infty]}, \ u^3_{[0\infty]}, \ u^3_{[\infty\infty]}, \ u^1_{1[00]}, \ u^1_{1[\infty 0]}.$$

Hence, this function may depend on $u^1_{[00]}$ and t only. Thus, equation (6.85) acquires the form

$$\frac{\partial \varphi^9}{\partial t} - u^1_{[00]} u^3_{[0\infty]} \frac{\partial \varphi^9}{\partial u^1_{[00]}} = 0,$$

from where it follows that φ^9 is constant.

Step 8. Differentiating equation (6.52) over $u^1_{[s0]}$, we obtain

$$X(\varphi^{13}) = (\varphi^{10}_{[s0]} - \varphi^9 g - D_3(\varphi^5)) K, \tag{6.88}$$

where

$$X = K \frac{\partial}{\partial u^3} + K_1 \frac{\partial}{\partial u_1^3} + K_{[0s]} \frac{\partial}{\partial u^3_{[0s]}} + K_{1[0s]} \frac{\partial}{\partial u^3_{1[0s]}}, \qquad K_1 = \frac{\partial K}{\partial x}.$$

Equations (6.50), (6.52) and (6.55), together with their differential consequences, imply

$$D_3(\varphi^{10}) + \varphi^{13}_{[x\infty]} = 0, \tag{6.89}$$

$$\frac{\partial \varphi^{13}}{\partial s} = 0,$$

$$\varphi^{13}_{[x0]} = 0. \tag{6.90}$$

Thus, the function φ^{13} depends on the variables

$$u^3, \ u_1^3, \ u_3^3, \ u^3_{[0s]}, \ u^3_{1[0s]}, \ u^3_{3[0s]}, \ x, \ t, \ u^1_{[00]}, \ u^1_{[\infty 0]},$$
$$u^1, \ u^2_{[0\infty]}, \ u^2_{[0x]}, \ u^2_{[\infty\infty]}, \ u^2_{[x\infty]}, \ u^3_{[0\infty]}, \ u^3_{[0x]}, \ u^3_{[\infty\infty]},$$
$$u^3_{[x\infty]}, \ u_1^1, \ u^1_{1[00]}, \ u^1_{1[\infty 0]}, \ u^2_{1[0\infty]}, \ u^2_{1[0x]}, \ u^2_{1[\infty\infty]}, \tag{6.91}$$
$$u^2_{1[x\infty]}, \ u^2_{3[0\infty]}, \ u^2_{3[0x]}, \ u^2_{3[\infty\infty]}, \ u^2_{3[x\infty]}, \ u^3_{1[0\infty]}, \ u^3_{1[0x]},$$
$$u^3_{1[\infty\infty]}, \ u^3_{1[x\infty]}, \ u^3_{3[0\infty]}, \ u^3_{3[0x]}, \ u^3_{3[\infty\infty]}, \ u^3_{3[x\infty]}.$$

Equality (6.90) means that the function φ^{13} vanishes on the submanifold $N = \mathrm{im}\, g_{[x0]}$. In the coordinate space (6.91), the submanifold N is described as the zero set for the functions

$$u^3, u_1^3, u_3^3, u^3_{[0s]}, u^3_{1[0s]}, u^3_{3[0s]} \tag{6.92}$$

(see the last equation in system (6.49)). Therefore, from the first part of Lemma 6.13 we obtain the representation

$$\varphi^{13} = u^3 a^1 + u_1^3 a^2 + u_3^3 a^3 + u^3_{[0s]} a^4 + u^3_{1[0s]} a^5 + u^3_{3[0s]} a^6, \tag{6.93}$$

where a^i, $i = 1, \ldots, 6$, are smooth functions in the variables (6.91).

Substituting expression (6.93) to equation (6.89), we obtain the equality

$$D_3(\varphi^{10}) + u^3_{[x\infty]}a^1_{[x\infty]} + u^3_{1[x\infty]}a^2_{[x\infty]} + u^3_{3[x\infty]}a^3_{[x\infty]}$$
$$+ u^3_{[0\infty]}a^4_{[x\infty]} + u^3_{1[0\infty]}a^5_{[x\infty]} + u^3_{3[0\infty]}a^6_{[x\infty]} = 0.$$

Since the function $D_3(\varphi^{10})$ is independent of the variables

$$u^3_{[x\infty]}, \ u^3_{1[x\infty]}, \ u^3_{3[x\infty]}, \ u^3_{[0\infty]}, \ u^3_{1[0\infty]}, \ u^3_{3[0\infty]},$$

from the last equality we obtain the equation $D_3(\varphi^{10}) = 0$. Hence (see solution of equation (6.85) at Step 7 and of equations (6.82) and (6.89)), the function φ^{10} depends on x only, and we have the equality $\varphi^{13}_{[x\infty]} = 0$. The latter means that the function φ^{13} vanishes on the submanifold $N_1 = \operatorname{im} g_{[x\infty]}$. This submanifold is the zero set for the functions

$$u^3 - u^3_{[x\infty]}, \ u^3_1 - u^3_{1[x\infty]}, \ u^3_3 - u^3_{3[x\infty]},$$
$$u^3_{[0s]} - u^3_{[0\infty]}, \ u^3_{1[0s]} - u^3_{1[0\infty]}, \ u^3_{3[0s]} - u^3_{3[0\infty]}. \tag{6.94}$$

Thus, from the second part of Lemma 6.13 we obtain the representation

$$\varphi^{13} = \sum_{i,j} f_i g_j b_{ij},$$

where by f_i, $i = 1, \ldots, 6$, the functions (6.92) are denoted, g_j, $j = 1, \ldots, 6$, are the functions (6.94), and b_{ij} are smooth functions in the variables (6.91).

Substituting the obtained representation for the function φ^{13} to equation (6.88), we have at the left-hand side of this equation the following expression:

$$\sum_i f_i \left(\sum_j X(g_j b_{ij}) \right) + \sum_j g_j \left(\sum_i X(f_i) b_{ij} \right).$$

The functions at the right-hand side of (6.88) do not depend of the variables of which the functions $f_1, \ldots, f_6, g_1, \ldots, g_6$ depend. Therefore,

$$\varphi^{10}_{[s0]} - \varphi^9 g - D_3(\varphi^5) = 0. \tag{6.95}$$

The functions $\varphi^{10}_{[s0]}$ and φ^5 are independent of x. Hence, from equality (6.7) for $\varphi^9 \neq 0$ it follows that g is also independent of x. But since this function is symmetric in x and s (this is a consequence of its definition (6.78) and of symmetry of the function $K(x,s)$), the function g is also independent of s. Therefore, the product $\varphi^9 g$ is a constant. From (6.7) it follows that $\varphi^{10}_{[s0]}$ does not depend on s and consequently φ^{10} is a constant.

Introduce the notation

$$\varphi^{14} = \varphi^{10} - \varphi^9 g, \qquad \varphi^{15} = \varphi^5 - \varphi^{14}t.$$

Equation (6.7) is rewritten in the form $D_3(\varphi^{15}) = 0$. From here and from the results of Step 7 it follows that φ^{15} is also a constant. Thus,

$$\varphi^1 = \varphi^9(xu^1_1 + u^1 g) + \varphi^{14}(u^1 + tu^1_3) + \varphi^{15}u^1_3. \tag{6.96}$$

Note that if φ^1, φ^2, φ^3 is a solution of system (6.50)–(6.55), then the functions φ^2 and φ^3 are uniquely determined by the function φ^1. Indeed, the functions φ^2 and φ^3 determine evolution of the variables u^2 and u^3. The variables u^2 and u^3 are uniquely determined by the variable u^1 (see (6.49)). Therefore, it suffices to find

some functions φ^2 and φ^3 together with φ^1 satisfying system (6.50)–(6.55). Setting $\varphi^{13} = 0$, we obtain

$$
\begin{aligned}
\varphi^2 &= \varphi^9(su_2^2 + xu_1^2 + u^2g) + \varphi^{14}(2u^2 + tu_3^2) + \varphi^{15}u_3^2, \\
\varphi^3 &= \varphi^9(su_2^3 + u^3g + xu_1^3) + \varphi^{14}(u^3 + u_3^3t) + \varphi^{15}u_3^3.
\end{aligned}
\tag{6.97}
$$

It is easily seen that the functions (6.96)–(6.97) satisfy system (6.50)–(6.55) and determine the classical symmetries corresponding to the liftings of the fields

$$
\varphi^9\left(u^1 g \frac{\partial}{\partial u^1} - x\frac{\partial}{\partial x}\right) + \varphi^{14}\left(u^1 \frac{\partial}{\partial u^1} - t\frac{\partial}{\partial t}\right) + \varphi^{15}\frac{\partial}{\partial t},
$$

where φ^9, φ^{14}, φ^{15} are constants. If φ^9 does not vanish, then g is a constant, while from equation (6.78) it follows that $K(x, s)$ is a homogeneous function with degree of homogeneity $\sigma = g - 1$.

EXERCISE 6.14. Reconstruct computations of the above example in all details.

In paper [**18**] another system of boundary differential equations was studied, also equivalent to equation (6.12), but possessing a larger semigroup. The symmetry algebra for this system is the same. In paper [**19**] the result of symmetry computations for a generalization of equation (6.12) is given. The symmetries, which were searched for, were of the closest type to the above found (6.96)–(6.97). Computations were not too cumbersome in spite of much more complicated equations, and were fulfilled without computer.

REMARK 6.5. The choice of a semigroup for a given equation is not unique. One can always extend it and in some cases make it smaller (compare the semigroups for the coagulation equation (6.12) given in [**18**] and in Example 6.4). Changing the semigroup may lead to changes in the symmetry algebra (see the remark in [**19**, §5]). Thus, a problem of a semigroup choice for obtaining the largest symmetry algebra arises. It can be shown that in the differential case, extension of the semigroup consisting of one element (the identity) may lead only to the "trivial" extension of the symmetry algebra. For example, let our equation possess a symmetry $\varphi(c)$ depending on an arbitrary constant c and assume that there exists an integral (or boundary differential) variable v independent of unknown functions in the equation. For example, v is the value of a dependent variable at some point. Then the change of c to v is a new symmetry $\varphi(v)$. Suppose that the symmetry $\varphi(c)$ depends on an arbitrary function c of some independent variable z. Then, exactly in the same way, we can change c by a variable v depending on z only and obtain a new symmetry $\varphi(v)$. We call such an extension of the symmetry algebra *extension by a fake constant*. It is trivial in the following sense: in the fiber $v = c$ of our equation, the symmetries $\varphi(v)$ and $\varphi(c)$ coincide.

EXAMPLE 6.8. The Khokhlov–Zabolotskaya equation (see §5.3 of Ch. 3) possesses the symmetry

$$
\varphi = q_3 p_{2g} p_1 + 2p_g p_3 + q_3 p_{22g},
$$

where $g\colon (q_1, q_2, q_3, q_4) \mapsto (a, q_2, b, c)$, a, b, c are arbitrary numbers. This symmetry is obtained from the symmetry $f(A)$ (see (5.16) of Ch. 3) by changing the function $A(q_2)$ by p_g.

REMARK 6.6. Assume that we are interested not in all solutions of a given equation, but only in the ones satisfying additional conditions (boundary or integral). Then a semigroup extension may lead to extension of the symmetry algebra. An example of integral symmetry the reader can find in the book [**29**]. The authors are looking for symmetries of the Maxwell equations. They compute classical symmetries of the equations obtained by the Fourier transformation of the initial system. The inverse Fourier transformation delivers integral symmetries of the system consisting of the Maxwell equations and of the existence conditions for the corresponding Fourier transformation.

In conclusion, we consider an example of using the Laplace transformation for obtaining integral symmetries of integro-differential equations.

EXAMPLE 6.9. Consider equation (6.12) in the case $K(x,s) = x + s$. The Laplace transformation with respect to the variable x takes this equation to the boundary differential equation

$$\Phi_2 = \Phi_A \Phi_1 + \Phi_{1A}\Phi - \Phi\Phi_1, \tag{6.98}$$

where

$$\Phi = \Phi(p,t) = \int_0^\infty u(x,t)e^{-px}\,dx,$$

$$\Phi_A = \int_0^\infty u(x,t)\,dx = \Phi(0,t),$$

$$\Phi_{1A} = -\int_0^\infty xu(x,t)\,dx = \frac{\partial \Phi}{\partial p}(0,t),$$

Φ_1 are Φ_2 the derivatives of Φ over p and t respectively. The semigroup of this equation consists of the mappings: the identity and $A\colon (p,t) \mapsto (0,t)$. The equation for the generating function of a symmetry ψ is of the form

$$X(\psi) = \Phi_1 \widehat{A}^*(\psi) + \Phi\widehat{A}^*(D_1(\psi)) + (\Phi_{1A} - \Phi_1)\psi, \tag{6.99}$$

where $X = D_2 - \Phi_A D_1 + \Phi D_1$. The 0-prolongation of equation (6.98) is the space of variables p, t, Φ_A, Φ_{1A}, Φ, Φ_1. If ψ is a function on the 0-prolongation, then X may be considered as a vector field on the same space, and it is easy to find the first integrals of this field:

$$q_1 = \Phi_{1A}, \qquad q_2 = \Phi_A e^{-q_1 t}, \qquad q_3 = \Phi e^{-q_1 t},$$

$$q_4 = q_1 p + \Phi_A - \Phi, \qquad q_5 = \left(\frac{q_1}{\Phi_1} - 1\right)e^{q_1 t}.$$

Equation (6.99) is easily solved in this case. The general solution, modulo summands corresponding to the trivial extension of the classical symmetry algebra (see Example 6.7), is

$$\psi = (f - \widehat{A}^*(f))\frac{\Phi_1}{q_1} + \frac{\Phi_1}{q_1}e^{q_1 t}\widehat{A}^*\left(\frac{\partial f}{\partial q_5}\right), \tag{6.100}$$

where f is a function in q_1, q_2, q_3, q_4, q_5, such that $\widehat{A}^*(\partial f/\partial q_3) = 0$. Substituting $f = q_1 q_3 q_4$ to formula (6.100), we obtain

$$\psi = q_3 q_4 \Phi_1 = \Phi e^{-q_1 t}(q_1 p + \Phi_A - \Phi)\Phi_1.$$

The corresponding symmetry of equation (6.12) for $K(x,s) = x + s$ is of the form

$$\varphi = e^{-q_1 t}\left[\frac{x}{3}\int_0^x\int_0^{s_1} u(x-s_1,t)u(s_1-s_2,t)u(s_2,t)\,ds_2\,ds_1\right.$$

$$-q_1\int_0^x u(x-s,t)su_s'(s,t)\,ds - q_1\int_0^x u(x-s,t)u(s,t)\,ds$$

$$\left.-\frac{x}{2}\int_0^\infty u(s,t)\,ds\int_0^x u(x-s,t)u(s,t)\,ds\right],$$

where $q_1 = -\int_0^\infty su(s,t)ds$.

EXERCISE 6.15. Prove formula (6.100). Using this formula, find all symmetries of equation (6.12) for $K(x,s) = x + s$ containing double integrals.

From symmetries of partial differential equations towards Secondary ("Quantized") Calculus

> But science is not yet just a catalogue of ascertained facts about the universe; it is a mode of progress, sometimes tortuous, sometimes uncertain. And our interest in science is not merely a desire to hear the latest facts added to the collection, we like to discuss our hopes and fears, probabilities and expectations.
>
> Sir A. Eddington

Introduction

The pre-history of rational mechanics was the study of so-called simple mechanisms. A number of attempts to explain the whole Nature as a machine composed of these mechanisms was made in that period. The "standard schemes" and "models" of the modern quantum field theory (QFT) look much alike these simple mechanisms.

This analogy maybe clarifies the reasons of an almost common feeling that quantum field theory, in its present form, is not yet a "true" well-established theory. Below we undertake an attempt to analyze why this is so and what ingredients are to be added to the solution to get the desired crystallization.

Having this in mind, we start the paper with some general observations on genesis of long-scale theories. These introductory pages furnish our subsequent considerations with the necessary initial impulse. Following it, we eventually arrive at Secondary or, more speculatively, Quantized Differential Calculus, which seems to have some chances to provide the passage from "standard models" to the "true" theory with the necessary mathematical background.

From the very beginning, we would like to stress that Secondary Calculus is only a language with which, hopefully, QFT can be developed smoothly, i.e., without "renormalizations", "anomalies", etc. If this is the case, the fundamental problem to translate QFT systematically into Secondary Calculus remains to be carried out separately. Of course, results and experiences accumulated in the study of concrete models up to now are indispensable to this purpose.

This text is neither a review nor a research account, but a long motivation for this Secondary Calculus. We describe informally some principle ideas and results already obtained in this field and also indicate some problems and perspectives which seem promising at this moment.

It was not our intention to present here a systematic and rigorous exposition of Secondary Calculus. That would be hardly possible within the limits of such a text. So, we restrict ourselves to a general panorama, which could help the interested reader to enter the subject by consulting the bibliography. A lot of details and techniques can be found in the main text of this book (see also [**131, 135, 130**]). It should be followed by [**3, 60, 132, 117, 138, 118, 119, 57, 62, 133, 76**].

And, finally, the first "philosophical" pages of this Appendix are to be read half-seriously keeping an eye on the uncertainty principle: making more precise the meaning of words used will kill the motivating impulses we hope they emit.

1. From symmetries to conceptions

It is banal to say that every theory has its origin in rather simple things. But what are they? The word "simple" of common language incorporates many meanings. In linear approximation, they can be displayed by the following diagram:
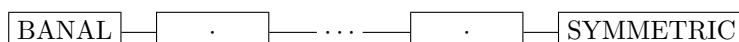
$$\boxed{\text{BANAL}}\!-\!\!\boxed{\quad\cdot\quad}\!-\!\cdots\!-\!\boxed{\quad\cdot\quad}\!-\!\!\boxed{\text{SYMMETRIC}}$$

in which the dots indicate the "intermediate states". In other words, we find enough reasons to interpret "symmetric" as "simple but not banal". Details are just obstructions to symmetry. So, the models manifesting only the essence of the phenomena in question are necessarily symmetric. Recall Euclidean geometry, Copernicus' planetary system, Newton's laws in mechanics or special relativity to illustrate this idea. Hence, we accept as the leading principle that the initial stage in the genesis of theories is the study of symmetric models. (Of course, the above remarks are applicable only to rather long-scale situations.)

Symmetry considerations replace conceptual thinking in studying symmetric models quite well. That is why they work well at first, especially for mathematically based theories, owing to the fact that "symmetry" implies "solvability" and "integrability" in this case.

At this point, the theory passes to the next stage of its development when the dominating paradigm states that everything can be composed of simple (symmetric) elements studied earlier and the only thing to be understood is how. Schematically, this period can be characterized as the time when operative concepts of the future "true" theory, not yet discovered, are substituted by their "morphemes" and when more or less mechanical mosaics of the latter replace the calculus of these conceptions. This is the reason to call this stage "morphological".

A serious deficiency of these morphological compositions is that many of them are to be corrected constantly to be in agreement with new experimental data and theoretical demands. This produces numerous perturbation-like schemes which are very characteristic to the morphological era.

Ptolemy's planetary system with its numerous epi- and hypo-cycles and quantum electrodynamics with its renormalizations illustrate this quite well. Also, one can learn from these examples that the incredibly exact correspondence to experiments is not all what is needed to be a "true" theory. Of course, there is nothing bad in using a perturbation scheme for technical purposes. But it would be hardly reasonable to erect a skyscraper on a perturbed foundation.

Darwin's selection theory seems to be applicable to this selection of concepts as well. For example, one can see many fantastic creations appearing during troubled times (for example, look at the history of QFT in recent decades). This is typical for situations when the expressive powers of the language do not correspond to the subject to be described. Summing up, we represent our idea on genesis of mathematically based theories by the scheme:

$$
\boxed{\begin{array}{c}\text{symmetric origin} \\ (\text{"beautiful times"})\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{morphological} \\ \text{era}\end{array}}
$$

$$
\Longrightarrow \boxed{\begin{array}{c}\text{conceptual self-organization} \\ (\text{"troubled times"})\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{conceptual} \\ \text{happy end}\end{array}} \quad (1)
$$
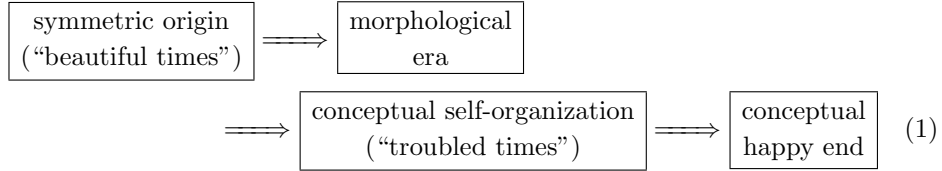
Of course, in reality, the indicated periods get mixed and this can happen, sometimes, in a very curious way. For example, nowadays synthetic geometries, typical creations of the morphological era, have almost left the land, being substituted by differential geometry. On the other hand, measure theory, being a morphological realization of the idea of integration, coexists peacefully with its future conqueror, namely, the de Rham-like cohomologies.

The passage from attempts to model the scope of new phenomena in terms of the "old", already existing mathematical language to a new one of a higher level, whose expressive potentials are just adequate to the new demands, is the essence of scheme (1). Here we use "mathematical language" in the spirit of "programming language". This enables us to take into account anthropomorphic elements present implicitly in the theories due to the fact that individual brains and scientific communities are something like computers and computer networks, respectively. The history of metric geometry from Hellenistic symmetric form based on common logic up to its modern Riemannian form based on Calculus gives an ideal illustration of the above scheme.

2. "Troubled times" of quantum field theory

Assuming scheme (1) to be true, it becomes quite clear that nowadays QFT passes through its "troubled times". Even some key words of QFT's current vocabulary, such as "renormalizations", "broken symmetries", "anomalies", "ghosts", etc., indicate a deep discrepancy of its physical content and the mathematical equipment used. Also, one can see too many Lie groups, algebras, etc., up to quantum and quasi-quantum ones, and symmetry considerations based on them, which play a fundamental role in the structure of modern QFT. This shows that the theory is not sufficiently distinct from its symmetric origin. In fact, the strongest and most obvious argument in favor of these "troubled times" comes from the perturbation type structure of the existing theory. However, the absence of real alternatives and long-time habits have reduced the value of this argument almost to zero.

We realize that the sceptic reader, even convinced of these "troubled times", will prefer to follow the current research activity in expectation of times when the aforementioned natural selection mechanisms will have accomplished their work. So, this text is mainly meant for those who would be interested to seek some possible artificial selection mechanisms, which, as is well known, work much faster.

At this point, we start to look for this "programming language" for QFT, being motivated by the above "evolution theory". Of course, the latter should be exposed

in more details to be perceived correctly. But we do not take the risk to go more in this direction, remembering the attitude toward any philosophy at the end of the "set-theoretic" epoch we are living in. Instead we invite the reader to return to this point once again after having read the whole text. Also, a development of above general ideas can be found in [**3**, Ch. 1]. In particular, there we touch such topics as: what anthropomorphic factor stands behind the idea to put set theory in the foundations of the whole mathematics and why Calculus is the language of classical physics.

3. "Linguization" of the Bohr correspondence principle

We find the initial data in the following two general postulates, which seem to be beyond doubt:

I. Calculus is the language of classical physics.
II. Classical mechanics is the limit case as $h \to 0$ of quantum mechanics (the Bohr correspondence principle).

These are our initial position and momentum, respectively.

To avoid misunderstanding, we would like to stress that the word "Calculus" is used here, and later on, in its direct sense, i.e., as a system of concepts (say, vector fields, differential forms, differential operators, jets, de Rham's, Spencer's, ... , cohomologies, etc., governed by general rules, or formulas like $d^2 = 0$, $L_X = i_X \circ d + d \circ i_X$, etc.). As we showed in [**123**], they all constitute a sort of "logic algebra" due to the fact that differential calculus can be, in fact, developed in a pure algebraic way over an arbitrary (super)commutative algebra A (see also [**60**, Ch. 1]). This algebraically constructed Calculus coincides with the standard one for smooth function algebras $A = C^\infty(M)$. Also, one can learn from this algebraic approach, and this is very important to emphasize, that there are many things to discover and to perceive in order to close this logic algebra, i.e., to get the whole Calculus. Higher-order analogs of the de Rham complexes [**128**] give such kind of an example.

Thus, the first postulate suggests to look for an extension of Calculus while the second one defines more precisely the direction to aim at. Having this in mind, we need to extract the mathematical essence of Bohr's correspondence principle and the following diagram illustrates how it can be done:

$$
\boxed{\begin{array}{c} \text{equations of quantum mechanics} \\ \text{(Schrödinger equations)} \end{array}} \quad \begin{array}{c} \xrightarrow{\;h \to 0\;} \\ \xleftarrow{\;\;\;\;\;\;\;\;} \\ \text{\scriptsize quantization} \end{array} \quad \boxed{\begin{array}{c} \text{equations of classical mechanics} \\ \text{(Hamilton equations)} \end{array}}
$$

$$\Downarrow \text{ "mathematization"}$$

$$
\boxed{\begin{array}{c} \text{partial differential} \\ \text{equations} \end{array}} \quad \xrightarrow{\;\text{CHAR}_Q\;} \quad \boxed{\begin{array}{c} \text{ordinary differential} \\ \text{equations} \end{array}}
$$
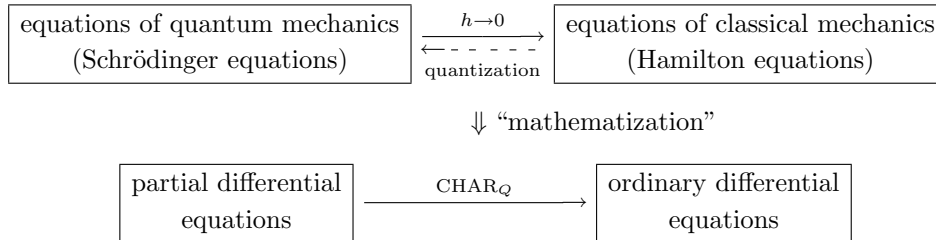
Here CHAR_Σ denotes the map which assigns to a given system of partial differential equations $\mathcal{E}$ a system $\mathcal{E}_\Sigma^0$ of ordinary equations describing how Σ-type singularities of solutions of $\mathcal{E}$ propagate. What is meant by solutions singularity types Σ and what is, in particular, the above singularity type Q will be discussed later on, see also [**124, 133, 74, 76**]. But now we shall explain what are the reasons for suspecting CHAR_Q to be behind the Bohr correspondence principle.

First, note that the mathematical background of the passage from wave to geometric optics can be naturally presented in the form $\mathcal{E} \to \mathcal{E}^0_{\mathrm{FOLD}}$, where FOLD stands for the folding type singularity of multi-valued solutions of $\mathcal{E}$ (see §9). On the other hand, multivalence of solutions is related to nonuniqueness of the Cauchy problem and, therefore, to the theory of (bi)characteristics.

REMARK 1. There exists a dual way to pass to geometric optics proposed by Luneburg [71] and based on the study of discontinuous solutions. However, the choice of Luneburg's approach instead of that we have adopted does not lead us to essential changes in our subsequent arguments.

Second, recollecting that Schrödinger discovered his famous equations proceeding from the analogy with wave-geometric optics, one can expect a similar mechanism in the passage from quantum to classical mechanics [102]. More precisely, it seems natural to hypothesize the equations of classical mechanics to be the Q-characteristic equations of the corresponding equations of quantum mechanics. These hypothetical Q-characteristic equations should play a similar role with respect to an appropriate "quantum" solution singularity type as the standard characteristic equations do with respect to the singular Cauchy problem. This hypothesis becomes almost evident in the framework of Maslov's approach to semi-classical asymptotics [83]. We refer also to the lectures by Levi-Cività [67] and the work by Racah [96], one of the first attempts to go this way.

This all motivates us to take the formula

$$\text{QUANTIZATION} = \text{CHAR}_Q^{-1} \tag{2}$$

as the leading principle and we go to seek for its consequences.

First of all, the direct attempt to extend (2) to QFT leads immediately to the problem illustrated by the following diagram:

$$
\boxed{\text{quantum fields}} \xrightarrow{h \to 0} \boxed{\text{classical fields}}
$$
$$
\Downarrow \quad \text{``mathematization''} \quad \Downarrow \tag{3}
$$
$$
\boxed{\;?\;} \xrightarrow{\text{CHAR}_Q} \boxed{\begin{array}{c}\text{partial differential}\\\text{equation}\end{array}}
$$

In other words, we have to answer the question: what kind of mathematical objects are to be placed into the left lower rectangle of (3) or, more precisely, what is the mathematical nature of the equations whose solution singularity propagation is described by means of partial differential equations? The scheme

$$
\boxed{\begin{array}{c}\text{ordinary differential}\\\text{equations}\end{array}} \xleftarrow{\text{CHAR}} \boxed{\begin{array}{c}\text{partial differential}\\\text{equations}\end{array}} \xleftarrow{\text{CHAR}} \boxed{\;?\;}
$$

motivates us to call these, yet unknown, mathematical objects *secondary quantized differential equations*.

Thus, the problem to consider next is

What are secondary quantized differential equations? $\qquad$ (4)

All the preceding discourses do not furnish us with the necessary impulse to attack it. In searching such an impulse we consider the simplest situation when a

CHAR-type mapping appears:

$$\sum_i a_i(x) \frac{\partial u}{\partial x_i} = b(x) \xrightarrow{\text{CHAR}} \{\dot{x}_i = a_i(x)\}.$$

In other words, we shall examine the passage from vector fields to ordinary differential equations making an attempt to understand what secondary ("quantized") vector fields should be.

We can profit from simplicity of this situation, which, in turn, comes from the symmetry of the context in full accordance with §1. More exactly, infinitesimal symmetries of the system $\dot{x}_i = a_i(x)$ are vector fields $Y = \sum c_i \partial/\partial x_i$ commuting with the field $X = \sum a_i \partial/\partial x_i$ and, as is well known, any vector field admits locally a lot of fields commuting with it. For our purpose it is important to observe that symmetries of the system $\dot{x}_i = a_i(x)$ are objects of the same nature as the differential operator (namely, $X = \sum a_i \partial/\partial x_i$) defining the first-order part of the initial equation $X(u) = b$. For this reason, it seems very likely that secondary quantized vector fields are identical to symmetries of partial differential equations.

So, proceeding to check this hypothesis it is wise to consider the theory of differential equations from the category theory viewpoint, which furnishes a means for drawing an analogy with the category of smooth manifolds.

4. Differential equations are diffieties

A natural generalization of the concept of infinitely prolonged equation motivated by the factorization procedure (see Ch. 3), nonlocal symmetries (see Ch. 6), etc., is given by the following

DEFINITION 1. A manifold $\mathcal{O}$ supplied with an n-dimensional distribution $\mathcal{C}$ satisfying the Frobenius complete integrability condition is called a *diffiety*[1], if it is locally of the form $\mathcal{E}^\infty$.

The integer n is called the *diffiety dimension* of $\mathcal{O}$ and is denoted by $\mathrm{Dim}\,\mathcal{O}$. Of course, it differs generally from the usual dimension of $\mathcal{O}$, which is equal to infinity as a rule.

EXAMPLE 1 (the projective diffiety). Let $E = E^{n+m}$ be a smooth $(n+m)$-dimensional manifold. The k-jet $[M]_\theta^k$ of an n-dimensional submanifold M at a point $\theta \in E$ is the class of n-dimensional submanifolds of E tangent to M with order k at $\theta \in E$. The totality of k-jets of all possible n-dimensional submanifolds M at all points $\theta \in E$ is denoted by E_n^k. If we recollect the construction of $J^k(\pi)$, $k = 0, \ldots, \infty$, we shall easily prove that

1. The set E_n^k is supplied with a natural structure of smooth manifold.
2. There exist the natural projections $E_n^{k+1} \to E_n^k$.
3. The space $J^\infty(E, n) = E_n^\infty$ defined as the inverse limit of these projections is equipped with the natural completely integrable distribution.

It is also obvious that $J^\infty(E, n)$ has locally the form $J^\infty(\pi)$ for an appropriate m-dimensional bundle π over an n-dimensional manifold and, thus, is a diffiety of dimension n.

---

[1]From "DIFFerential varIETY".

Diffieties are objects of a category called the *category of differential equation*, with the morphisms being the maps that preserve the distribution.

All natural constructions of the theory of differential equations can be extended to arbitrary diffieties with minor or no modifications. In particular, any local isomorphism $\tau\colon \mathcal{O}' \to \mathcal{O}$ is called a *covering* of $\mathcal{O}$, while a symmetry of $\mathcal{O}'$ is said to be a τ-*nonlocal symmetry* of $\mathcal{O}$. Evidently, all τ-symmetries of $\mathcal{O}$ constitute a Lie algebra for a fixed $\tau\colon \mathcal{O}' \to \mathcal{O}$, which coincides with $\mathrm{sym}\,\mathcal{O}'$. But, at first glance, it seems to be absurd to look for the commutator of two nonlocal symmetries defined on two different coverings. However, it turns out to be possible to find the desired commutator on a suitable third covering. Therefore, in order to organize all nonlocal symmetries of $\mathcal{O}$ into something like a Lie algebra one must take into consideration all coverings of $\mathcal{O}$ simultaneously.

All coverings of a given diffiety constitute in a natural way a category which we called a cobweb (see [**62**]). This construction implies many important consequences for Secondary Calculus. But at the moment this is only a beautiful perspective to be explored systematically.

We remark also that, while diffieties are analogs of affine varieties of algebraic geometry, cobwebs are analogs of fields of rational functions on them.

There are two different natural inclusions of the category of finite-dimensional manifolds to the category of differential equations:

1. $M^n \Rightarrow J^\infty(M^n, n)$.
2. $M^n \Rightarrow J^\infty(M^n, 0)$.

In the first case an n-dimensional manifold M goes to an n-Dimensional diffiety, i.e., $\dim M = \mathrm{Dim}\,M$, whereas in the second case $\mathrm{Dim}\,M = 0$. These two ways are, in a sense, dual to each other, and therefore, lead to a kind of duality in the theory of differential equations. Also, one can see that the traditional "differential" mathematics (Calculus, geometry, equations, etc.) viewed as a part of diffiety theory becomes conceptually closed only if the underlying manifolds are understood to be zero-dimensional diffieties. In other words, the standard "differential" mathematics forms zero-dimensional part of the diffiety theory. So, from this point of view it would be quite natural to suspect that the relevant mathematics necessary to quantize smoothly classical fields has not yet been discovered to a great extent.

Let us now return to the problem posed at the end of the previous section and ask ourselves: what do the notion of symmetry means when applied to zero-dimensional diffieties, i.e., to usual manifolds. Obviously, $\mathrm{sym}\,M = \mathrm{D}(M)$, where $\mathrm{D}(M)$ stands for the Lie algebra of all vector fields on M. In an arbitrary coordinate system $u^1, \dots, u^m$ on M, the standard coordinate expression

$$X = \sum_i \varphi_i(u) \frac{\partial}{\partial u^i}, \qquad u = (u^1, \dots, u^m),$$

for a vector field X on M is a particular case of formula (2.15) of Ch. 4 for evolutionary derivation

$$\vartheta_\varphi = \sum_{i,\sigma} D_\sigma(\varphi_i) \frac{\partial}{\partial u^i}.$$

Recall that to each evolutionary derivation ϑ_φ there corresponds the evolution equation (2.19) of Ch. 4

$$\frac{\partial u^i}{\partial t} = \varphi_i(x, u, \dots, u^j_\sigma, \dots), \qquad i = 1, \dots, m. \tag{5}$$

In the case $\vartheta_\varphi = X$, system (5) takes the form

$$\frac{du^i}{dt} = \varphi_i(u^1, \ldots, u^m), \qquad i = 1, \ldots, m. \tag{6}$$

But in (6) we recognize the ordinary differential equations of characteristics for the first-order partial differential operator $X = \sum_i \varphi_i \partial/\partial u^i$. Now the analogy $X \mapsto \vartheta_\varphi$, (6) $\mapsto$ (5) motivates the following principal statement:

> If $\varphi = (\varphi_1, \ldots, \varphi_m)$ is the generating function of a symmetry $\chi \in$ sym $\mathcal{E}^\infty$, then system (5) of partial differential equations can be naturally considered as the characteristic system corresponding to the operator χ.

By virtue of (3), this gives the desired solutions of the main problem (4) for first-order equations:

> Symmetries of partial differential equations are first-order secondary quantized differential operators.

This is our starting point when we look for secondary quantized differential operators of higher order.

5. Secondary ("quantized") functions

It seems natural to define higher-order secondary quantized differential operators as compositions of first-order ones. But going this way we meet immediately the following difficulty.

Recall that first-order secondary differential operators are elements of the Lie algebra sym $\mathcal{O}$. On the other hand, they are not proper differential operators but cosets (equivalence classes) of them. So the question arises: how to compose the cosets? We leave to the interested reader the task of verifying that a direct attack to this problem fails.

Another aspect of this problem can be extracted from a similar question: on what kind of objects do secondary differential operators act? No doubt, secondary differential operators should be proper operators, i.e., act on some kind of objects. The usual functions cannot be taken as such. One can see this by trying to define an action of the algebra sym $\mathcal{O}$ on $C^\infty(\mathcal{O})$. The only natural way to do this is to put $\chi(f) = X(f)$ for $\chi \in$ sym $\mathcal{O}$, $X \in \mathrm{D}_\mathcal{C}(\mathcal{O})$, $\chi = X \mod \mathcal{C}\mathrm{D}(\mathcal{O})$, and $f \in C^\infty(\mathcal{O})$. But this definition is clearly not correct. Namely, if $X_1, X_2 \in \mathrm{D}_\mathcal{C}(\mathcal{O})$ are two different vector fields and $X_1 \equiv X_2 \mod \mathcal{C}\mathrm{D}(\mathcal{O})$, then, generally, $X_1(f) \neq X_2(f)$.

However, it is clear that, linguistically, secondary operators should act on secondary functions and we shift this question by asking what they are. The following analogy will help us to answer.

Let $\Lambda^i(M)$ denote the space of differential i-forms on the manifold M. The map

$$C^\infty(M) \xrightarrow{\ d\ } \Lambda^1(M) \tag{7}$$

provides extremal problems on smooth functions on M with the "universal solution". Treating smooth manifolds as zero-dimensional diffieties we see that the analog of (7) should be a map providing variational problems for multiple integrals

with the universal solution. But this is the well-known Euler–Lagrange map:

$$
\boxed{\begin{array}{c}\text{variational functionals,}\\ \text{or "actions"}\end{array}} \xrightarrow{\ \mathsf{E}\ } \boxed{\begin{array}{c}\text{differential}\\ \text{operators}\end{array}} \tag{8}
$$

i.e., E associates with an "action" $\int_\Omega L\, dx_1 \cdots dx_n$ the left-hand side of the corresponding Euler–Lagrange equation. Therefore, this analogy between (7) and (8) suggests to adopt "actions" as secondary (or "quantized") functions. This idea is to be corrected because "actions", understood in the standard way, contain a detail, malignat for our aim and necessary to be eliminated. This detail is the explicit reference to the domain of integration Ω. So, our next problem is to find a meaning for hieroglyphs of the form $\int L\, dx_1 \cdots dx_n$ (without "Ω"!). We shall solve it by interpreting them as some kind of cohomology classes (for more motivations see [**38**]). But before that, we need some preliminaries (cf. Ch. 4).

Let $\mathcal{O}$ be a diffiety, $\operatorname{Dim}\mathcal{O} = n$. Observe that there is the horizontal de Rham complex on $\mathcal{O}$, which is constructed similarly to that on an infinitely prolonged equation. Namely, consider the submodule $\mathcal{C}\Lambda^i(\mathcal{O})$ of $\Lambda^i(\mathcal{O})$ consisting of forms whose restrictions to the contact distribution of $\mathcal{O}$ vanish:

$$
\omega \in \mathcal{C}\Lambda^i(\mathcal{O}) \iff (Y_1, \ldots, Y_i) = 0 \qquad \forall Y_1, \ldots, Y_i \in \mathcal{C}\mathrm{D}(\mathcal{O}).
$$

We put

$$
\Lambda^i_0(\mathcal{O}) = \Lambda^i(\mathcal{O}) / \mathcal{C}\Lambda^i(\mathcal{O}).
$$

Elements of $\Lambda^i_0(\mathcal{O})$ are called *horizontal differential forms* on $\mathcal{O}$. Evidently, $\Lambda^i_0(\mathcal{O}) = 0$ for $i > n$.

It is easy to see that $d(\mathcal{C}\Lambda^i(\mathcal{O})) \subset \mathcal{C}\Lambda^{i+1}(\mathcal{O})$ and, therefore, the standard differential d induces the *horizontal differential*

$$
\widehat{d}\colon \Lambda^i_0(\mathcal{O}) \to \Lambda^{i+1}_0(\mathcal{O}).
$$

Of course, $\widehat{d}^2 = 0$ and this enables us to introduce the horizontal de Rham complex of $\mathcal{O}$. Its cohomology is called the *horizontal de Rham cohomology* of $\mathcal{O}$ and is denoted by $\bar{H}^i(\mathcal{O})$, $i = 0, \ldots, n$.

Finally, we accept the following basic interpretation:

Secondary (or "quantized") functions on $\mathcal{O}$ are elements of the cohomology group $\bar{H}^n(\mathcal{O})$.

In other words, we consider the cohomology group $\bar{H}^n(\mathcal{O})$, $\operatorname{Dim}\mathcal{O} = n$, to be an analog of the smooth function algebra in Secondary Calculus.

To justify the choice made, we shall describe in coordinates the "horizontal" constructions just given, for $\mathcal{O} = J^\infty(E, n)$ (see Ch. 4 and 5). First of all, we observe that the coset of a differential form $\omega \in \Lambda^i(J^\infty(E, n))$ contains only one element of the form

$$
\rho = \sum_{1 \le k_1 < \ldots k_i \le n} a_{k_1, \ldots k_i}(x, u, \ldots, u^j_\sigma, \ldots)\, dx_{k_1} \wedge \cdots \wedge dx_{k_i},
$$

where $a_{k_1, \ldots, k_i} \in C^\infty(J^\infty(E, n))$. The characteristic feature of such a form is that the differentials du^j, du^j_σ do not enter into its coordinate expression. Therefore, the module $\Lambda^i_0(J^\infty(E, n))$ can be identified locally with the module of the form of this type.

Under the identification made, the horizontal de Rham differential $\widehat{d}$ looks as

$$\widehat{d}\rho = \sum_{s,k_1,\ldots,k_i} D_s(a_{k_1,\ldots k_i})\, dx_s \wedge dx_{k_1} \wedge \cdots \wedge dx_{k_i},$$

where D_s is the s-th total derivative. In particular, every horizontal $(n-1)$-form can be uniquely represented as

$$\rho = \sum_i (-1)^{i-1} a_i\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n, \qquad a_i \in C^\infty(J^\infty(E,n)),$$

and

$$\widehat{d}\rho = \mathrm{div}(A)\, dx_1 \wedge \cdots \wedge dx_n,$$

where $A = (a_1,\ldots,a_n)$ and $\mathrm{div}(A) = \sum_i D_i(a_i)$. Also, horizontal n-forms look as

$$L(x,u,\ldots,u^j_\sigma,\ldots)\, dx_1 \wedge \cdots \wedge dx_n, \qquad L \in C^\infty(J^\infty(E,n)),$$

and one can recognize Lagrangian densities in them. So, we see that the horizontal cohomology $\bar{H}^n(J^\infty(E,n))$ can be identified locally with the linear space of equivalence classes of Lagrangian densities on $J^\infty(E,n)$ with respect to the following relation:

$$L_1 \sim L_2 \iff L_1 - L_2 = \mathrm{div}(A) \text{ for some } A.$$

On the other hand, actions $\int_\Omega L_i\, dx_1 \wedge \cdots \wedge dx_n$, $i = 1,\ 2$, are equivalent in the sense that they lead to identical Euler–Lagrange equations if and only if $L_1 \sim L_2$. This is independent of the choice of Ω. For these reasons, it is natural to identify hieroglyphs $\int L\, dx_1 \wedge \cdots \wedge dx_n$ with n-dimensional horizontal cohomology classes.

We conclude this section by noting that similar reasonings are valid for arbitrary diffieties as well.

6. Higher-order scalar secondary ("quantized") differential operators

First of all, we must justify the above adopted definition of secondary functions demonstrating that first-order differential operators really act naturally on them. In other words, we must look for a natural action of the algebra $\mathrm{sym}\,\mathcal{O}$ on the space $\bar{H}^n(\mathcal{O})$. This, however, can be done straightforwardly.

First, note that if $X \in \mathrm{D}_\mathcal{C}(\mathcal{O})$, $\omega \in \mathcal{C}\Lambda^i(\mathcal{O})$, and L_X denotes the Lie derivative along X, then $L_X(\omega) \in \mathcal{C}\Lambda^i(\mathcal{O})$, as it results from definitions and the fact that

$$\mathcal{C}\Lambda^i(\mathcal{O}) = \mathcal{C}\Lambda^1(\mathcal{O}) \wedge \Lambda^{i-1}(\mathcal{O}), \qquad i > 1.$$

This allows us to define the Lie derivative on horizontal forms by passing to quotients:

$$L_X : \Lambda^i_0(\mathcal{O}) \to \Lambda^i_0(\mathcal{O}).$$

Next, let

$$\chi \in \mathrm{sym}\,\mathcal{O}, \qquad \chi = X \mod \mathcal{C}\mathrm{D}(\mathcal{O}) \qquad \text{for } X \in \mathrm{D}_\mathcal{C}(\mathcal{O}),$$
$$\theta \in \bar{H}^n(\mathcal{O}), \qquad \theta = \omega \mod \widehat{d}\Lambda^{n-1}_0(\mathcal{O}) \quad \text{for } \omega \in \Lambda^n_0(\mathcal{O}).$$

We define now the action of χ on θ by putting

$$\chi(\theta) = L_X(\omega) \mod \widehat{d}\Lambda^{n-1}_0(\mathcal{O}) \in \bar{H}^n(\mathcal{O}).$$

The fact that the action is well defined is implied by the following two statements:

1. $L_Y(\omega) \in \widehat{d}\Lambda^{n-1}_0(\mathcal{O})$ if $\omega \in \Lambda^n_0(\mathcal{O})$ and $Y \in \mathcal{C}\mathrm{D}(\mathcal{O})$.

2. $L_X \circ \widehat{d} = \widehat{d} \circ L_X$.

They both are direct consequences of definitions.

Now we see that the above definition of secondary functions correlates nicely with other "secondary" constructions and, therefore, can serve as an example in proceeding to more complicated "secondary" notions. For example, let us observe that we have succeeded to define a correct action of one quotient (namely, $\operatorname{sym}\mathcal{O} = \mathrm{D}_{\mathcal{C}}(\mathcal{O})\big/\mathcal{C}\mathrm{D}(\mathcal{O})$) on another (namely, $\bar{H}^n(\mathcal{O}) = \Lambda_0^n(\mathcal{O})/\widehat{d}\Lambda_0^{n-1}(\mathcal{O})$) owing to:

1. The quotient $\mathrm{D}_{\mathcal{C}}(\mathcal{O})$ consists of first-order differential operators which act on the $C^\infty(\mathcal{O})$-module $\Lambda_0^n(\mathcal{O})$ leaving $\widehat{d}\Lambda_0^{n-1}(\mathcal{O})$ invariant.
2. The images of $\Lambda_0^n(\mathcal{O})$ under the action of first-order operators belonging to $\mathcal{C}\mathrm{D}(\mathcal{O})$ are contained in $\widehat{d}\Lambda_0^{n-1}(\mathcal{O})$.

We shall get the necessary generalization to higher-order secondary differential operators simply by replacing the words "first-order" by "k-th order" in 1 and 2 above. More exactly, let $\mathrm{Diff}_k(\Lambda_0^n(\mathcal{O}))$ denote the $C^\infty(\mathcal{O})$-module of all ("usual") differential operators of order $\le k$ acting on $\Lambda_0^n(\mathcal{O})$ and put

$$\overline{\mathrm{Diff}}_k(\mathcal{O}) = \{\, \Delta \in \mathrm{Diff}_k(\Lambda_0^n(\mathcal{O})) \mid \Delta(\widehat{d}\Lambda_0^{n-1}(\mathcal{O})) \subset \widehat{d}\Lambda_0^{n-1}(\mathcal{O}) \,\},$$

$$\underline{\mathrm{Diff}}_k(\mathcal{O}) = \{\, \Delta \in \mathrm{Diff}_k(\Lambda_0^n(\mathcal{O})) \mid \Delta(\Lambda_0^n(\mathcal{O})) \subset \widehat{d}\Lambda_0^{n-1}(\mathcal{O}) \,\}.$$

Then the space of all *scalar secondary* ("*quantized*") *operators* of order $\le k$ on $\mathcal{O}$ is defined to be the quotient

$$\mathfrak{Diff}_k(\mathcal{O}) = \overline{\mathrm{Diff}}_k(\mathcal{O})\big/\underline{\mathrm{Diff}}_k(\mathcal{O}). \tag{9}$$

Of course, every secondary operator $\Delta \in \mathfrak{Diff}_k(\mathcal{O})$ can be understood as an operator

$$\Delta \colon \bar{H}^n(\mathcal{O}) \longrightarrow \bar{H}^n(\mathcal{O}),$$

acting on secondary functions. In fact, if

$$\Delta = \delta \quad \mathrm{mod}\ \underline{\mathrm{Diff}}_k(\mathcal{O}) \qquad \text{for } \delta \in \overline{\mathrm{Diff}}_k(\mathcal{O}),$$

$$\Theta = \omega \quad \mathrm{mod}\ \widehat{d}\Lambda_0^{n-1}(\mathcal{O}) \qquad \text{for } \omega \in \Lambda_0^n(\mathcal{O}),$$

then the horizontal cohomology class

$$\Delta(\Theta) = \delta(\omega) \quad \mathrm{mod}\ \widehat{d}\Lambda_0^{n-1}(\mathcal{O})$$

is well defined, i.e., does not depend on the choice of the representatives δ and ω.

For $\mathcal{O} = J^\infty(E, n)$, thus defined secondary differential operators admit the following coordinate description. Operators of the form

$$\sum_{s=1}^{k} \sum_{\substack{i_1,\dots,i_s \\ \sigma_1,\dots,\sigma_s}} a_{\sigma_1,\dots,\sigma_s}^{i_1,\dots,i_s} \frac{\partial^s}{\partial u_{\sigma_1}^{i_1} \dots \partial u_{\sigma_s}^{i_s}} + \mathrm{const},$$

where $\sigma_1, \dots, \sigma_s$ are multi-indices, are called *vertical* (with respect to chosen coordinate system). Then it can be proved that every coset

$$\Delta = \delta \quad \mathrm{mod}\ \underline{\mathrm{Diff}}_k(J^\infty(E, n)) \in \mathfrak{Diff}(J^\infty(E, n))$$

for $\delta \in \overline{\mathrm{Diff}}_k(J^\infty(E, n))$, contains only one vertical operator. So, the quotient (9) representing secondary differential operators can be identified locally with the set of

all *vertical secondary operators*. These operators (of order $\leq k$) can be represented in the form

$$\vartheta_\nabla = \sum_{i=1}^{m} \sum_{\sigma} \mathcal{L}(\nabla^i) \circ \frac{\partial}{\partial p_\sigma^i}, \qquad \nabla = (\nabla^1, \ldots, \nabla^m),$$

where $\nabla^i \in \mathrm{Diff}_{k-1}(C^\infty(J^\infty(E,n)))$ are arbitrary $C^\infty(J^\infty(E,n))$-vertical operators and

$$\mathcal{L}_\sigma(\nabla^i) = [D_{i_1}, \ldots, [D_{i_s}, \nabla^i] \ldots]$$

where $\sigma = (i_1, \ldots, i_s)$. The generating operator ∇ is the higher-order analog of the generating functions for evolutionary derivations but, unlike the latter, it is not defined uniquely if $k > 1$.

Secondary operators of order > 1 are not reduced to compositions of first-order secondary operators. This fact is instructive in connection with the discussion at the beginning of §5.

We note also that an explicit description of secondary differential operators on arbitrary diffieties is a much more difficult problem.

Further details, results, and alternative views concerning secondary differential operators can be found in [**38**].

Finally, turning back to question (4), we can exhibit the simplest k-th order linear secondary (quantized) differential equation as

$$\Delta(H) = 0, \qquad \Delta \in \mathfrak{Diff}_l(\mathcal{O}), \qquad H \in \bar{H}^n(\mathcal{O}).$$

It must be emphasized, however, that these equations form a very special class of secondary differential equations. For instance, differentials $d_1^{p,q} = d_1^{p,q}(\mathcal{O})$ of the $\mathcal{C}$-spectral sequence (see Ch. 5 and §7) give us other examples of secondary quantized differential operators and, therefore, secondary differential equations. One of them looks as

$$\mathsf{E}\left(\int L\,dx\right) = 0,$$

where $\int L\,dx \in \bar{H}^n(\mathcal{O})$ and E is the Euler operator assigning to an action $\int L\,dx$ the corresponding Euler–Lagrange equation. This is due to the fact that $\mathsf{E} = d_1^{0,n}$. Note also that operators $d_1^{p,q}$ are of finite order, say $\mathrm{ord}(k)$, when restricted to elements of the k-th filtration, but $\mathrm{ord}(k) \to \infty$ when $k \to \infty$. For example, $\mathrm{ord}(k) = 2k$ for the operator E.

7. Secondary ("quantized") differential forms

In this section, we shall consider another aspect of Secondary Calculus, namely, secondary ("quantized") differential forms. What are they? This is a more difficult question than the one about secondary differential operators we have already discussed. By this and other reasons we shall omit here preliminary motivations showing, as before, how to arrive at exact definitions. However, some *a posteriori* justifications will be given.

Let $\mathcal{O}$ be a diffiety. Adopting the notations of §6, we consider the algebra

$$\Lambda^*(\mathcal{O}) = \sum_{i \geq 0} \Lambda^i(\mathcal{O})$$

of all differential forms on $\mathcal{O}$ and its ideal

$$\mathcal{C}\Lambda^*(\mathcal{O}) = \sum_{i \geq 0} \mathcal{C}\Lambda^i(\mathcal{O}).$$

Denote by $\mathcal{C}^k\Lambda^*(\mathcal{O})$ the k-th power of this ideal. Then all ideals $\mathcal{C}^k\Lambda^*(\mathcal{O})$ are stable with respect to the exterior differential d. So, we get the following filtration:

$$\Lambda^*(\mathcal{O}) \supset \mathcal{C}\Lambda^*(\mathcal{O}) \supset \mathcal{C}^2\Lambda^*(\mathcal{O}) \supset \cdots \supset \mathcal{C}^k\Lambda^*(\mathcal{O}) \supset \cdots$$

of the de Rham complex of $\mathcal{O}$ by its subcomplexes $\{\mathcal{C}^k\Lambda^*(\mathcal{O}), d\}$. The ideals $\mathcal{C}^k\Lambda^*(\mathcal{O})$ are naturally graded,

$$\mathcal{C}^k\Lambda^*(\mathcal{O}) = \sum_{s \geq 0} \mathcal{C}^k\Lambda^{k+s}(\mathcal{O}),$$

where

$$\mathcal{C}^k\Lambda^{k+s}(\mathcal{O}) = \mathcal{C}^k\Lambda^*(\mathcal{O}) \cap \Lambda^{k+s}(\mathcal{O}).$$

Thus, as in Ch. 5 for infinitely prolonged equations, we can construct the $\mathcal{C}$-spectral sequence

$$E_r(\mathcal{O}) = \sum_{p,q} E_r^{p,q}(\mathcal{O}), \qquad d_r = \sum_{p,q} d_r^{p,q},$$

of the diffiety $\mathcal{O}$.

If $\mathrm{Dim}\,\mathcal{O} = n$, then all nontrivial terms of $E_r^{p,q}(\mathcal{O})$ are situated in the region $p \geq 0$, $0 \leq q \leq n$, i.e., as in the case of infinitely prolonged equations.

DEFINITION 2. Elements of $E_1(\mathcal{O})$ are called *secondary* ("*quantized*") *differential forms* on the diffiety $\mathcal{O}$.

Some reasons in favor of this interpretation are as follows. Let M be a finite-dimensional manifold considered as a zero-dimensional diffiety (see §4). Then the term E_1 of its $\mathcal{C}$-spectral sequence degenerates to one line

$$0 \longrightarrow \Lambda^0(M) \xrightarrow{d_1^{0,0}} \Lambda^1(M) \xrightarrow{d_1^{1,0}} \Lambda^2(M) \longrightarrow \cdots \longrightarrow \Lambda^m(M) \longrightarrow 0.$$

Moreover, $d_1^{p,0} = d$. In other words, we see that the de Rham complex of M coincides with the (generally) nontrivial part of the first term of its $\mathcal{C}$-spectral sequence.

We can observe that standard constructions and formulas connecting "usual" vector fields and differential forms are also valid for their secondary ("quantized") analogs. For instance, the inner product operator of secondary vector fields ("symmetries") with secondary differential forms as well as corresponding Lie derivatives are well defined. Moreover, they are related by means of the secondary analog of the infinitesimal Stokes formula

$$L_X = i_X \circ d + d \circ i_X,$$

in which the exterior differential d is to be replaced by its secondary analog, i.e., by $d_1(\mathcal{O})$.

Finally, we remark that secondary differential forms are bigraded objects unlike the "usual" ones, which are only monograded. The reason is clearly seen from the above diagrams. This is an illustration of the fact that secondary objects are richer and more complicated structures than their "primary" analogs. The same idea can be expressed alternatively by saying that the "usual" (or "primary") mathematical objects are degenerate forms of the secondary ones. This statement can be

also viewed as the following mathematical paraphrase by the Bohr correspondence principle:

$$\boxed{\text{Secondary Calculus}} \xrightarrow{\text{Dim}\rightarrow 0} \boxed{\text{Calculus}}$$

The cobweb theory (see §4) allows us to give an exact meaning to "Dim $\rightarrow 0$". This is because the Dimension (not dimension!) is an $\mathbb{R}$-valued function in the framework of this theory.

We saw in Ch. 5 how the $\mathcal{C}$-spectral sequence works. Let us make some further remarks on the term E_2. In the case of infinitely prolonged equations, $E_2(\mathcal{E}^\infty)$ consists of characteristic classes of bordisms composed of solutions of $\mathcal{E}$. The standard "differential characteristic classes" theories can be obtain in this way under a suitable choice of $\mathcal{E}$ (see [**138, 121**]). This approach leads, however, to finer characteristic classes, for instance, *special characteristic classes*. We illustrate this topic for solutions of the (vacuum) Einstein equations, or Einstein manifolds. Let $\mathcal{E}$ be the Einstein system on a manifold M. Then it is possible to show that

$$\mathcal{E}^\infty \big/ \operatorname{Diffeo}(M) = (\mathcal{E}')^\infty,$$

where $\operatorname{Diffeo}(M)$ is the diffeomorphism group of M acting naturally on $\mathcal{E}^\infty$ and $\mathcal{E}'$ is a certain system of partial differential equations. In fact, $\mathcal{E}'$ does not depend on M and, therefore, its solutions are just diffeomorphism classes of Einstein manifolds. The corresponding characteristic classes are elements of $E_2((\mathcal{E}')^\infty)$.

8. Quantization or singularity propagation? Heisenberg or Schrödinger?

In the preceding pages, we have reached the coasts of "terra incognita", i.e., diffieties and Secondary Calculus on them, whose existence was predicted by the linguistic version of the Bohr correspondence principle as formulated in §3. Being the exact analog of algebraic geometry for partial differential equations, this branch of pure mathematics deserves to be explored systematically, maybe much more than algebraic geometry itself and independently of the possible physical applications that stimulated the expedition. Later on we shall discuss briefly some other topics related to Secondary Calculus. But now it would be timely to reexamine how close to the solution of the quantization problem for quantum fields we are after having got Secondary Calculus at our disposal.

It should be stressed from the very beginning that the passage to the "linguistic" version of the Bohr principle inevitably costs us the loss of its original physical context. On the other hand, the accumulated experience in Secondary Calculus convinces us that every *natural* construction in the classical Calculus has its secondary analog, which can be found by means of a more or less regular procedure. So, one can expect to deduce fundamental QFT equations by "secondarizing" sample situations in which both the source and the target of the Bohr principle belong to the area of classical Calculus.

Evidently, quantum mechanics of particles is exactly such a sample due to the fact that the Bohr correspondence principle here starts from differential (Schrödinger) equations and finishes also at differential (Hamilton) equations. However, the Bohr principle in this case is to be reinterpreted exclusively in terms of Calculus to become secondarizable. This is the key point.

The desired reinterpretation is not obvious and, in particular, should not be based on the "$h \rightarrow 0$" trick, formal series on H, deformations, Hilbert spaces, and

similar things. We accept formula (2) to be the first approximation. Then our approach to QFT can be summarized as

$$\text{FIELD QUANTIZATION} = \text{CHAR}^{-1}_{\mathcal{S}(Q)},$$

where $\mathcal{S}$ stands for "secondarization". Hence, the question to be answered first is: what is the solution singularity type (or types) outlined in §3?

The last problem belongs to the theory of solution singularities of partial differential equations, which has not been elaborated enough up to now to provide us with the immediate answer. So, we postpone the direct attack for the future and limit ourselves here to a quick trip through the theory of some special solution singularities called *geometric*. In addition, the reader can conceive from this model more precise ideas on the general theory as well as more detailed motivation for formula (2). But first we shall make some remarks of a historical nature.

As it is well known, two different approaches, one by Heisenberg and the other by Schrödinger, were at the origin of quantum mechanics. In modern terms, the first is based on a formal noncommutative deformation of the commutative algebra of classical observables while the latter proceeds from an analogy with optics. Both were proclaimed and even proved equivalent and this is just the point we would like to doubt now. Namely, it seems that a more exact formulation of this equivalence theorem would be:

> *The Schrödinger point of view becomes equivalent to the Heisenberg one after being reduced appropriately.*

Below some brief general justifications for this assertion are given and the reader is asked not to confuse "approach" with "picture" in what follows.

First, the Heisenberg approach is "programmed" in the language of operator algebras while that by Schrödinger is in Calculus. The former is nonlocalizable in principle and this is its great disadvantage in what concerns applications to fundamental (nontechnical) problems of physics. In particular, the passage from one space-time domain to another cannot be expressed in terms of this language only (see, e.g., [**39**]). But, evidently, fundamental physical theories, both at classical and quantum levels, must be localizable in this sense by their nature. On the other hand, Calculus is the only localizable language due to the fact that localizable operators are just differential ones.

Second, in the Heisenberg approach classical mechanics appears to be a limit case of quantum mechanics or, vice versa, the latter is viewed to be a noncommutative deformation of the former. In particular, this means that they both are treated to be things of the same nature differing from each other by a parameter. This is not so in the framework of the Schrödinger approach. In fact, as it follows from the general mathematical background of the passage from wave to geometric optics, the latter appears to be a particular aspect of the former. So, applying the analogy between quantum mechanics and optics discovered by Schrödinger one can conclude that

> *classical mechanics is a particular aspect of quantum mechanics.*

In this connection it would be relevant to note that Plank's constant is a true constant and, therefore, "$h \to 0$" can serve as a trick but not as a ground-stone of the theory.

Thus these are, shortly, the reasons in favor of the Schrödinger alternative. On the other hand, it is clearly seen that it had no chances to be realized mathematically

in the building period of quantum electrodynamics and other quantum field theories. So, the Heisenberg alternative remained, due to its formality and abstraction, the only possible way for progress of these theories. This was its invaluable historical merit and it seems to be going to be exhausted now. Finally, we add that this text can be regarded also as an attempt to provide the Schrödinger approach with the mathematical tools which are necessary to extend it to QFT.

9. Geometric singularities of solutions of partial differential equations

In this section we present geometric singularities of solutions of (nonlinear) partial differential equations and some general results on them relevant to our discussion of the quantization problem. Some examples illustrating the general theory and, in particular, the mechanism connecting wave and geometric optics are collected in the next section.

Solution singularities which we called geometric arise naturally in the context of the theory of multi-valued solutions of (nonlinear) partial differential equations. There are different ways to realize strictly the idea of multivalence and we choose the one based on the notion of R-manifold. This is as follows.

Recall (see Ch. 3 and 4) that a submanifold $W \subset J^k(E, n)$, $0 \le k \le \infty$, is called *integral*, if $T_\theta W \subset \mathcal{C}_\theta$ for every $\theta \in J^k(E, n)$ ($\mathcal{C}_\theta$ stands for the Cartan plane at θ). An integral submanifold W is called *locally maximal*, if no open part of it belongs to another integral submanifold of greater dimension.

DEFINITION 3. A locally maximal n-dimensional integral submanifold of $J^k(E, n)$ is said to be an *R-manifold*.

In particular, the graphs $L_{(k)} = M_f^k$ of jets of sections of a bundle $\pi \colon E \to M$ are R-manifolds. They are characterized by the following two properties:

1. $L_{(k)}$ is a locally maximal integral manifold.
2. The restriction of the projection

$$\pi_{k,k-1} \colon J^k(E, n) \to J^{k-1}(E, n)$$

   to $L_{(k)}$ is an immersion.

So, omitting property 2, we get the multi-valued analogs of submanifolds $L_{(k)}$, i.e., R-manifolds.

REMARK 2. Recall that there exist different types of locally maximal integral submanifolds of $J^k(E, n)$ which differ from each other by their dimensions. One of these types is formed by fibers of the projection $\pi_{k,k-1}$. These are integral submanifolds of the greatest possible dimension.

Informally, R-manifolds can be treated, generally, as nonsmooth n-dimensional submanifolds of $E = J^0(E, n)$ whose singularities can be resolved by lifting them to a suitable $J^k(E, n)$.

Now we define a *multi-valued solution* of a partial differential equation $\mathcal{E} \subset J^k(E, n)$ to be an R-submanifold, say W, belonging to one of its prolongations $\mathcal{E}^{(s)} \subset J^{k+s}(E, n)$, $0 \le s < \infty$.

If $W \subset J^k(E, n)$ is an R-manifold, then its *singular* (or *branch*) *points* are defined to be the singular points of the projection $\pi_{k,k-1} \colon J^k(E, n) \to J^{k-1}(E, n)$ restricted to W (see Figure 1).

We stress here that W is a smooth (=nonsingular) submanifold of $J^k(E, n)$ and the adjective "singular" refers to the projection $\pi_{k,k-1}$.
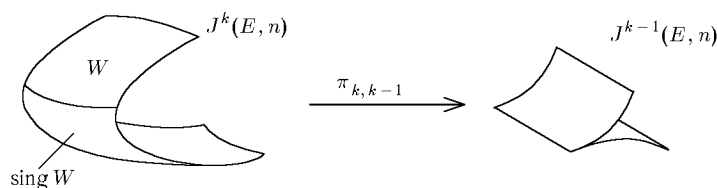
FIGURE 1

A very rich and interesting structural theory stands behind these simple definitions. This cannot be reduced to the standard singularity (or catastrophe) theory. On the contrary, the latter is a particular degenerate case of the former.

We start with a classification of geometric singularities, which is, of course, the first structural problem to be considered. According to "the general principles" we have to classify s-jets of R-manifolds in $J^k(E, \pi)$ for a prescribed integer s under the group of contact transformations of this jet space. The simplest case $s = 1$ is sufficient for our purposes.

Let $W \subset J^k(E, n)$ be an R-manifold and $\theta \in \operatorname{sing} W$. The subspaces of the tangent space $T_\theta J^k(E, n)$ which are of the form $T_\theta W$ are called *singular R-planes* (at θ). So, our problem is to classify singular R-planes.

Let $P = T_\theta W$ be a singular R-plane at θ. The subspace P_0 of P that consists of vectors annihilated by $(\pi_{k,k-1})_*$ is called the *label of P*. It turns out that singular R-planes are equivalent if and only if their labels are equivalent. So, the classification problem in question is reduced to the label classification problem. We define the *type* of a (singular or not) R-plane to be the dimension of its label:

$$\operatorname{type} P = \dim P_0, \qquad 0 \le \operatorname{type} P \le n.$$

Obviously, $\operatorname{type} P = 0$ if and only if P is nonsingular.

EXAMPLE 2. Branched Riemannian surfaces are identical to multi-valued solutions of the classical Cauchy–Riemann equation. Let W be one of them. Then the set $\operatorname{sing} W$ consists of a number of isolated points, say θ_α. In this case, $\operatorname{type} P_\alpha = 2$ for $P_\alpha = T_{\theta_\alpha} W$.

The final result of the label classification is as follows [140]:

THEOREM 1. *Label equivalent classes of geometric singularities are in one-to-one correspondence with isomorphic classes of unitary commutative $\mathbb{R}$-algebras, so that the dimension of a label is equal to that of the algebra corresponding to it.*

Recall that every unitary commutative finite-dimensional algebra splits into a direct sum of algebras $\mathbb{F}_{(k)}$, $k = 1, 2, \ldots$, where $\mathbb{F}_{(k)}$ denotes the unitary $\mathbb{F}$-algebra generated by one element ξ such that $\xi^k = 0$, $\xi^{k-1} \ne 0$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Such a splitting is not unique but the multiplicity numbers showing how many times a given algebra $\mathbb{F}_{(k)}$ enters it do not depend on the splitting. So, these multiplicity numbers determine completely the isomorphism class of the algebra in question.

Below we speak of A-type geometric singularities referring to the commutative algebra A corresponding to it by the above theorem.

EXAMPLE 3. Since the only one-dimensional $\mathbb{R}$-algebra is $\mathbb{R}$ itself, there exists only one label type of geometric singularities with one-dimensional label. This type is realized by R-manifolds projected to the manifold of independent variables

as foldings. By this reason, it is denoted by FOLD. The standard theory of characteristic covectors takes a natural part in the FOLD-singularity theory.

EXAMPLE 4. There are just three isomorphic classes of two-dimensional unitary commutative algebras, namely, that of $\mathbb{C}$, $\mathbb{R}_{(2)}$, and $\mathbb{R} \oplus \mathbb{R}$, where (see above) $\mathbb{R}_{(2)} = \{\, 1, \xi \mid \xi^2 = 0 \,\}$. For equations with two independent variables, $\mathbb{C}$-type geometric singularities look as ramification points of Riemannian surfaces and as $(n-2)$-dimensional families of such ones for n independent variables. In four-dimensional space-time, a $\mathbb{C}$-type singularity can be viewed as a vortex around a moving curve. This sets fire to suspicion that $\mathbb{C}$-singularities could play an important role in the future turbulence theory.

The next question that arises immediately when studying concrete equations is:

> *What label type geometric singularities does a given system of partial differential equations admit?*

This is an essentially algebraic problem which we illustrate with the following examples to omit the general discussion.

EXAMPLE 5. A system of partial differential equations admits FOLD-type singularities only if it admits nonzero characteristic covectors. For instance, solutions of elliptic equations do not admit FOLD-singularities.

EXAMPLE 6. Let $\mathcal{E}$ be a second-order scalar differential equation with two independent variables. Then it admits only one of the three types of two-dimensional singularities mentioned above. This is the $\mathbb{C}$-type for elliptic equations, the $\mathbb{R}_{(2)}$-type for parabolic equations, and the $\mathbb{R} \oplus \mathbb{R}$-type for hyperbolic equations.

A more delicate problem is to describe submanifolds of the form $\mathrm{sing}_\Sigma W$ for multi-valued solutions W of a given differential equation and a given solution singularity type Σ. Here $\mathrm{sing}_\Sigma W \subset W$ stands for the submanifold of Σ-singular points of W. In other words, we are interested in determining the shape of Σ-singularities admitted by a given equation.

The solution of this problem can be sketched as follows: Let a label solution singularity type Σ be fixed; then it is possible to associate with a given system of partial differential equations $\mathcal{E}$ another system $\mathcal{E}_\Sigma$ such that submanifolds of the form $\mathrm{sing}_\Sigma W$, W being a multi-valued solution of $\mathcal{E}$, satisfy $\mathcal{E}_\Sigma$ and, conversely, every solution of $\mathcal{E}_\Sigma$ is of the form $\mathrm{sing}_\Sigma W$ for (possibly formal) multi-valued solutions of $\mathcal{E}$.

If $\mathcal{E}$ is of n independent variables, then $\mathcal{E}_\Sigma$ is of $n - s$ independent variables, where s is the dimension of the label Σ. The construction of equations $\mathcal{E}_\Sigma$ is not simple enough to be reproduced here. Instead, in the next section we exhibit some examples from which the reader can conceive an idea of them. Informally speaking, if $\mathcal{E}$ describes a physical substance, say a field or a continuous medium, then $\mathcal{E}_\Sigma$ describes behavior of a certain kind of singularities of this substance, that can be characterized by the label singularity type Σ. In the case when $\mathcal{E}$ refers to independent space-time variables the equation $\mathcal{E}_\Sigma$ describes propagation of Σ-type singularities in the substance in question.

Denote by CHAR_Σ the functor that associates the equation $\mathcal{E}_\Sigma$ with a given equation $\mathcal{E}$. The problem:

> *To what extend does the behavior of singularities of a given type of*
> *physical system determine the system itself?*

is evidently of a fundamental importance and the search for the domain of invertibility of the functor CHAR_Σ is maybe the most significant aspect of it. The following result gives an instructive example of this nature.

THE FOLD-RECONSTRUCTION THEOREM. *Every hyperbolic system of partial differential equations $\mathcal{E}$ is determined completely by the associated system $\mathcal{E}_{\text{FOLD}}$.*

In other words, to write down explicitly a hyperbolic system $\mathcal{E}$ it is sufficient to know the system $\mathcal{E}_{\text{FOLD}}$.

The above theorem can be reformulated by saying that the functor CHAR_Σ is invertible on the class of hyperbolic equations. On the other hand, this functor is not invertible on the class of elliptic equations due to the fact that $\mathcal{E}_{\text{FOLD}}$ is empty for any elliptic $\mathcal{E}$.

The general "singularity reconstruction problem" we are discussing may have various flavors depending on the chosen, not necessary geometric, solution singularity type. For instance, the classical problem of fields and sources can be viewed as its particular case. Another remarkable example can be found in the history of electrodynamics. Observing that the elementary laws of electricity and magnetism such as that by Coulomb or Faraday describe the behavior of some kind of singularities of electromagnetic fields, we see Maxwell's equations to deliver a solution to the corresponding singularity reconstruction problem.

The importance of multi-valued solution theory comes in evidence also due to its relations with Sobolev–Schwartz theory of generalized solutions of linear partial differential equations. These relations are based on the observation that one can get a generalized solution of a given linear differential equation simply by summing up branches of multi-valued solution. As a matter of fact, the procedure assigning the generalized solution to a given multi-valued one is more delicate than a simple summation and is based on the choice of a de Rham type cohomology theory and a suitable class of test functions. Maslov-type characteristic classes then arise as obstructions to perform this procedure and their nature depends on the cohomology theory chosen (see [**74, 133, 76**]).

It is worth stressing that generalized solutions assigned to multi-valued ones with no FOLD-singularities are, in fact, smooth, i.e., not properly generalized functions. This correlates nicely with the well-known fact that generalized solutions of elliptic equations are exhausted by smooth solutions, i.e., single-valued ones, while such equations admit nontrivial multi-valued solutions (say, branched Riemannian surfaces for the Cauchy–Riemann equation) with non-FOLD-type singularities. These and other similar facts show multi-valued solutions to be a satisfactory substitution for generalized ones for nonlinear differential equations, where generalized solutions cannot even be defined. Moreover, the former are a finer tool in the framework of the linear theory also.

Further details and results on the topics touched upon in this section the reader finds in the book [**60**] and in the lecture [**133**]. For a systematic exposition see [**140**]. Many other interesting aspects of solution singularity theory are presented in the recent review by V. Lychagin [**76**].

Multi-valued solutions were introduced in the work [**124**], followed by a technically simple but instructive work [**63**] by A. Krishchenko. Afterwards, a significant

series of works by V. Lychagin appeared. Unfortunately, these names almost exhaust the list of contributions in this field. For a full bibliography see [**60, 76, 140**][2].

10. Wave and geometric optics and other examples

In this section we illustrate the generalities of the previous one with some simple examples taken from [**69**]. We enter here neither into technical details nor into interpretations of exhibited equations, referring the reader to [**69**].

10.1. Σ-characteristic equations. Let $\pi\colon E \to M$ be a fiber bundle, $\mathcal{E} \subset J^k(\pi)$ be a system of differential equations, and Σ be a label solution singularity type. The Σ-*characteristic system* of Σ is the system of differential equations whose solutions are of the form $\pi_k(\mathrm{sing}_\Sigma W)$, where $\pi_k\colon J^k(\pi) \to M$ is the natural projection and W is a multi-valued solution of $\mathcal{E}$.

Denote the Σ-characteristic equation of $\mathcal{E}$ by $\mathcal{E}_\Sigma^0$ and observe that the whole system $\mathcal{E}_\Sigma$ is obtained by adding to $\mathcal{E}_\Sigma^0$ some other equations called *complementary*. If $\mathcal{E}$ refers to independent space-time variables, then $\mathcal{E}_\Sigma^0$ governs motions of Σ-singularities locuses of the physical system in question while the complementary equations describe the evolution of the internal structures of Σ-singularities.

Classical characteristic equations, whose theory was initiated by Hugoniot and then developed systematically by Hadamar (see [**40**]), arise naturally in the study of uniqueness of the initial data problem. As we have already mentioned, the uniqueness problem is included in the theory of FOLD-singularities. So, it is not surprising that FOLD-characteristic equations coincide with classical ones. We recommend the works [**67, 66, 96**], in which first attempts to apply classical characteristic equations to quantum mechanics and relativity were made.

The coordinate-wise representation of FOLD-characteristic equations looks as follows. Let the basic equation $\mathcal{E}$ be given by

$$F_j(x, u, \dots, u_\sigma^i, \dots) = 0, \qquad j = 1, \dots, l,$$

where $F_j \in \mathcal{F}_k(\pi)$. Introduce the *characteristic matrix* of $\mathcal{E}$ to be

$$\mathcal{M}_F = \begin{pmatrix} \sum_{|\sigma|=k} \partial F_1/\partial u_\sigma^1 p^\sigma & \dots & \sum_{|\sigma|=k} \partial F_1/\partial u_\sigma^m p^\sigma \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_{|\sigma|=k} \partial F_l/\partial u_\sigma^1 p^\sigma & \dots & \sum_{|\sigma|=k} \partial F_l/\partial u_\sigma^m p^\sigma \end{pmatrix},$$

where $p^\sigma = p_1^{i_1} \cdot \ldots \cdot p_n^{i_n}$ for $\sigma = (i_1, \dots, i_n)$. The FOLD-characteristic equation becomes trivial, i.e., $0 = 0$, for $l < m$. If $l \geq m$, we get the FOLD-characteristic equation $\mathcal{E}_{\mathrm{FOLD}}^0$ of Σ-singular locuses representable in the form $x_n = \varphi(x_1, \dots, x_{n-1})$ by substituting $\partial\varphi/\partial x_i$ for p_i, $i = 1, \dots, n-1$, and -1 for p_n in $\mathcal{M}_F$ and then equating to zero all m-th order minors of the matrix so obtained.

REMARK 3. Strictly speaking, the above procedure is valid only for formally integrable $\mathcal{E}$.

10.2. Maxwell's equations and geometric optics. Consider the vacuum Maxwell equations (="wave optics"):

$$\mathrm{div}\, E = 0, \qquad \mathrm{curl}\, E = -\frac{1}{c}\frac{\partial H}{\partial t},$$

$$\mathrm{div}\, H = 0, \qquad \mathrm{curl}\, H = \frac{1}{c}\frac{\partial E}{\partial t}.$$

---

[2]See also a recent paper by A. Givental [**34**].

In this case, $n = 4$, $m = 6$, $l = 8$. So, the characteristic matrix is 8×6 rectangular and a direct computation shows that all its sixth-order minors are of the form

$$\lambda \left(p_1^2 + p_2^2 + p_3^2 - \frac{1}{c^2} p_4^2 \right)^2,$$

with $\lambda = 0$ or $\pm p_i p_j$. So, putting $x_4 = t$ we see that the equation $\mathcal{E}_{\mathrm{FOLD}}^0$ coincides with the standard eikonal equation

$$\left(\frac{\partial \varphi}{\partial x_1} \right)^2 + \left(\frac{\partial \varphi}{\partial x_2} \right)^2 + \left(\frac{\partial \varphi}{\partial x_3} \right)^2 = \frac{1}{c^2}. \tag{10}$$

In such a way we obtain the interpretation of this well-known fact in terms of the solution singularity theory. However, this gives us something more, namely, the complementary equations that compose, together with the eikonal equation, the whole system $\mathcal{E}_{\mathrm{FOLD}}$. They look as follows:

$$\operatorname{div} h_E = \operatorname{grad} \varphi \cdot \operatorname{curl} h_H,$$
$$\operatorname{curl} h_E + \operatorname{div} h_H \cdot \operatorname{grad} \varphi = \operatorname{grad} \varphi \times \operatorname{curl} h_H.$$

Here h_E and h_H are singular values, i.e., values on the singular surface $t = \varphi(x_1, x_2, x_3)$, of the electric and magnetic fields, respectively.

10.3. On the complementary equations. It is obvious from the procedure of §10.1 that very different equations can have the same characteristic equation. For example, the eikonal equation (10) is also the characteristic equation for the Klein–Gordon equation

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \Delta u - m^2 u = 0.$$

So, it is not possible to reconstruct the original equation knowing only its characteristic equation. In view of the Reconstruction Theorem of §9, the only information one needs for the reconstruction is contained exactly in the complementary equations. Therefore, an independent and direct physical interpretation of quantities in these equations would allow one to make up information that lacks to solve the corresponding singularity reconstruction problem. One can see now that this *singularity interpretation problem* becomes very important. For example, a solution of this problem for continuous media would provide us with a regular method

> *to deduce equations governing a given continuous medium proceeding from observation of how a given type (or types) of singularities propagate in it.*

This would be an attractive alternative to the present phenomenological status of mechanics of continuous media.

It is clear that quantization "*à la* Schrödinger" can also be treated as such a kind of interpretation problem. In this context the Hamilton–Jacobi equations of classical mechanics considered as Q-characteristic equations are to be completed by suitable complementary equations. It is natural to think that the standard formal quantization methods "*à la* Heisenberg" cover just the remaining gap of these hypothetical complementary equations.

10.4. Alternative singularities via the homogenization trick. The classical, i.e., FOLD-characteristic equation for the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \Delta \psi - V\psi = 0, \tag{11}$$

and for singular locuses given in the form $t = x_4 = \varphi(x_1, x_2, x_3)$, is

$$\left(\frac{\partial \varphi}{\partial x_1}\right)^2 + \left(\frac{\partial \varphi}{\partial x_2}\right)^2 + \left(\frac{\partial \varphi}{\partial x_3}\right)^2 = 0.$$

This demonstrates that geometric singularities are not adequate for the correspondence between quantum and classical mechanics. For the hypothetical "quantum" singularity type (see §3) the Q-characteristic equation $\mathcal{E}_Q^0$ should be

$$\frac{\partial \varphi}{\partial t} - \frac{1}{2m} \sum_{i=1}^{3} \left(\frac{\partial \varphi}{\partial x_i}\right)^2 - V = 0. \tag{12}$$

It is possible, however, to interpret (12) as the classical characteristic equation for the "homogenized" Schrödinger equation

$$\frac{\partial^2 \widetilde{\psi}}{\partial t \partial s} - \frac{1}{2m} \sum_{i=1}^{3} \frac{\partial^2 \widetilde{\psi}}{\partial x_i^2} + V\frac{\partial^2 \widetilde{\psi}}{\partial s^2} = 0, \tag{13}$$

in five-space with coordinates x_1, x_2, x_3, t, s assuming that singular locuses are given in the form $s - \varphi(x_1, x_2, x_3, t) = 0$. On the other hand, (13) reduces to (11) on the functions

$$\widetilde{\psi} = \psi(x, t) \exp\left(\frac{i}{\hbar} s\right). \tag{14}$$

This motivates us to define Q-singularities as the reduction of FOLD-singularities on the functions (14). This is not, however, very straightforward and we refer the reader to [**69**] for some results of this approach.

10.5. $\mathbb{R}_{(k)}$-characteristic equations. In this subsection, some analogs of the Hamilton–Jacobi equation for extended (i.e., not point-like) singular locuses are exhibited. For simplicity, we chose the wave equation

$$\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

as a basis. Since $\mathbb{R}_{(1)} = $ FOLD, the $\mathbb{R}_{(1)}$-characteristic equation coincides with the standard eikonal equation (10).

For $k = 2$ and the singularity locuses given by

$$x_2 = \varphi(s, t), \qquad x_3 = \psi(s, t) \quad \text{with } s = x_1,$$

the $\mathbb{R}_{(2)}$-characteristic equation looks as

$$\left(\frac{\partial \psi}{\partial t}\frac{\partial \varphi}{\partial s} - \frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial s}\right)^2 + \left(\frac{\partial \varphi}{\partial t}\right)^2 + \left(\frac{\partial \psi}{\partial t}\right)^2 - c^2\left(\frac{\partial \varphi}{\partial s}\right)^2 - c^2\left(\frac{\partial \psi}{\partial s}\right)^2 - c^2 - 0.$$

Its solutions are two-dimensional surfaces tangent to the light cone.

Finally, the $\mathbb{R}_{(3)}$-characteristic equation for singularity curves of the form $x_i = x_i(t)$, $i = 1, 2, 3$, is

$$\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 = c^2.$$

For other examples, results, and discussions see [**69**].

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